Estimation of Tails and Related Quantities Using the Number of Near-Extremes

Astin Topic: Other

Enkelejd Hashorva and Jürg Hüsler*

*University of Bern and Allianz Suisse

enkelejd.hashorva@stat.unibe.ch, juerg.huesler@stat.unibe.ch

Abstract: In an insurance context, consider $\{X_n, n \geq 1\}$ random claim sizes with common distribution function F and $\{N(t), t \geq 0\}$ an integer valued stochastic process that counts the number of claims occurring during the time interval $[0, t)$. Based on the number of near-extremes which are the observations X_i near the largest or the m-th largest observation we derive in this paper a strongly consistent estimator of upper tails of X_1. Further, estimators for both the tail index and the upper endpoint are introduced when F is a generalised Pareto distribution. Asymptotic normal law for the proposed estimators is additionally presented.

Keywords: Number of near-extremes, Generalised Pareto Distribution, estimation of tail index, estimation of the upper endpoint, asymptotic normality.

1 Introduction

In an insurance context, consider $\{X_n, n \geq 1\}$ random claim sizes and $\{N(t), t \geq 0\}$ an integer valued stochastic process that counts the number of claims that occur during the time interval $[0, t)$. Denote by $X_{n-m+1:n}$ $(n \geq n \geq 1)$ the nth largest order statistics of $X_1, X_2, \ldots, X_n$ and let further $X_{N(t)-m+1:N(t)}$ be the corresponding quantity for $X_1, X_2, \ldots, X_{N(t)}$. Counting the number of extreme points that fall in a random window defined by the sample maximum, Pakes and Steutel (1997) introduce the following discrete random variable

$$L_n(a) := \sum_{i=1}^{n} \mathbf{1}(X_i \in (X_{n:n} - a, X_{n:n}]), \quad a > 0, n \geq 1.$$

Asymptotic and distributional properties investigated by these authors turned out to be interesting in various context. In an insurance situation, Li and Pakes (2001) show that $L_{N(t)}(a)$ is an interesting quantity. Hashorva (2003) extents their results for dependent claim sizes, and shows further that $L_{N(t)}(a)/N(t)$ is a strongly consistent estimator of certain tail probabilities related to claim sizes.

Statistical applications of the number of near-extremes for iid claim sizes are discussed in Müller (2003). Making use of previous results of Hashorva and Hüsler (2001a) a strongly consistent estimator of the tail index of the generalised Pareto distribution is derived therein. Such distributions are widely used in statistics and specifically in insurance applications.

In this paper we consider the problem of estimating specific tail quantities in an insurance context, by further allowing claim sizes to be dependent.

Restricting our attention to generalised Pareto distribution functions we introduce a more general estimator that the one given by Müller (2003). Let us now motivate this novel estimator: Assume that claim sizes have common continuous distribution function F. Define further

$$K_n(a, b, m) := \sum_{i=1}^{n} \mathbf{1}(X_i \in (X_{n-m+1:n} - b, X_{n-m+1:n} - a]), \quad n > m \geq 1$$

and

$$\mathcal{K}_t(a, b, m) := \sum_{i=1}^{N(t)} \mathbf{1}(X_i \in (X_{N(t)-m+1:N(t)} - b, X_{N(t)-m+1:N(t)} - a]), \quad t \geq 0$$

with $0 \le a < b < \infty$. Put $\mathcal{K}_t(a,b,m) = 0$ if $N(t) < m$. Distributional and asymptotic properties of $\mathcal{K}_t(a,b,m)$ are investigated in Hashorva and Hüsler (2001b). The role of the weak convergence of the normalised sample maxima is strongly highlighted therein.
Now, by Theorem 4.1 of Hashorva and Hüsler (2001a) we have that

$$\frac{K_n(a,b,m)}{n} \stackrel{a.s.}{\to} F(\omega - a) - F(\omega - b), \quad n \to \infty$$

holds where $\omega := \sup\{u : F(u) < 1\}$ is the upper endpoint of the distribution function F. Further by Lemma 2.5.3 of Embrechts et al. (1997) also

$$\frac{\mathcal{K}_t(a,b,m)}{N(t)} \stackrel{a.s.}{\to} F(\omega - a) - F(\omega - b), \quad t \to \infty,$$

holds if

$$N(t) \stackrel{p}{\to} \infty, \quad t \to \infty. \tag{1}$$

So we have a strongly consistent estimator of the region in the upper tail $F(\omega-a)-F(\omega-b)$ if $\omega < \infty$. Now if we consider for instance F given by

$$F(x) = 1 - (1 - x/\omega)^{-1/\gamma}, \quad x \in [0,\omega), \gamma < 0, \omega < \infty \tag{2}$$

we have for some constant $\tau > 1$ and $0 \le a < b < \tau b < \omega$

$$\frac{F(\omega - a) - F(\omega - b)}{F(\omega - \tau a) - F(\omega - \tau b)} = \frac{b^{-1/\gamma} - a^{-1/\gamma}}{(\tau b)^{-1/\gamma} - (\tau a)^{-1/\gamma}} = \tau^{1/\gamma},$$

implying thus that

$$\hat{\gamma}_t(a,b,\tau,m) := \frac{\ln \tau}{\ln(\mathcal{K}_t(a,b,m)/\mathcal{K}_t(\tau a, \tau b, m))}$$

is a strongly consistent estimator of γ, i.e.

$$\boldsymbol{P}\Big\{\lim_{t\to\infty} \hat{\gamma}_t(a,b,\tau,m) = \gamma\Big\} = 1.$$

In this article we deal with both estimators $\mathcal{K}_t(a,b,m)/N(t)$ and $\hat{\gamma}_t(a,b,\tau,m)$ by extending further these results for the case that claim sizes are dependent and also allowing m to vary with t. Further, a new estimator for ω is also introduced.

Since it is of practical interest, we consider additionally CLT (central limit theorem) results for both estimators, generalising thus previous results of Müller (2003).

2 Results

We consider in the following claim sizes $\{X_n, n \geq 1\}$ that have common continuous distribution function F. The case F being not continuous can be handled along the same lines, therefore omitted here. The counting process $\{N(t), t \geq 0\}$ is assumed to take integer values, and moreover it is nondecreasing in t. For a, b positive put in the following

$$\mathbf{P}(a,b) := F(\omega - a) - F(\omega - b).$$

Clearly if $\omega = \infty$ then $\mathbf{P}(a,b) = 0$. If $\omega < \infty$ we will consider only constants such that $0 \leq a < b < \infty$ so that $\mathbf{P}(a,b) \in (0,1)$ holds. Further let $\{m(n), n \in \mathbb{N}\}$ be a nondecreasing sequence of integers satisfying

$$\lim_{n\to\infty} m(n)/n \quad = \quad 0. \tag{3}$$

Considering $m(n)$ instead of a constant m, allows us to deal with random regions that are to a certain extent far from the sample maximum $X_{n:n}$. To keep the notation simple, we write in the following $m_t := m(N(t))$.

Next, we discuss first the strong consistency of $\mathcal{K}_t(a,b,m_t)/N(t)$.

2.1 Strong consistency

In the next proposition the strong consistency of the estimator of $\mathbf{P}(a,b)$ is derived.

Proposition 2.1. *Let $\{X_n, n \geq 1\}$ be claim sizes with underlying distribution function F and $N(t)$ a counting process. Assume that for the empirical distribution function F_n of F we have*

$$\boldsymbol{P}\Big\{\lim_{n\to\infty} F_n(u) \quad \overset{a.s.}{\to} \quad F(u)\Big\}, \quad u \in \mathbb{R}. \tag{4}$$

Then we have

$$\boldsymbol{P}\Big\{\lim_{t\to\infty} \mathcal{K}_t(a,b,m_t)/N(t) = \mathbf{P}(a,b)\Big\} = 1. \tag{5}$$

If further

$$\frac{N(t)}{t} \quad \overset{a.s.}{\to} \quad \mathcal{Z}, \quad t \to \infty \tag{6}$$

holds with $\mathcal{Z}$ a positive random variable, then we get

$$\frac{\mathcal{K}_t(a,b,m_t)}{t} \stackrel{a.s.}{\to} \mathcal{Z}\mathbf{P}(a,b) \tag{7}$$

as $t \to \infty$.

Remark: (i) Convergence in (4) (which is well-known as Glivenko-Cantelli Theorem) holds for stationary ergodic sequences (see e.g. Ergodic Theorem in Kallenberg (1997)). (ii) Clearly $\mathbf{P}(a,b)$ equals zero for a distribution function with infinite endpoint, i.e. $\omega = \infty$.

As mentioned above, if $\omega = \infty$ then the estimator above converges to 0, therefore it is of no use for estimating purposes. However, if $\omega < \infty$, which can be justified for most of insurance claims, it is interesting to calculate other quantities apart from $\mathbf{P}(a,b)$. So it is possible then to derive estimators when the distribution function is parametrised as in the definition (2). For the latter case, both parameters γ and ω can be unknown. As motivated above, $\hat{\gamma}_t(a,b,\tau,m_t)$ can be used for estimating the unknown parameter γ. Similarly, we motivate an estimator for the unknown upper endpoint ω. Let for the insurance model discussed above, claim sizes have distribution function F so that (2) is satisfied in a ν-neighbourhood of ω. Since we showed that for appropriate constants a, b

$$\mathcal{K}_t(a,b,m_t)/N(t) \stackrel{a.s.}{\to} \mathbf{P}(a,b) = [b^{-1/\gamma} - a^{-1/\gamma}]\omega^{1/\gamma}$$

we may take as an estimator of the unknown endpoint ω

$$\hat{\omega}_t(a,b,m_t) := \left(\frac{\mathcal{K}_t(a,b,m_t)}{N(t)[b^{-1/\hat{\gamma}_t(a,b,\tau,m_t)} - a^{-1/\hat{\gamma}_t(a,b,\tau,m_t)}]} \right)^{\hat{\gamma}_t(a,b,\tau,m_t)}.$$

In the next proposition we discuss the strong consistency of both estimators $\hat{\gamma}_t(a,b,\tau,m_t)$ and $\hat{\omega}_t(a,b,m_t)$.

Proposition 2.2. *Let claim sizes and counting process be as in the above proposition. Assume further that F has a finite upper endpoint ω, and both* (1), (3) *and* (4) *hold. Then we have for constants $0 \le a < b < \infty, \tau > 1$*

$$\hat{\gamma}_t(a,b,\tau,m_t) \stackrel{a.s.}{\to} \frac{\ln \tau}{\ln(\mathbf{P}(a,b)/\mathbf{P}(\tau a,\tau b))}, \quad t \to \infty. \tag{8}$$

If the distribution function F satisfies (2) *for all $x \in (\omega - \nu, \omega)$ with $\nu > 0$ and $\gamma < 0$, then also*

$$\boldsymbol{P}\Big\{\lim_{t\to\infty} \hat{\gamma}_t(a,b,\tau,m_t) = \gamma\Big\} = 1, \tag{9}$$

and

$$\boldsymbol{P}\Big\{\lim_{t\to\infty}\hat{\omega}_t(a,b,m_t)=\omega\Big\}=1, \tag{10}$$

hold if further $\tau b<\nu$.

Taking $a=0$ our estimator $\hat{\gamma}_t(a,b,\tau,m_t)$ reduces to the one introduced in Müller (2003). In the literature there are many different estimators for γ. As pointed out in the aforementioned paper, this estimator is very robust, and outperforms other known estimators for particular values of γ. See Müller (2003) for various simulation results and comparisons with other estimators.

2.2 CLT Results

In this section we obtain weak convergence results by restricting the distribution function of the claim sizes and also assuming a certain asymptotic behaviour of the number of claims as $t\to\infty$. Initially we consider the estimator for the upper tails.

Proposition 2.3. *Let $\{X_n, n\geq 1\}$, $N(t)$ be as in Proposition 2.2 and suppose that $N(t)$ is independent of the claim sizes for all $t\geq 0$. Assume that there exists a positive sequence $\{\varepsilon_n, n\geq 1\}$ such that*

$$\lim_{n\to\infty}\boldsymbol{P}\{X_{n-m(n)+1:n}<\omega-\varepsilon_n\} \;=\; 0, \tag{11}$$

and

$$\sqrt{n}\boldsymbol{P}\{X_{n-m(n)+1:n}\in[\omega-s-\varepsilon_n,\omega-s]\} \;=\; 0, \quad \textit{for } s\in\{a_1,a_2,b_1,b_2\} \tag{12}$$

hold, with constants $0\leq a_i<b_i<\infty, i=1,2$. Suppose further that there exists a stochastic process $\{Y(s), s\in I\!R\}$ such that

$$\sqrt{n}[F_n(s)-F(s)]\xrightarrow{d}Y(s), \quad s:F(s)\in(0,1), \tag{13}$$

so that the convergence in distribution in (13) holds also for the fidis (finite dimensional distributions) of the stochastic process $\{\sqrt{n}[F_n(s)-F(s)], s\in I\!R\}$. Then we have

$$\sqrt{N(t)}\Bigg(\mathcal{K}_t(a_1,b_1,m_t)/N(t)-\mathbf{P}(a_1,b_1),\mathcal{K}_t(a_2,b_2,m_t)/N(t)-\mathbf{P}(a_2,b_2)\Bigg)$$

$$\xrightarrow{d} \Big(Y(\omega - a_1) - Y(\omega - b_1), Y(\omega - a_2) - Y(\omega - b_2)\Big). \tag{14}$$

Remark: (i) In Hashorva (1999) weak convergence of $\sqrt{n}[K_n(0,b,1)/n - (1 - F(\omega - b))]$, $b > 0$, is shown if F satisfies

$$\lim_{s \uparrow \omega - b} \frac{F(\omega - b) - F(s)}{[1 - F(s+b)]^{1/2}} = 0. \tag{15}$$

This is a slightly weaker condition than the ones imposed above for the distribution function F. Note further that (15) is satisfied if F is the Beta distribution function $B(\alpha, \beta)$ with shape parameter $\beta < 2$.

(ii) If claim sizes are independent and m is bounded (or does not depend on n) then the following condition

$$\lim_{n \to \infty} n \ln F(\omega - \varepsilon_n) \quad = \quad 0 \tag{16}$$

implies (11). □

Proposition 2.3 and the central limit theorem for the empirical process of iid random variables imply immediately:

Corollary 2.4. *Under the assumptions of Proposition 2.3 assuming independent claim sizes, we have*

$$\sqrt{N(t)}\Bigg(\mathcal{K}_t(a_1, b_1, m_t)/N(t) - \mathbf{P}(a_1, b_1),\ \mathcal{K}_t(a_2, b_2, m_t)/N(t) - \mathbf{P}(a_2, b_2)\Bigg)$$
$$\xrightarrow{d} \Big(B_F(\omega - a_1) - B_F(\omega - b_1), B_F(\omega - a_2) - B_F(\omega - b_2)\Big) \tag{17}$$

holds with B_F a F-Brownian Bridge.

Let in the following

$$\hat{\eta}_t := \frac{\hat{\gamma}_t(a, b, \tau, m_t)^2 \sqrt{N(t)}}{\ln(\tau)} \quad \left(\frac{\mathcal{K}_t(\tau a, \tau b, m_t) - \mathcal{K}_t(a, b, m_t) + 2\mathcal{K}_t(a, \tau a \wedge b, m_t)}{\mathcal{K}_t(a, b, m_t)\mathcal{K}_t(\tau a, \tau b, m_t)}\right)^{1/2}.$$

We can now present a CLT-result for $\hat{\gamma}_t(a, b, \tau, m_t)$. Along the same lines, a similar result can be shown for $\hat{\omega}_t(a, b, m_t)$ in the particular case with F satisfying (2).

Proposition 2.5. *Let claim sizes $\{X_n, n \geq 1\}$ be as in Proposition 2.3. Suppose that $N(t)$ is a counting process, independent of claim sizes for all $t > 0$, and that there exists a positive sequence $\{\varepsilon_n, n \geq 1\}$ such that* (11) *and* (12) *hold. Suppose further that the convergence in distribution of fidi's*

$$\sqrt{n}[F_n(s) - F(s)] \overset{d}{\to} Y(s) = B_F(s), \quad n \to \infty \; s : F(s) \in (0,1), \tag{18}$$

holds, with B_F a F-Brownian Bridge. Then we have

$$\sqrt{N(t)}\left[\hat{\gamma}_t(a,b,\tau,m_t) - \frac{\ln \tau}{\ln(\mathbf{P}(a,b)/\mathbf{P}(\tau a, \tau b))}\right]/\hat{\eta}_t \overset{d}{\to} U, \quad t \to \infty, \tag{19}$$

with U a standard Gaussian random variable.

If further the distribution function F satisfies (2) *for all $x \in (\omega - \nu, \omega)$ with $\nu > 0$ and $\gamma < -1/2$, and $m(n)$ is bounded, then*

$$\sqrt{N(t)}[\hat{\gamma}_t(a,b,\tau,m_t) - \gamma]/\hat{\eta}_t \overset{d}{\to} U, \quad t \to \infty. \tag{20}$$

Indeed the quantity $\hat{\eta}_t$ plays a normalisation role. It converges almost surely to the asymptotic variance of the estimator. So it is interesting to find out the explicit form of the asymptotic variance for special F satisfying (2) because constants a, b, τ can be chosen such that the variance is minimal.
Note first that for $t \to \infty$ we have

$$\begin{aligned}
\hat{\eta}_t^2 &= \frac{(\hat{\gamma}_t)^4}{\ln^2(\tau)}\left(\frac{\mathcal{K}_t(\tau a, \tau b, m_t)/N(t) - \mathcal{K}_t(a,b,m_t)/N(t) + 2\mathcal{K}_t(a, \tau a \wedge b, m_t)/N(t)}{[\mathcal{K}_t(a,b,m_t)/N(t)][\mathcal{K}_t(\tau a, \tau b, m_t)/N(t)]}\right) \\
&\overset{a.s.}{\to} \left(\frac{\ln \tau}{\ln^2(\mathbf{P}(a,b)/\mathbf{P}(\tau a, \tau b))}\right)^2 \frac{\mathbf{P}(\tau a, \tau b) - \mathbf{P}(a,b) + 2\mathbf{P}(a, \tau a \wedge b)}{\mathbf{P}(a,b)\mathbf{P}(\tau a, \tau b)}.
\end{aligned} \tag{21}$$

If F satisfies (2) for all $x \in (\omega - \nu, \omega)$ with $\nu > 0$, then

$$\mathbf{P}(\tau a, \tau b) = \tau^{-1/\gamma}\mathbf{P}(a,b) = \tau^{-1/\gamma}\omega^{1/\gamma}[b^{-1/\gamma} - a^{-1/\gamma}],$$

hence we get

$$\begin{aligned}
&(\hat{\eta}_t)^2 \\
&\overset{a.s.}{\to} \frac{\gamma^4 \tau^{1/\gamma}\left\{\tau^{-1/\gamma} - 1 + 2\mathbf{P}(a, \tau a \wedge b)/\mathbf{P}(a,b)\right\}}{(\ln \tau)^2 \mathbf{P}(a,b)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\gamma^4 \tau^{1/\gamma} \left\{ \tau^{-1/\gamma} - 1 + 2[(\tau a \wedge b)^{-1/\gamma} - a^{-1/\gamma}]/[b^{-1/\gamma} - a^{-1/\gamma}] \right\}}{(\ln \tau)^2 \omega^{1/\gamma} [b^{-1/\gamma} - a^{-1/\gamma}]} \\
&= \frac{\gamma^4 \tau^{1/\gamma} \left\{ \tau^{-1/\gamma} - 1 + 2[\tau^{-1/\gamma} - 1] a^{-1/\gamma}/[b^{-1/\gamma} - a^{-1/\gamma}] \mathbf{1}(\tau a < b) + 2\mathbf{1}(\tau a \geq b) \right\}}{(\ln \tau)^2 \omega^{1/\gamma} [b^{-1/\gamma} - a^{-1/\gamma}]} \\
&= \frac{\gamma^4 \tau^{1/\gamma} \left\{ [\tau^{-1/\gamma} - 1][1 + 2a^{-1/\gamma}/[b^{-1/\gamma} - a^{-1/\gamma}] \mathbf{1}(\tau a < b)] + 2\mathbf{1}(\tau a \geq b) \right\}}{(\ln \tau)^2 \omega^{1/\gamma} [b^{-1/\gamma} - a^{-1/\gamma}]}
\end{aligned}
$$

It is straightforward to find the minimum of the above function. See Müller (2003) for the case $a = 0$.

We consider next an example, where the above techniques can be applied.

Example 1. Let claim sizes have distribution function $F_{\lambda,\omega}$ be such that in a neighbourhood of the upper finite endpoint ω we have

$$
F_{\lambda,\omega}(x) = \frac{1 - \exp(-\lambda x)}{1 - \exp(-\lambda \omega)}, \quad \lambda > 0, \tag{22}
$$

holds for all $x \in (\omega - \nu, \omega)$ with ν a positive constant. $F_{\lambda,\omega}$ is the distribution function of a truncated exponential random variable.

In reinsurance, it is of particular importance to estimate the unknown parameters λ and the upper endpoint ω. By the above results we have for any $m \in I\!N$ (as $t \to \infty$)

$$
\begin{aligned}
\frac{\mathcal{K}_t(\tau a, \tau a + c, m)}{N(t)} &\overset{a.s.}{\to} \frac{\exp(-\lambda \tau a) - \exp(-\lambda(\tau a + c))}{1 - \exp(-\lambda \omega)} \\
&= \frac{\exp(-\lambda \tau a)[1 - \exp(-\lambda c)]}{1 - \exp(-\lambda \omega)},
\end{aligned}
$$

if $N(t) \to \infty$ as $t \to \infty$ and a, c are positive constants, $\tau \geq 1$ with $0 < \tau a + c < \nu$.

Hence we get

$$
\frac{\mathcal{K}_t(a, a + c, m)}{\mathcal{K}_t(\tau a, \tau a + c, m)} \overset{a.s.}{\to} \exp(\lambda a(\tau - 1)), \quad t \to \infty,
$$

thus the following statistics

$$
\hat{\lambda}_t := \frac{\ln(\mathcal{K}_t(a, a + c, m)/\mathcal{K}_t(\tau a, \tau a + c, m))}{a(\tau - 1)} \overset{a.s.}{\to} \lambda, \quad t \to \infty
$$

is a strongly consistent estimator of λ. Along the lines of Hashorva (2003) we have further that for $a \in (0, \nu)$ also

$$
\hat{\omega}_t := (\hat{\lambda}_t)^{-1} \ln((1 + \psi(a, \hat{\lambda}_t))/\psi(a, \hat{\lambda}_t)) \overset{a.s.}{\to} \omega, \quad t \to \infty, \tag{23}
$$

holds with $\psi(a, \hat{\lambda}_t) := \mathcal{K}_t(0, a, m_t)/(t[\exp(\hat{\lambda}_t a) - 1])$; thus $\hat{\omega}_t$ is a strongly consistent estimator of ω.

3 Comments

We provide next some comments on the counting process $N(t)$ and the distribution function F satisfying (2).

Typically, in insurance models used by actuaries, the stochastic process $\{N(t), t \geq 0\}$ is some renewal counting process defined as follows

$$N(t) \quad := \quad \sup\Big\{j : \sum_{i=1}^{j} V_i \leq t\Big\}, \quad t \geq 0, \tag{24}$$

with $\{V_n, n \geq 1\}$, iid positive random variables with finite first moment $\boldsymbol{E}\{V_1\} = 1/\lambda > 0$. It is well known (e.g. Rolski (1999) or Embrechts et al. (1997)) that

$$N(t)/t \overset{a.s.}{\to} \lambda^{-1}, \quad t \to \infty,$$

hence our asymptotic conditions on $N(t)$ for this case are satisfied.

Distribution functions satisfying condition (2) are called power-function distributions. These belong to the class of generalised Pareto distributions denoted here by $W_\gamma, \gamma \in I\!R$. It is well-known (see Pickands (1975)) that W_γ is a good approximation of the excess distribution function

$$F_u(x) := (F(x) - F(u))/(1 - F(u)), \quad u : F(u) \in (0, 1).$$

More precisely, Pickands (1975) shows that

$$\lim_{u \to \omega} \sup_{x \in (0, \omega - u)} \Big| F_u(x) - W_\gamma(x/\sigma(u)) \Big| = 0$$

holds for some fixed tail index γ. When $\gamma < 0$, then W_γ is the power distribution.

It should be noted further, that the choice of parameters a, b, τ and $m(n)$ is very crucial for the above estimation procedure. Some explorative graphics can help in deciding which set of constants should be used. However, theoretical considerations need a deeper investigation and are not the topic of this paper.

Another important development is to assume that the power-function relation in the underlying distribution function F holds for ν-neighbourhoods asymptotically. Estimation of γ for this asymptotic model can be developed similarly to the case that ν is a fixed constant not vanishing to 0. This is however scheduled for a forthcoming research paper.

4 Proofs

Proof of Proposition 2.1. For all $n > m \geq 1, u \in I\!R$ we have

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}(X_{n-m+1:n}-X_i<u, X_i\leq X_{n-m+1:n}) &= F_n(X_{n-m+1:n})-F_n(X_{n-m+1:n}-u)\\ &= (n-m+1)/n-F_n(X_{n-m+1:n}-u),\end{aligned}$$

with F_n the empirical distribution function of the claims sizes, thus we get

$$\begin{aligned}\frac{\mathcal{K}_t(a,b,m_t)}{N(t)} &= \frac{1}{N(t)}\sum_{i=1}^{N(t)}\mathbf{1}(a_i\leq X_{N(t)-m_t+1:N(t)}-X_i<b_i)\\ &= F_{N(t)}(X_{N(t)-m_t+1:N(t)}-a_i)-F_{N(t)}(X_{N(t):N(t)}-b_i).\end{aligned}$$

By the monotonicity of the empirical distribution function we have that (4) holds locally uniformly and also uniformly in $I\!R$. Lemma 2.5.3 of Embrechts et al. (1997) implies

$$F_{N(t)}(u) \overset{a.s.}{\to} F(u), \quad u\in I\!R, \tag{25}$$

holds locally uniformly in $I\!R$. Hence in order to show the first claim it suffices to prove that

$$X_{N(t)-m(N(t))+1:N(t)} \overset{a.s.}{\to} \omega, \quad t\to\infty. \tag{26}$$

By Theorem 4.1.14 of Embrechts et al. (1997) we have

$$\boldsymbol{P}\Big\{\lim_{n\to\infty} X_{n-m(n)+1:n}=\omega\Big\}=1,$$

thus applying again the afore-mentioned lemma, we get that (26) is satisfied.

Note that a direct proof for case $m(N(t))=m\in I\!N$ can be obtained by the following lines: Since the random sequence $\{X_{N(t)-m+1:N(t)}, t\geq 0\}$ is monotone in t, it suffices to show actually that

$$X_{N(t)-m:N(t)} \overset{p}{\to} \omega, \quad t\to\infty$$

holds. Now for any $\varepsilon>0$ we have

$$\begin{aligned}\boldsymbol{P}\{X_{N(t)-m+1:N(t)}<\omega-\varepsilon\} &= \boldsymbol{P}\{F_{N(t)}(\omega-\varepsilon)>1-m/N(t)\}\\ &= \boldsymbol{P}\{F(\omega-\varepsilon)>1-o_p(1)\}\\ &\to 0, \quad t\to\infty\end{aligned}$$

by the continuity of the distribution function F at the upper endpoint ω. $\square$

Proof of Proposition 2.2. The assumptions and Proposition 2.1 imply

$$\frac{\mathcal{K}_t(a,b,m_t)}{\mathcal{K}_t(\tau a,\tau b,m_t)} = \frac{\mathcal{K}_t(a,b,m_t)/N(t)}{\mathcal{K}_t(\tau a,\tau b,m_t)/N(t)} \stackrel{a.s.}{\to} \frac{\mathbf{P}(a,b)}{\mathbf{P}(\tau a,\tau b)}, \quad t\to\infty,$$

therefore for the choice of constants a,b,τ

$$\begin{aligned}\hat{\gamma}_t(a,b,\tau,m_t) &= \frac{\ln\tau}{\ln(\mathcal{K}_t(a,b,m_t)/\mathcal{K}_t(\tau a,\tau b,m_t))}\\ &\stackrel{a.s.}{\to} \frac{\ln\tau}{\ln(\mathbf{P}(a,b)/\mathbf{P}(\tau a,\tau b))}\\ &= \frac{\ln\tau}{\ln\tau^{1/\gamma}}\\ &= \gamma\end{aligned}$$

if F satisfies (2). Further since $\hat{\gamma}_t(a,b,\tau,m_t)\stackrel{a.s.}{\to}\gamma$ we get as $t\to\infty$

$$\begin{aligned}\hat{\omega}_t(a,b,m_t) &= \left(\frac{\mathcal{K}_t(a,b,m_t)}{N(t)[b^{-1/\hat{\gamma}_t(a,b,\tau,m_t)}-a^{-1/\hat{\gamma}_t(a,b,\tau,m_t)}]}\right)^{\hat{\gamma}_t(a,b,\tau,m_t)}\\ &\stackrel{a.s.}{\to} \left(\frac{\mathbf{P}(a,b)}{b^{-1/\gamma}-a^{-1/\gamma}}\right)^{\gamma}\\ &= \omega,\end{aligned}$$

hence the proof. $\square$

Proof of Proposition 2.3. For notational simplicity we write m instead of $m(n)$ in the following. Further we assume that $a_i>0, i=1,2$. The case $a_i=0$ for $i=1$ or $i=2$ follows along the same lines. As above we have for all $n>m\geq 1, u\in\mathbb{R}$

$$\frac{K_n(a,b,m)}{n} = F_n(X_{n-m+1:n}-a_i)-F_n(X_{n:n}-b_i).$$

Next, since further $X_{n-m+1:n}\leq\omega$ holds almost surely, we have

$$F_n(X_{n-m+1:n}-b_i) \leq F_n(\omega-b_i)$$

and for n large (since $\lim_{n\to\infty}\varepsilon_n=0$)

$$\begin{aligned}F_n(X_{n-m+1:n}-b_i)-F_n(\omega-b_i) &\geq F_n(\omega-b_i-\varepsilon_n)-F_n(\omega-b_i)\\ &\quad +\mathbf{1}(X_{n-m+1:n}<\omega-\varepsilon_n).\end{aligned}$$

Hence with (11) and (12)

$$
\begin{aligned}
&\sqrt{n}[F_n(X_{n-m+1:n} - b_i) - F(\omega - b_i)] \\
&\geq \sqrt{n}[F_n(\omega - b_i - \varepsilon_n) - F_n(\omega - b_i)] + \sqrt{n}\mathbf{1}(X_{n-m+1:n} < \omega - \varepsilon_n) \\
&\quad + \sqrt{n}[F_n(\omega - b_i) - F(\omega - b_i)] \\
&= \sqrt{n}[F_n(\omega - b_i) - F(\omega - b_i)] + o_p(1).
\end{aligned}
$$

Along the same lines we get further

$$
\sqrt{n}[F_n(X_{n-m+1:n} - a_i) - F(\omega - a_i)] = \sqrt{n}[F_n(\omega - a_i) - F(\omega - a_i)] + o_p(1).
$$

Finally, (13) implies for $n \to \infty$

$$
\begin{aligned}
&\sqrt{n}\Big(K_n(a_1, b_1, m)/n - \mathbf{P}(a_1, b_1), K_n(a_2, b_2, m)/n - \mathbf{P}(a_2, b_2)\Big) \\
&\quad \stackrel{d}{\to} \Big(Y(\omega - a_1) - Y(\omega - b_1), Y(\omega - a_2) - Y(\omega - b_2)\Big),
\end{aligned}
$$

hence the statement follows applying further Lemma 2.5.3 of Embrechts et al. (1997). □

Proof of Proposition 2.5. First note that the joint weak convergence

$$
\begin{aligned}
&\sqrt{n}\Big(K_n(a, b, m)/n - \mathbf{P}(a, b), K_n(\tau a, \tau b, m)/n - \mathbf{P}(\tau a, \tau b)\Big) \\
&\quad \stackrel{d}{\to} \Big(Y(\omega - a) - Y(\omega - b), Y(\omega - \tau a) - Y(\omega - \tau b)\Big),
\end{aligned}
$$

holds under the assumptions of the proposition. Using now the well-known delta method we get

$$
\sqrt{n}[\hat{\gamma}(a, b, \tau, m(n)) - \tilde{\gamma}] \stackrel{d}{\to} V, \quad n \to \infty
$$

with $\tilde{\gamma} := (\mathbf{P}(a, b)/\mathbf{P}(\tau a, \tau b))^{-1} \ln \tau$ and V a Gaussian random variable with mean zero and variance given by

$$
\begin{aligned}
\mathbf{Var}\{V\} &= \left(\frac{\ln \tau}{\ln^2(\mathbf{P}(a, b)/\mathbf{P}(\tau a, \tau b))}\right)^2 \\
&\quad \times \left(\frac{1}{\mathbf{P}(a, b)}, \frac{-1}{\mathbf{P}(\tau a, \tau b)}\right)^\top \mathbf{Var}\{(B_1, B_2)\} \left(\frac{1}{\mathbf{P}(a, b)}, \frac{-1}{\mathbf{P}(\tau a, \tau b)}\right),
\end{aligned}
$$

where B_1, B_2 are two random variables such that

$$
\mathbf{Var}\{B_1\} = \mathbf{P}(a, b)(1 - \mathbf{P}(a, b)), \quad \mathbf{Var}\{B_2\} = \mathbf{P}(\tau a, \tau b)(1 - \mathbf{P}(\tau a, \tau b))
$$

and

$$\begin{aligned}\mathbf{Cov}\{B_1, B_2\} &= \boldsymbol{P}\{X_1 \in [\omega - b, \omega - \tau a \wedge b]\} - \mathbf{P}(a,b)\mathbf{P}(\tau a, \tau b) \\ &= \mathbf{P}(a,b) - \mathbf{P}(a,b)\mathbf{P}(\tau a, \tau b) - \boldsymbol{P}\{X_1 \in [\omega - \tau a \wedge b, \omega - a]\} \\ &= \mathbf{P}(a,b)(1 - \mathbf{P}(\tau a, \tau b)) - \mathbf{P}(a, \tau a \wedge b)\end{aligned}$$

hold. So putting together we obtain

$$\begin{aligned}&\mathbf{Var}\{V\} \\ &= \left(\frac{\ln \tau}{\ln^2(\mathbf{P}(a,b)/\mathbf{P}(\tau a, \tau b))}\right)^2 \left[\frac{\mathbf{Var}\{B_1\}}{\mathbf{P}(a,b)^2} - \frac{2\mathbf{Cov}\{B_1, B_2\}}{\mathbf{P}(a,b)\mathbf{P}(\tau a, \tau b)} + \frac{\mathbf{Var}\{B_2\}}{\mathbf{P}(\tau a, \tau b)^2}\right] \\ &= \left(\frac{\ln \tau}{\ln^2(\mathbf{P}(a,b)/\mathbf{P}(\tau a, \tau b))}\right)^2 \\ &\quad \left[\frac{1 - \mathbf{P}(a,b)}{\mathbf{P}(a,b)} - \frac{2[\mathbf{P}(a,b)(1 - \mathbf{P}(\tau a, \tau b)) - \mathbf{P}(a, \tau a \wedge b)]}{\mathbf{P}(a,b)\mathbf{P}(\tau a, \tau b)} + \frac{1 - \mathbf{P}(\tau a, \tau b)}{\mathbf{P}(\tau a, \tau b)}\right] \\ &= \left(\frac{\ln \tau}{\ln^2(\mathbf{P}(a,b)/\mathbf{P}(\tau a, \tau b))}\right)^2 \Bigg[(1 - \mathbf{P}(a,b))\mathbf{P}(\tau a, \tau b) - 2[\mathbf{P}(a,b)(1 - \mathbf{P}(\tau a, \tau b)) \\ &\quad - \mathbf{P}(a, \tau a \wedge b)] + (1 - \mathbf{P}(\tau a, \tau b))\mathbf{P}(a,b)\Bigg] / (\mathbf{P}(a,b)\mathbf{P}(\tau a, \tau b)) \\ &= \left(\frac{\ln \tau}{\ln^2(\mathbf{P}(a,b)/\mathbf{P}(\tau a, \tau b))}\right)^2 \frac{\mathbf{P}(\tau a, \tau b) - \mathbf{P}(a,b) + 2\mathbf{P}(a, \tau a \wedge b)}{\mathbf{P}(a,b)\mathbf{P}(\tau a, \tau b)}.\end{aligned}$$

Since further (21) holds as $t \to \infty$, the first claim follows. Along the lines of Müller (2003) the second can be established easily, thus the proof. □

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