

# Valuation of Equity-Indexed Annuities under Stochastic Interest Rate

**X. Sheldon Lin**

Department of Statistics, University of Toronto  
Toronto, Ontario M5S 3G3, Canada  
Tel: (416) 946-5969, fax (416) 867-5133  
sheldon@utstat.utoronto.ca

**Ken Seng Tan**

Department of Statistics and Actuarial Science,  
University of Waterloo, Waterloo, Ontario N2L 3G1, Canada  
Tel: (519) 888-4567 × 6688, fax: (519) 746-1875  
Email: kstan@uwaterloo.ca

## **Abstract**

In this paper, we consider the pricing and hedging issues of equity-indexed annuities (EIAs). Traditionally the values of the guarantees embedded in these contracts are priced by modeling the underlying index fund while keeping the interest rates constant. The assumption of constant interest rates becomes unrealistic in pricing and hedging the EIA since the embedded guarantees are often of much longer maturity. To solve this problem, we propose an economic model which has the flexibility of modeling the underlying index fund as well as the interest rates jointly. Some popular EIA are evaluated to assess the implication of the proposed model over the traditional model.

---

Both authors are grateful to the Committee for Knowledge Extension and Research of the Society of Actuaries for financial support. The authors also thank J. Marrion for providing his survey on EIAs for 99-2000.

# 1 Introduction

Insurance companies traditionally offer fixed annuities and invest their premium incomes heavily in bonds and mortgages to meet their liability obligations. While this strategy had provided a steady cash flow and performed well in the past when interest rates were relatively high and the cost of investment in the equity market was high, this investment product however has lost its attractiveness in recent years. This is in part due to the gigantic swing in the economy. The financial market in recent years is experiencing a bullish equity market and low interest rate environment. Furthermore, with the accessibility (at a relatively low cost) of other investment instruments, such as mutual funds, and the increasingly sophistication of the investors, the investors are demanding higher return than those provided by conventional annuities. All these have led actuaries to design a new type of annuity products known as equity-indexed annuities (EIAs). Ever since the first offering in 1995 by Keyport Life Insurance Co., EIAs have enjoyed increasingly popularity in both U.S. and Canada. In fact, EIAs have been called the most significant individual product development since universal life. The importance of this product is also evidenced from the rapidly growing sales that have passed over 5 billions in 1999, according to the survey conducted by the Advantage Group (Marrion, 2000a, 2000b).

EIAs appeal to investors because they not only offer some of the benefits underlying conventional annuities, they also offer participation in the stock market while limiting downside risk of the stock market. A typical EIA guarantees a minimum return (normally 3%) on a portion of the initial amount invested, which is required by nonforfeiture laws. In addition to this minimum guarantee, the annuitant receives some participation in the appreciation of a pre-determined stock index such as S&P 500. The indexing feature extends over a fixed term, typically ranging from one to ten years. There are several indexing methods for EIAs. In order of increasing sales volumes, they are Annual Reset, Point-to-Point, Annual Yield Spread, High-Water Mark, and Term Yield Spread. The index growth on an EIA with annual reset option is measured each year by comparing the index level at the beginning and the end of the year. The index growth with point-to-point

indexing is based on the growth between two time points. As in the Annual Reset method, the Annual Yield Spread method resets the index growth annually but a yield spread is deducted from the stock index. The index growth with high-water mark (also called high point or point-to-point with discrete lookback) feature is calculated to the highest index anniversary value over the entire term of the annuity. The Term Yield Spread method is similar to the Annual Yield Spread method except that a yield spread is deducted for the entire term of the EIA. In addition to the methods above, an averaging scheme is often used to calculate the index growth in order to reduce the costs of the guarantees and to be partly immunized from the market volatility.

There have been several researches on this subject. See Tiong (2000) and the references therein. It is generally assumed in these researches that the stock index and interest rates are within a Black-Scholes framework; i.e. the stock index follows a lognormal process and the interest rates are constant. In this research, we consider a more general economic model and we assume that the underlying stock index and the interest rates are stochastic and follow certain Ito processes. We present our framework in Sections 2 and 3. In Section 4, we examine the implication of our proposed framework to the conventional model by conducting a detailed analysis on the most popular type of embedded equity guarantees. Section 5 concludes the paper.

## 2 Economic Model Selections

Two crucial economical factors in the valuation and hedging of an EIA are the term structure of interest rates and the level of the stock index. The research to-date has primarily focused on modeling just one of the key variables. For example, in the Black-Scholes framework, the index is stochastic while other variables such as volatility or interest rates are constant. While these assumptions might be adequate for most options offered by the exchanges and banks, it is dangerous to extrapolate that these assumptions are also applicable to the guarantees embedded in EIAs. Most of the options offered by the exchanges and banks typically are short-dated with maturity less than one year and thus a Black-Scholes framework would provide a reasonable approximation for pricing purposes. In contrast, the embedded

guarantees associated with EIAs have maturities ranging from 1 to 10 years. It is therefore unreasonable to assume that the interest rates would remain level for such a long duration.

Our approach in this section is to jointly model the term structure of interest rates and the stock index with stochastic processes governed by stochastic differential equations. The basic model consists of a stochastic differential equation for the short term interest rate and a stochastic differential equation for the stock index. The essential feature in our model has the following properties:

- The interest rate process reproduces today's market term structure. In particular, the model reproduces the current yield curve as well as bond prices at different maturities. This is essential when we consider the hedging strategy against the liability of an EIA.
- The model incorporates the correlation between the interest rate and the stock index. This is accomplished by explicitly introducing the correlation between the diffusion processes.
- The model is mathematically or computationally tractable in the sense it can be implemented using sophisticated mathematical or numerical tools such as the Monte Carlo simulation.

We now begin with an interest rate model. For our purpose, a short rate model is considered. Let  $r(t)$  be the short rate at time  $t$ . It is assumed that the short rate process  $\{r(t)\}, 0 \leq t \leq \mathcal{T}$ , where  $\mathcal{T}$  is the time until maturity of an EIA, satisfies the following stochastic differential equation

$$dr(t) = \mu_r(t, r)dt + \sigma_r(t, r)d\hat{W}_r(t), \quad (2.1)$$

where  $\mu_r(t, r)$  and  $\sigma_r(t, r)$  are the drift and volatility of the the short rate process and  $\{\hat{W}_r(t)\}$  is a standard Brownian motion. Let  $S(t)$  be the stock index level at time  $t$  which similarly is governed by a stochastic differential equation of the form:

$$dS(t) = \mu_S(t)S(t)dt + \sigma_S(t)S(t)d\hat{W}_S(t). \quad (2.2)$$

Here  $\mu_S(t)$  represents the instantaneous rate of return of the index at time  $t$ ,  $\sigma_S(t)$  is the volatility of the index at time  $t$ , and  $\{\hat{W}_S(t)\}$  is a standard Brownian motion that is correlated with  $\{\hat{W}_r(t)\}$  with correlation coefficient  $\rho$ ; i.e.

$$\text{corr}(\hat{W}_r(t), \hat{W}_S(t)) = \rho.$$

We further assume that the volatility  $\sigma_S(t)$  is a positive deterministic function.<sup>1</sup>

In order to price the embedded guarantees with the index level  $S(t)$  and the interest rate  $r(t)$  being stochastic, we first need to identify a risk-neutral probability measure  $Q$  associated with these correlated processes. This implies that the present value process

$$V(t) = e^{-\int_0^t r(u)du} S(t) \quad (2.3)$$

is a martingale under the probability measure  $Q$ .<sup>2</sup> Using the Girsanov Theorem, it can be shown that there exists a unique probability measure, denoted again by  $Q$ , such that under  $Q$  the present value process  $\{V(t)\}$  (2.3) is a martingale. Furthermore, the short rate process  $\{r(t)\}$  and the index process  $\{S(t)\}$  satisfy the following stochastic differential equations

$$dr(t) = \mu_r(t, r)dt + \sigma_r(t, r)dW_r(t), \quad (2.4)$$

and

$$dS(t) = r(t)S(t)dt + \sigma_S(t)S(t)dW_S(t), \quad (2.5)$$

where  $W_r(t)$  and  $W_S(t)$  are again correlated standard Brownian motions with the same correlation coefficient  $\rho$ . The derivation of this result is presented in Appendix A.

---

<sup>1</sup>In our formulation, we have intentionally skipped rigorous mathematical description when we introduce these stochastic differential equations. To be more precise, it should be understood that there is an underlying filtered probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ , where  $\Omega$  is the sample space,  $\mathcal{F}$  is the associated information structure or  $\sigma$ -algebra,  $\{\mathcal{F}_t\}$  is the natural information structure or filtration generated by  $\{\hat{W}_r(t), \hat{W}_S(t)\}$ , and  $P$  is the probability measure. The probability measure  $P$  represents the probability of an event occurring in the real world and hence is often referred to as the physical measure or the  $P$ -measure. For rigorous treatment of this subject, see Lin (2001).

<sup>2</sup>A stochastic process  $\{V(t)\}$  is a martingale if, for any  $t > s$ ,  $E[V(t) | \mathcal{F}_s] = V(s)$ . See Lin (2001) for example.

Given the risk-neutral probability measure  $Q$ , we are able to value any payoff or claim contingent on the values of the index level  $S(t)$  and the interest rate  $r(t)$  using the Fundamental Theorem of Asset Pricing. For instance, if  $C(s, r, S)$ ,  $s > t$ , is the amount of a contingent payoff at time  $s$  where  $r(s) = r$  and  $S(s) = S$ , then the time- $t$  price of this contingent claim is given by

$$P(t, s) = E_Q \left[ e^{-\int_t^s r(u) du} C(s, r(s), S(s)) \mid \mathcal{F}_t \right], \quad (2.6)$$

where  $\mathcal{F}_t$  is the time- $t$  information structure. Intuitively, the time- $t$  price is simply the expected discounted payoff under the risk-neutral probability measure  $Q$  with the discounting function depends explicitly on the short rate process. As a special case, the time-0 price is then given by

$$P(0, t) = E_Q \left[ e^{-\int_0^t r(u) du} C(t, r(t), S(t)) \right]. \quad (2.7)$$

It is easy to see that using the Law of Iterated Expectations, the above expression is equivalent to

$$P(0, t) = E_Q \left[ e^{-\int_0^s r(u) du} P(s, t) \right]. \quad (2.8)$$

Note that the price function  $P(s, t)$  depends implicitly on the functions  $r(t)$  and  $S(t)$  at time  $t$  but not  $r(u)$  and  $S(u)$  for all  $u < t$ . This is because the processes  $\{r(t)\}$  and  $\{S(t)\}$  satisfy (2.4) and (2.5) and hence are Markovian. As a result, we may differentiate  $P(t, s)$  with respect to  $S(t)$  and we use the notation  $\Delta(t, s)$  to denote the resulting derivative; i.e.

$$\Delta(t, s) = \frac{\partial P(t, s)}{\partial S(t)}. \quad (2.9)$$

The quantity  $\Delta(t, s)$  may be interpreted as follows: consider a portfolio composed of stock index and a money market account that earns interest at rate  $r(t)$  at time  $t$ . Suppose further that this portfolio is rebalanced continuously over the time period  $[0, s]$  in such a way that (a) there is no money either injected into or taken out from the portfolio, and (b) the value of the portfolio at time  $t$  is equal to  $P(t, s)$ . It can be shown by the Fundamental Theorem of Asset Pricing that the construction of such a portfolio is possible under the

model specification (2.4) and (2.5). Let  $\{\phi(t), \theta(t)\}$  be the trading strategy associated with the portfolio above; i.e.  $\phi(t)$  is the amount in the money market account and  $\theta(t)$  is the number of units for the stock index at time  $t$ . This trading strategy is then self-financing. In other words, we have

$$dP(t, s) = r(t)\phi(t)dt + \theta(t)dS(t), \text{ with } \phi(t) = P(t, s) - \theta(t)S(t), \quad (2.10)$$

or equivalently

$$d[e^{-\int_0^t r(u)du}P(t, s)] = \theta(t)dV(t), \quad (2.11)$$

where  $V(t)$  is given in (2.3). Moreover,  $\theta(t) = \Delta(t, s)$ , i.e.  $\Delta(t, s)$  is the number of units of the stock index held in the replicating portfolio. Using finance terminology, the symbol  $\Delta(t, s)$  is called the delta of the contingent claim and is a measure of the sensitivity of the price to the underlying asset value.

As the guarantees in an EIA are functional of the index level and/or the interest rate, the Fundamental Theorem of Asset Pricing may apply with some modifications as we will see in the following section. We end this section by discussing two plausible choices for the interest rate models. As mentioned earlier, a basic requirement for an interest rate model is that it reproduces the current term structure of interest rates. For this purpose, we will consider two interest rate models: the extended Vasicek (1990) model (also called the Hull and White model) and the Cox, Ingersoll and Ross (1985a, 1985b) (CIR) model. In the former, the drift term  $\mu_r(t, r) = \kappa[\theta(t) - r]$  and  $\sigma_r(t, r) = \sigma_r$ , where  $\theta(t)$  is a deterministic function that will be determined by the current term structure of interest rates and  $\sigma_r$  is a positive constant. It can be shown that a closed-form solution for  $r(t)$  exists and is given by

$$r(t) = r(0)e^{-\kappa t} + \kappa \int_0^t e^{-\kappa(t-u)}\theta(u)du + \sigma_r \int_0^t e^{-\kappa(t-u)}dW_r(u). \quad (2.12)$$

It is easy to see that the short rate process  $\{r(t)\}$  is a Gaussian process, and therefore a closed-form expression for the price of a default-free zero coupon bond can be obtained. For a further analysis, see Lin (Chapter 5, 2001). One drawback of this model is that the model could generate negative short rates since  $r(t)$  is a normal random variable for each  $t$ . However, in most practical applications the probability of having a negative interest rate is

very small and hence it is still a reasonable model, especially in lieu of its tractability. The CIR model has the same drift function as in the extended Vasicek model but the volatility function is given by  $\sigma_r(t, r) = \sigma_r \sqrt{r}$ . This model always produces positive interest rates but the closed-form expression for  $r(t)$ , although exists, is quite complex. It is expressed in terms of the Bessel functions and can only be solved numerically.

In our simulation results presented in Section 4, we consider the Vasicek model together with the stock index process that has a constant volatility; i.e.  $\sigma_S(t) = \sigma_S$ .

### 3 Pricing and Hedging with Mortality Risk

In this section, we incorporate mortality risk into our analysis. We will discuss how to price a contingent claim whose payoff depends not only on certain financial assets but also on the survivability of its holder. A fundamental idea of pricing a contingent claim in the financial market using the risk-neutral probability measure is that one can perfectly replicate the payoff of the contingent claim by rebalancing a portfolio consisting of the underlying risky asset(s) and the money market account in a self-financing strategy. A necessary condition for this pricing principle to hold is that all financial assets involved must be tradable. When the payoff of a claim depends not only on risky assets but also contingents on the mortality of the holder, this condition is violated and the pricing principle no longer applies. Consequently, a perfect hedging using a self-financing trading strategy becomes impossible. In this case, we consider the risk-minimizing hedging strategy suggested by Schweizer (1994). (See also Moller (1998, 2001)). We now present this approach in greater details.

We begin by introducing necessary actuarial symbols. As in Bowers et al. (1997), let  $(x)$  be an annuitant who purchases an EIA at age  $x$  and  $T(x)$  the future lifetime of  $(x)$ . Also, let  ${}_t p_x$  and  ${}_t q_x$  be, respectively, the probability of survival and death; i.e.  ${}_t p_x = P(T(x) > t)$  and  ${}_t q_x = 1 - {}_t p_x$  and with the convention that  ${}_1 q_x = q_x$ . Furthermore, the force of mortality is defined as  $\mu(x + t) = -\frac{1}{{}_t p_x} \frac{d{}_t p_x}{dt}$ .

We assume that the future lifetime  $T(x)$  is stochastically independent<sup>3</sup> of the Brownian motions  $\hat{W}_r(t)$  and  $\hat{W}_S(t)$  under the physical probability measure  $P$ . Consequently,  $T(x)$  is also stochastically independent of  $r(t)$  and  $S(t)$  since both  $r(t)$  and  $S(t)$  satisfy equations (2.1) and (2.2), respectively. Intuitively, this means that the event of death is independent of the interest rates and the stock index and vice versa. An immediate implication is that the mortality risk is diversifiable by increasing the size of an insurance portfolio. Furthermore,  $T(x)$  is also independent of  $r(t)$  and  $S(t)$  under the risk-neutral probability measure  $Q$ . This is due to the fact that the Radon-Nikodym derivative  $\frac{dQ}{dP}$  used to obtain the probability measure  $Q$  is a function of  $r(t)$ ,  $S(t)$  and the stochastic processes given in (2.2), and hence is independent of  $T(x)$ .<sup>4</sup>

Consider now a contingent claim that pays  $C(t, r(t), S(t))$  at time  $t$  if a death occurs between times  $t - 1$  and  $t$ , where  $t = 1, 2, \dots, \mathcal{T}$ , and  $C_f(\mathcal{T}, r(\mathcal{T}), S(\mathcal{T}))$  otherwise. The present value of the payoff function for this claim is then

$$C = \sum_{t=1}^{\mathcal{T}} e^{-\int_0^t r(u) du} C(t, r(t), S(t)) I(t-1 < T(x) \leq t) + e^{-\int_0^{\mathcal{T}} r(u) du} C_f(\mathcal{T}, r(\mathcal{T}), S(\mathcal{T})) I(T(x) > \mathcal{T}), \quad (3.1)$$

where  $I(A)$  is the indicator function of event  $A$ ; i.e. if an outcome  $\omega$  is in  $A$ , then  $I(A) = 1$ , otherwise  $I(A) = 0$ .

---

<sup>3</sup>More precisely, this means that the information structure or  $\sigma$ -algebra generated by  $T(x)$  is independent of the information structure  $\{\mathcal{F}_t\}$  generated by  $\{\hat{W}_r(t)\}$  and  $\{\hat{W}_S(t)\}$ .

<sup>4</sup>We remark that we compromise the mathematical rigor in the above discussion in order to present the intuitive idea on how to integrate the mortality risk into the model. Two steps are needed to precisely describe the risk-neutral probability measure  $Q$  when the mortality risk is taken into account. First, we expand the information structure  $\{\mathcal{F}_t\}$  to include the information structure generated by  $T(x)$ . The new information structure  $\{\hat{\mathcal{F}}_t\}$  is such that for each  $t$ ,  $\hat{\mathcal{F}}_t$  is the smallest information structure containing  $\mathcal{F}_t$  and the information structure generated by  $T(x)$ . Then we expand the risk-neutral probability measure  $Q$  to all events in  $\{\hat{\mathcal{F}}_t\}$  while maintaining the independence between  $\{\mathcal{F}_t\}$  and the information structure generated by  $T(x)$ , and we denote this new risk-neutral probability measure  $\hat{Q}$ . The procedure of extending a probability measure to a larger information structure is somewhat complex but is standard in measure theory. For detailed description, see Moller (1998), Section 2.3. For notational simplicity, we again denote  $\{\hat{\mathcal{F}}_t\}$  and  $\hat{Q}$  by  $\{\mathcal{F}_t\}$  and  $Q$ , respectively.

The intrinsic value process  $\{IV_C(t)\}$  associated with  $C$  is defined as the expected value of  $C$  under the risk-neutral probability measure conditioning on the time- $t$  information structure; i.e.,

$$IV_C(t) = \mathbb{E}_Q[ C \mid \mathcal{F}_t ]. \quad (3.2)$$

Obviously the intrinsic value process  $\{IV_C(t)\}$  is a martingale. The price process  $\{P_C(t)\}$  associated with  $C$  is defined as

$$P_C(t) = e^{\int_0^t r(u)du} IV_C(t). \quad (3.3)$$

In other words, the intrinsic value process is the present value of the price process. By definition, we have  $P_C(0) = IV_C(0)$ . Furthermore, it can be shown that a price system defined in this way does not admit arbitrage.

We now identify  $IV_C(t)$  and hence  $P_C(t)$ . Due to the independency between  $T(x)$  and  $e^{-\int_0^t r(u)du} C(t, r(t), S(t))$  for all  $t$ , we have, for  $t = 0, 1, \dots, \mathcal{T}$ ,

$$\begin{aligned} IV_C(t) &= \sum_{s=1}^{\mathcal{T}} \mathbb{E}_Q \left[ e^{-\int_0^s r(u)du} C(s, r(s), S(s)) I(s-1 < T(x) \leq s) \mid \mathcal{F}_t \right] \\ &\quad + \mathbb{E}_Q \left[ e^{-\int_0^{\mathcal{T}} r(u)du} C_f(\mathcal{T}, r(\mathcal{T}), S(\mathcal{T})) I(T(x) > \mathcal{T}) \mid \mathcal{F}_t \right] \\ &= \sum_{s=1}^t e^{-\int_0^s r(u)du} C(s, r(s), S(s)) I(s-1 < T(x) \leq s) \\ &\quad + I(T(x) > t) e^{-\int_0^t r(u)du} \sum_{s=t+1}^{\mathcal{T}} P(t, s) {}_{s-t-1}p_{x+t} q_{x+s-1} \\ &\quad + I(T(x) > t) e^{-\int_0^t r(u)du} P_f(t, \mathcal{T}) {}_{\mathcal{T}-t}p_{x+t}, \end{aligned} \quad (3.4)$$

where  $P(t, s)$  and  $P_f(t, \mathcal{T})$  are defined in (2.6) with payoffs being  $C(s, r(s), S(s))$  and  $C_f(\mathcal{T}, r(\mathcal{T}), S(\mathcal{T}))$ , respectively. Consequently, the price  $P_C(t)$  at time  $t$  is given by

$$\begin{aligned} P_C(t) &= \sum_{s=1}^t e^{\int_s^t r(u)du} C(s, r(s), S(s)) I(s-1 < T(x) \leq s) \\ &\quad + I(T(x) > t) \sum_{s=t+1}^{\mathcal{T}} P(t, s) {}_{s-t-1}p_{x+t} q_{x+s-1} \\ &\quad + I(T(x) > t) P_f(t, \mathcal{T}) {}_{\mathcal{T}-t}p_{x+t}. \end{aligned} \quad (3.5)$$

If time  $t$  is between two payment dates, i.e., there is an integer  $k$  such that  $k < t < k + 1$ , then  $IV_C(t)$  and  $P_C(t)$  are obtained by replacing  $t$  by  $k$  in the first summation and inserting  ${}_{k-t+1}p_{x+t}$  into the first term of the second summation in (3.4) and (3.5), respectively.

In the above formulation, we have assumed that the benefit is paid discretely at the end of the year of death. In some cases, the claim could be paid at the moment of death. The intrinsic value process  $\{IV_C(t)\}$  and the price process  $\{P_C(t)\}$  in these cases can similarly be derived. In fact, the expressions are simpler and it can be shown that they can be reduced to

$$\begin{aligned} IV_C(t) &= \int_0^t e^{-\int_0^s r(u)du} C(s, r(s), S(s)) dI(T(x) \leq s) \\ &\quad + I(T(x) > t) e^{-\int_0^t r(u)du} \int_t^T P(t, s) {}_{s-t}p_{x+t} \mu(x + s) ds \\ &\quad + I(T(x) > t) e^{-\int_0^t r(u)du} P_f(t, \mathcal{T}) {}_{\mathcal{T}-t}p_{x+t}, \end{aligned} \quad (3.6)$$

and

$$\begin{aligned} P_C(t) &= \int_0^t e^{\int_s^t r(u)du} C(s, r(s), S(s)) dI(T(x) \leq s) \\ &\quad + I(T(x) > t) \int_t^T P(t, s) {}_{s-t}p_{x+t} \mu(x + s) ds \\ &\quad + I(T(x) > t) P_f(t, \mathcal{T}) {}_{\mathcal{T}-t}p_{x+t}. \end{aligned} \quad (3.7)$$

The details of this derivation is omitted.

As mentioned earlier, a claim contingent on the survivability of  $(x)$  may not be hedged perfectly using a self-financing trading strategy. In this case, we employ risk-minimizing trading strategies of Schweizer (1994). Consider again a portfolio that consists of the stock index and a money market account that earns interest at rate  $r(t)$  at time  $t$  such that the time- $t$  value of the portfolio is equal to the price  $P_C(t)$  of the claim contingent. Let  $\theta(t)$  be the number of units of the stock index at time  $t$ . Consequently the amount in the money market account at time  $t$  is

$$\phi(t) = P_C(t) - \theta(t)S(t). \quad (3.8)$$

We now introduce a stochastic process as follows.

$$CO(t) = IV_C(t) - \int_0^t \theta(u) dV(u). \quad (3.9)$$

Intuitively,  $\int_0^t \theta(u) dV(u)$  represents the present value of the accumulated capital gains at time  $t$  so that  $CO(t)$  represents the present value of the accumulated costs up to time  $t$  for maintaining the portfolio. Thus the function  $CO(t)$  is typically referred to as the cost process associated with the trading strategy  $\{\theta(t)\}$ . If the trading strategy  $\{\phi(t), \theta(t)\}$  is self-financing, then it follows from (2.11) that

$$d[CO(t)] = d[IV_C(t)] - \theta(t)dV(t) = 0.$$

Hence,  $CO(t)$  is a constant and  $CO(t) = CO(0) = IV_C(0) = P_C(0)$ . Obviously, the trading strategy  $\{\phi(t), \theta(t)\}$  under consideration may not be self-financing due to the embedded mortality risk in the contingent claim  $C$ . To solve this problem, we seek a trading strategy  $\{\theta(t)\}$  which minimizes the variance of the cost process  $\{CO(t)\}$  under the risk-neutral probability measure. We denote the resulting strategy as the risk-minimizing trading strategy. More formally, the risk-minimizing trading strategy  $\{\theta(t)\}$  is the optimal stochastic process for the following optimization problem:

$$\text{Minimize } E_Q \left[ \left( CO(t) - E_Q[CO(t)] \right)^2 \right], \text{ for all } 0 \leq t \leq \mathcal{T}. \quad (3.10)$$

Since both the discount process  $\{V(t)\}$  and the intrinsic process  $\{IV_C(t)\}$  are martingales under the risk-neutral probability measure  $Q$ , equation (3.9) implies that the cost process  $\{CO(t)\}$  is also a martingale under  $Q$ . Thus  $E_Q[CO(t)] = CO(0) = IV_C(0) = E_Q(C)$  so that the time-0 price of the claim and the expectation in (3.10) can be rewritten as

$$E_Q \left[ \left( CO(t) - E_Q(C) \right)^2 \right].$$

We remark that if  $\{\phi(t), \theta(t)\}$  is a self-financing strategy, then as pointed out earlier the cost process  $\{CO(t)\}$  of (3.9) is a constant over time. In this case, we have zero variance and an optimal process is obtained. A trading strategy is called a mean-self-financing trading strategy if the associated cost process is a martingale. Thus, the trading strategy under consideration is a mean-self-financing trading strategy. As a result, the minimization is taken with respect to all mean-self-financing trading strategies.

We now present the risk-minimizing trading strategy introduced above. As shown in Moller (1998) that for the discrete contingent claim, the optimal number of units of the stock index  $\theta(t)$  at time  $t$  is given by

$$\theta(t) = I(T(x) > t) \left[ \sum_{s=t+1}^{\mathcal{T}} \Delta(t, s) {}_{s-t-1}p_{x+t} q_{x+s-1} + \Delta_f(t, \mathcal{T}) {}_{\mathcal{T}-t}p_{x+t} \right], \quad (3.11)$$

where  $\Delta(t, s)$  and  $\Delta_f(t, \mathcal{T})$  are obtained by the formula (2.9) with the payoffs being

$C(s, r(s), S(s))$  and  $C_f(\mathcal{T}, r(\mathcal{T}), S(\mathcal{T}))$ , respectively. Therefore the amount in the money market account at time  $t$  is

$$\phi(t) = P_C(t) - \theta(t)S(t), \quad (3.12)$$

where  $P_C(t)$  is the time- $t$  price of the claim given in (3.5). The optimal trading strategy for the continuous contingent claim can be obtained similarly. In this case, the optimal  $\theta(t)$  is

$$\theta(t) = I(T(x) > t) \left[ \int_t^{\mathcal{T}} \Delta(t, s) {}_{s-t}p_{x+t} \mu(x+s) ds + \Delta_f(t, \mathcal{T}) {}_{\mathcal{T}-t}p_{x+t} \right]. \quad (3.13)$$

The amount in the money market account at time  $t$  is given by the formula (3.12) with  $P_C(t)$  given in (3.7).

## 4 Numerical Illustrations

In this section, we provide further analysis on our proposed model. We achieve this by considering one particular type of EIA. We conduct an extensive simulation and address the implication of our proposed model as compared to the conventional framework.

The EIA of interest to us has an annual resetting feature. This is also the most popular type of contracts sold in the market. For this contract, the contingent claim  $C(t, r(t), S(t))$  in year  $t$  can be represented as

$$C(t, r(t), S(t)) = \prod_{s=1}^t \max[1 + \alpha R_s, 1 + G],$$

where the  $t$ -th year return process  $\{R_t\}$  is given by

$$R_t = \frac{S(t)}{S(t-1)} - 1.$$

The parameter  $\alpha$  is the participation rate on the appreciation of the appropriate index fund,  $1+G = e^g$  is the minimum guarantee, and  $C_f(\mathcal{T}, r(\mathcal{T}), S(\mathcal{T})) = C(\mathcal{T}, r(\mathcal{T}), S(\mathcal{T}))$ . Hence the rate of growth at each period is always guaranteed by a continuously compounded rate  $g$  while the appreciation due to the growth in the stock index is also limited to the proportion  $\alpha$ .

At the time the contract is initiated, the participation rate, the minimum guarantee rate and the index fund are specify. We consider the time-0 value of the contract so that (3.4) (or (3.5)) simplifies to

$$\begin{aligned} P_C(0) = IV_C(0) &= \sum_{s=1}^{\mathcal{T}} \mathbb{E}_Q[e^{-\int_0^s r(u)du} C(s, r(s), S(s))]_{s-1} p_x q_{x+s-1} \\ &\quad + \mathbb{E}_Q[e^{-\int_0^{\mathcal{T}} r(u)du} C(\mathcal{T}, r(\mathcal{T}), S(\mathcal{T}))] \mathcal{T} p_x. \end{aligned}$$

Under the Black-Scholes framework with constant interest rates and the absence of mortality risk, the value of the above contingent claims can be expressed analytically similar to the Black-Scholes type formula. (See Boyle and Tan (1997) or Tiong (2000) for details.) With the more realistic model allowing the interest rates to be stochastic, we lost the tractability and hence must resort to simulation in order to compute the value of the embedded guarantees.

In our simulation studies, we consider a 5-year EIA issues to a life ages 50, 60 and 70. The mortality of the annuitant is assumed to be governed by the 1979-1981 U.S. Life Table (see Table 3.3.1 of Bower et al. (1997)). The index fund in consideration follows a geometric Brownian motion with the initial value normalized to 1 unit. Furthermore, the volatility of the index is assumed to be constant which admits values of  $\{10\%, 20\%, 30\%\}$ . For the stochastic interest rate model, we use the popular Vasicek (1977) model of the form

$$dr = \kappa(\theta - r)dt + \sigma_r dW_r(t).$$

The parameter values used in our simulation results correspond to those estimated by Ait-Sahalia (1996); i.e.  $\kappa = 0.85837$ ,  $\theta = 0.089102$ ,  $\sigma_r =$

0.0021854 and initial interest rate  $r_0 = 0.08362$ . Furthermore, the correlation between the index and the interest rates can be  $\{0, -10\%, -20\%\}$ .

It is easy to see that for a fixed minimum guarantee level  $g$ , the value of the annual reset EIA increases monotonically with the participation rate  $\alpha$ . In fact for certain ranges of  $\alpha$ , the time-0 value of the EIA,  $P_C(0)$ , will be less than the initial value of the index. As we increase the participation rate, the EIA becomes more valuable and hence can be more expensive than the initial value of the index fund. In other words, there exists a critical value  $\alpha^*$  satisfying the following relationship:

$$1 = \sum_{s=1}^{\mathcal{T}} \mathbb{E}_Q[e^{-\int_0^s r(u)du} C(s, r(s), S(s))]_{s-1} p_x q_{x+s-1} + \mathbb{E}_Q[e^{-\int_0^{\mathcal{T}} r(u)du} C(\mathcal{T}, r(\mathcal{T}), S(\mathcal{T}))] \mathcal{T} p_x. \quad (4.1)$$

Alternatively, this implies that for arbitrary  $\alpha$ , we have

$$\begin{aligned} P_C(0) &< 1 && \text{if } \alpha < \alpha^*, \\ P_C(0) &> 1 && \text{if } \alpha > \alpha^*. \end{aligned}$$

In our numerical illustration, we compute the critical  $\alpha^*$  for each set of parameter values. We achieve this by simulating 200,000 trajectories where each trajectory corresponds to the joint processes  $\{S(t)\}$  and  $\{r(t)\}$  being simulated daily (assuming 250 trading days per year). The critical  $\alpha^*$  is then estimated from this set of trajectories using the bisection approach until (4.1) satisfies. The above procedure is replicated independently 10 times to provide an estimate of the standard errors for the  $\alpha^*$  estimator.

The results are reported in Table 1 for different values of age-at-entry, volatility of the index fund and the correlation coefficient. The column labeled “BS” is the corresponding critical participation rate under the Black-Scholes framework such that the interest rate  $r_0 = 8.362\%$  remains level for entire maturity of the contract. We modify the annual ratchet analytical pricing formula as derived in Boyle and Tan (1997) and Tiong (2000) in order to reflect the mortality risk. Consequently the critical participation rate implied from this model can be used to assess the impact on the interest rates being stochastic. The values in the last column of Table 1 provide a more appropriate benchmark. These values are generalization of the “BS”

Age	$\sigma_S$	Correlation, $\rho$			BS	Adjusted BS
		0	-10%	-20%		
50	10%	81.670(0.014)	81.704(0.014)	81.739(0.014)	79.629	81.638
	20%	57.728(0.016)	57.750(0.016)	57.695(0.016)	55.423	57.695
	30%	43.773(0.016)	43.787(0.016)	43.802(0.016)	41.728	43.741
60	10%	81.663(0.014)	81.698(0.014)	81.732(0.014)	79.629	81.631
	20%	57.720(0.016)	57.742(0.016)	57.764(0.016)	55.423	57.687
	30%	43.766(0.016)	43.780(0.016)	43.795(0.016)	41.728	43.734
70	10%	81.648(0.013)	81.683(0.013)	81.717(0.013)	79.629	81.617
	20%	57.703(0.015)	57.725(0.015)	57.747(0.016)	55.423	57.670
	30%	43.750(0.016)	43.765(0.016)	43.779(0.016)	41.728	43.719

Table 1: Impact of correlation, volatilities of index and ages at entry on the critical participation rates. The critical rates are expressed in percentage. The values in parenthesis are the corresponding estimates of the standard errors based on 10 independent replications with each replication requires generating 200,000 trajectories

values in the sense that the interest rates can be any arbitrary deterministic function. Hence the interest rates need not be flat for all maturities as in the Black-Scholes model. This additional flexibility is particularly useful in our present context since under the Vasicek model, the entire term structure of interest rates is completely specify from a given set of parameter values. Typically the implied term structure from Vasicek model is increasing and this feature is captured in the generalized pricing formula. The results in column “Adjusted BS” are computed based on the current term structure of interest rates generated from the Vasicek model. This should provides a better benchmark for comparison purposes. We make the following remarks:

- The first observation is that as we increase the volatility of the index fund, the critical participation rate declines. This is to be anticipated since the more volatile the fund is, the greater the appreciation of the index and hence the more valuable the reset guarantee. Consequently, this must be compensated by the lower participation rate in order to neutralize the gain from the reset guarantee.

- The “BS” critical values are consistently lower than the corresponding “adjusted BS” values. This in part is due to the increasing term structure implied from the Vasicek model.
- The critical values from both the “BS” and “adjusted BS”, on the other hand, are consistently lower than those assuming stochastic interest rates for the ranges of parameter values considered in this study. This suggests that relying on a model that does not explicitly capture the stochastic nature of the interest rates could lead to an underestimation of the critical value of the participation rates.
- It is also interesting to note that the “adjusted BS” values approximate reasonably well to the uncorrelated stochastic interest rates model. The approximation deteriorates as we decrease the correlation from 0 to  $-20\%$ .

## 5 Conclusion

In this paper, we introduce an economic model which not only captures the behavior of the stock index, but also the interest rates. This is an improvement over the traditional model which only permits the stock index to be stochastic while having the interest rates constant. The impact of allowing the interest rates to be stochastic is addressed by considering some numerical examples. For the EIA with annual resets, we found that assuming a deterministic interest rates consistently underestimates the critical participation rates

It should be noted our conclusion may only be relevant to the type of EIA we considered in this paper, particularly for the ranges of parameter values assumed. Our subsequent work would be to provide further analysis, especially across various kinds of EIAs.

## A The Derivation of the Risk-Neutral Probability Measure

As in Section 2, the short rate process and the index level satisfy the stochastic differential equations

$$dr(t) = \mu_r(t, r)dt + \sigma_r(t, r)d\hat{W}_r(t) \quad (\text{A.1})$$

and

$$dS(t) = \mu_S(t)S(t)dt + \sigma_S(t)S(t)d\hat{W}_S(t), \quad (\text{A.2})$$

where the correlation coefficient  $\text{corr}(\hat{W}_r(t), \hat{W}_S(t)) = \rho$ . We now find a probability measure  $Q$  such that the present value process  $\{V(t) = e^{-\int_0^t r(u)du} S(t)\}$  is a martingale.

Using Ito's lemma, it can be shown that

$$\begin{aligned} dV(t) &= e^{-\int_0^t r(u)du} dS(t) - r(t)e^{-\int_0^t r(u)du} S(t)dt \\ &= \mu_S(t)V(t)dt + \sigma_S(t)V(t)d\hat{W}_S(t) - r(t)V(t)dt \\ &= [\mu_S(t) - r(t)]V(t)dt + \sigma_S(t)V(t)d\hat{W}_I(t). \end{aligned} \quad (\text{A.3})$$

Introduce a Radon-Nikodym derivative  $\frac{dQ}{dP}$  as follows:

$$\frac{dQ}{dP} = e^{\int_0^T b(t)dW(t) - \frac{1}{2}\int_0^T b^2(t)dt}, \quad (\text{A.4})$$

where

$$b(t) = \frac{r(t) - \mu_S(t)}{\sigma_S(t)}.$$

For any event  $A \in \mathcal{F}$ , we define the probability  $Q(A)$  of  $A$  as

$$Q(A) = \int_A \left( \frac{dQ}{dP} \right) dP,$$

so that we obtain a probability measure  $Q$  over the space  $(\Omega, \mathcal{F})$ . Let

$$\begin{aligned} W_r(t) &= \hat{W}_r(t), \quad \text{and} \\ W_S(t) &= -\int_0^t b(u)du + \hat{W}_S(t). \end{aligned} \quad (\text{A.5})$$

Then, by the Girsanov Theorem, under the probability measure  $Q$ ,  $\{W_r(t)\}$  and  $\{W_S(t)\}$  are correlated Brownian motions with the same correlation coefficient  $\rho$  and

$$dr(t) = \mu_r(t, r)dt + \sigma_r(t, r)dW_r(t) \quad (\text{A.6})$$

and

$$dV(t) = \sigma_S(t)V(t)dW_S(t). \quad (\text{A.7})$$

Equation (A.7) implies that the present value process  $\{V(t)\}$  is a martingale since it presents no drift term. It follows from (A.5) that

$$d\hat{W}_S(t) = b(t)dt + dW_S(t).$$

Thus, we have

$$\begin{aligned} dS(t) &= \mu_S(t)S(t)dt + \sigma_S(t)S(t)d\hat{W}_S(t) \\ &= \mu_S(t)S(t)dt + \sigma_S(t)S(t)[b(t)dt + dW_S(t)] \\ &= \mu_S(t)S(t)dt + [r(t) - \mu_S(t)]S(t)dt + \sigma_S(t)S(t)dW_S(t) \\ &= r(t)S(t)dt + \sigma_S(t)S(t)dW_S(t). \end{aligned}$$

Therefore, under the probability measure  $Q$ , the index level  $S(t)$  satisfies the stochastic differential equation

$$dS(t) = r(t)S(t)dt + \sigma_S(t)S(t)dW_S(t)$$

and this completes the derivation.

## References

1. Ait-Sahalia, Y. (1996). Nonparametric pricing of interest rate derivative securities, *Econometrical*, 3(3):527–560.
2. Bowers, N., H. Gerber, J. Hickman, D. Jones and C. Nesbitt (1997). *Actuarial Mathematics*, Second Edition. Society of Actuaries, Schaumburg, Illinois.
3. Boyle, P.P. and K.S. Tan (1997). Valuation of ratchet options. Working paper, University of Waterloo.

4. Cox, J.C., J.E. Ingersoll and S. Ross (1985a). An intertemporal general equilibrium model of asset prices, *Econometrica*, 53, 363-384.
5. Cox, J.C., J.E. Ingersoll and S. Ross (1985b). A theory of the term structure of interest rates, *Econometrica*, 53, 385-407
6. Hull, J. and A. White (1990). Pricing interest rate derivative securities, *Review of Financial Studies*, 3:573-592.
7. Lin, X. (2001). *Stochastic Calculus with Financial Applications*, Society of Actuaries Finance Committee Project, in progress.
8. Marrion, J. (2000a). Advantage equity index 4th quarter 1999 sales report, March, 2000.
9. Marrion, J. (2000b). Advantage 2000 equity index report, November, 2000.
10. Moller, T. (1998). Risk-minimizing hedging strategies for unit-linked life insurance contracts, *ASTIN Bulletin*, 28, 17-47.
11. Moller, T. (2001). Hedging of equity-linked life insurance contracts, *North American Actuarial Journal*, 5(2):79-95.
12. Schweizer, M. (1994). Risk-minimizing hedging strategies under restricted information, *Mathematical Finance*, 4, 327-363.
13. Tiong, S. (2000). Valuing equity-indexed annuities, *North American Actuarial Journal*, 4, 149-170.
14. Vasicek, O.A. (1977). An equilibrium characterization of the term structure, *Journal of Financial Economics*. 5:177-188.