

Basis Risk and Cat Risk Management

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Abstract: Index-based securities can be used to manage the risk of losses from catastrophic events such as hurricanes and earthquakes. In contrast to traditional (indemnity-based) covers the distinction between index value and the company's own loss experience gives rise to some basis risk in transactions of this type. After a short motivation the distribution of the basis risk will be formulated in terms of the copula connecting the index and the losses. Next we will show how such a formulation in terms of a copula can be used to understand the efficiency of the securitization.

Keywords: Basis risk, copulas, natural catastrophes, risk management.

1. Introduction

For some time now we have seen reinsurance coverage that is based on some parametric trigger instead of the company's actual losses. This parametric trigger typically takes the form of an index which is derived from publicly available information.

The advantage of such a construction is clear. Any intransparency regarding loss regulation or loss development is avoided. The Northridge earthquake provides a reminder of just how long it can take to establish the ultimate loss amount even for cat business. In contrast, the index value is available without delay. Therefore products of this type can also be used to transfer insurance risk directly to the financial markets.

These notes try to show how a quantitative approach to some of the issues raised by the replacement of indemnity coverage through such an attempt at securitization might look like. Before setting up the formalism it might be helpful to go through a brief description of the information and processes involved. These „stylized facts“ should give some substance to the formulas that will follow.

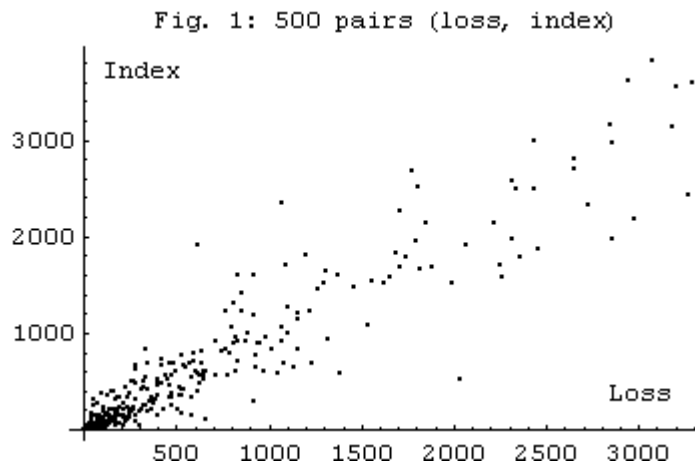
„Stylized facts“

Let's start with a look at a possible construction of the index. For example, it can be constructed as some well-defined function of wind speeds at certain sites which are determined by official meteorological services. The art of the index modeller then is to derive such a function of the set of wind speed readings that takes values as close as possible to the losses from the portfolio to be protected. Once such an index has been established, an event will not only lead to some loss X to the insured portfolio, but in addition will give rise to an index value Y .

The modelling of natural catastrophes is particularly suitable for this kind of products as here the portfolio loss is very often derived from models for certain physical triggers such as wind

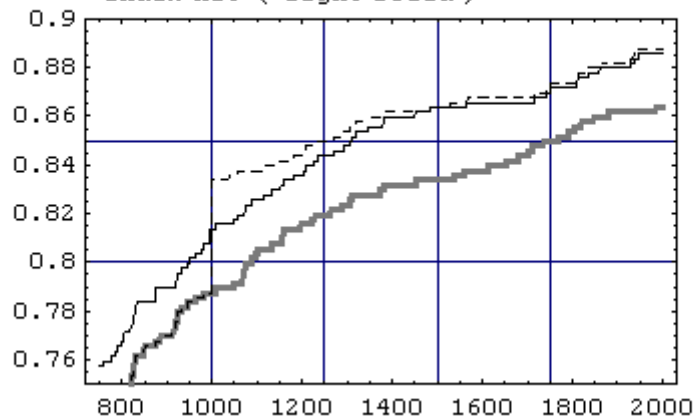
speeds. If this modelling is extended to produce the index value in addition it can be used to set up a catalog of pairs (X_i, Y_i) of possible event outcomes for the portfolio and the index.

The following graph [fig. 1] (which is based upon artificial data) should give an idea of the nature of such a catalog. It shows 500 equiprobable pairs of events (X_i, Y_i) . The plot clearly reveals the strong dependence between the two margins: for larger losses X_i we see most of the time larger index values Y_i as well.



Armed with such a catalog we are already in a position to compare the effect of an indemnity cover on the distribution of the net severity with that of the same cover on an index basis. We just have to evaluate for each pair (X_i, Y_i) the loss retained after the respective coverage. Fig. 2 exemplifies this for a cover 500 xs 1000 (placed 100%).

Fig. 2: Empirical distributions:
gross (bold), indemnitynet (broken)
index net (light solid)



The advantages of an index-based approach (as compared to a traditional indemnity cover) mentioned before will have to be put in relation to the basis risk incurred by replacing the „true“ loss by an index. The following section is going to provide a somewhat formal frame for this task; it will be applied to the sample data in the final section.

2. Formalism

We continue to denote the gross event loss from the company's portfolio by X and the value taken by the index by Y , their respective distributions being

$$F(x) = \text{Prob}[X \leq x], \quad G(y) = \text{Prob}[Y \leq y]$$

The copula describing the dependence structure between these random variables X and Y will be called C . Thus the joint distribution of X and Y can be written as

$$J(x, y) = \text{Prob}[X \leq x, Y \leq y] = C[F(x), G(y)]$$

Now a CatXL cover 100% of L vs R can be written as a monotonous function of the underlying loss X

$$r(X) = s(X - R)^+ - s(X - R - L)^+$$

i.e. the „share“ 100% of that part of the loss that exceeds the „retention“ R is paid by the reinsurer, but only up to a maximum payment of sL (where L is the „cover“ granted).

Applied to the company's loss X this describes the traditional indemnity cover; applied to the value Y of the index we obtain the payment $r(Y)$ from the index cover. Hence the retained event loss Z is obtained by deducting either the indemnity cover $r(X)$ or the index cover $r(Y)$ from the original loss X :

$$Z = X - r(X), \quad \text{or} \quad Z = X - r(Y)$$

N.B.: By setting $G=F$ and taking C to be the comonotonicity copula $C[u,v]=\min(u,v)$ the indemnity cover is formally a special case of the index protection. This may be understood as taking the company's actual loss experience as an index.

We can now start to calculate the distribution H of the net loss Z

$$\begin{aligned} H(z) &= \text{Prob}[Z \leq z] = \langle \Theta(z - Z) \rangle \\ &= \int_0^1 \int_0^1 \Theta(z - F^{-1}(u) + r(G^{-1}(v))) C^{(1,1)}[u, v] du dv \\ &= 1 - \int_{-\infty}^{+\infty} C^{(1,0)}[F(w), G(r^{-1}(w - z))] dF(w) \end{aligned}$$

Here, Θ denotes Heaviside's unit saltus, $\Theta(x < 0) = 0$ and $\Theta(x \geq 0) = 1$, whereas the superscripts appended to C denote the partial derivatives w.r.t. its first and second arguments in the usual way. Lastly, we have introduced the (generalized) inverse r^{-1} of the CXL mapping.

It is now easy to plug in different models for margin and copula and calculate the distribution of the net severity. E.g., the independence copula $C[u,v]=uv$, would lead us to convolution-like formulas. Rather than going into this we shall exploit the properties of the distributions (margins as well as copula) to simplify our result still further.

We observe that since F is a loss distribution we know that $F(x < 0) = 0$. Also, we can give a more explicit form for the second argument of the copula derivative:

$$G(r^{-1}(x)) = G\left(\frac{x}{s} + R\right) \Theta(x) \Theta(sL - x) + \Theta(x - sL)$$

Here, the explicit indication of the support achieved by the Θ terms is particularly helpful for carrying out the calculations. By evaluating the integral for those regions where the copula derivative vanishes ($w-z < 0$) or equals unity ($w-z > sL$) we obtain the following result:

$$H(z) = F(z + sL) - \int_{\max(0,z)}^{\max(0,z+sL)} C^{(1,0)} \left[F(w), G\left(\frac{w-z}{s} + R\right) \right] dF(w)$$

Remark

From this result we can (as mentioned in section 1) retrieve the indemnity case by setting $F=G$ and taking C to be the comonotonicity copula $C[u,v]=\min(u,v)$. We note that we can write the required partial derivative of that copula as $\Theta(v-u)$ and get after some simplification the following expression:

$$H_c(z) = F(z)\Theta(R-z) + F\left(\frac{z-Rs}{1-s}\right)\Theta(z-R)\Theta(R+(1-s)L-z) + F(z+sL)\Theta(z-R-(1-s)L)$$

Of course, this result can be obtained much more elegantly by observing that

$$H_c(z) = \text{Prob}[X - r(X) \leq z] = \text{Prob}[(id - r)(X) \leq z] = F((id - r)^{-1}(z))$$

Here, id denotes the identity mapping, $id(x)=x$.

3. Efficiency of securitization

We go back to the (artificial) data from the first section. In order to apply the formalism of the previous section to that problem, we have to extract the dependence information from the catalog of (loss, index)-pairs.

Now, a convenient way to represent the dependence between the two margins is provided by the empirical copula, which in turn we display graphically by plotting the ranks of the index values against those of the losses [fig. 3]:

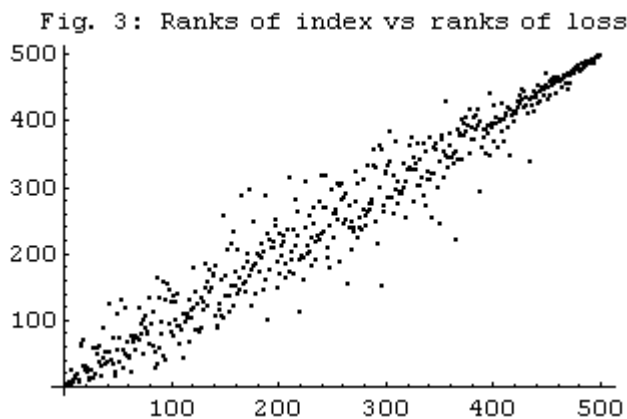


Fig. 3: Ranks of index vs ranks of loss

The tail dependence is even more clearly recognizable in this kind of graph than in those displayed previously. A convenient family of analytical copulas that displays tail dependence is the Gumbel-Hougaard family. It has a single parameter $\theta > 1$ which determines the degree of tail dependence that the copula implies. Here is the analytical form of the Gumbel-Hougaard copula:

$$C_\theta[u,v] = \exp\left(-\left((-\log u)^\theta + (-\log v)^\theta\right)^{1/\theta}\right)$$

For $\theta=1$ this becomes the independence copula $u v$, whereas if θ goes to infinity the copula approaches that of comonotonicity, $\min(u,v)$. It is a rather useful property of the Gumbel-Hougaard copula that it allows us to go smoothly from one of these extremes to the other.

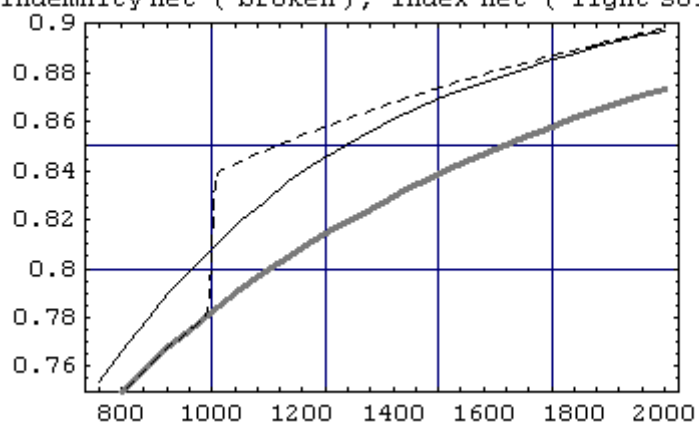
Maximum likelihood estimation can be used to fit a Gumbel-Hougaard copula to this empirical copula. From our data we determine a parameter $\theta=7.3$.

As the marginal distributions are the description of our (hypothetical) book, we do not attempt to estimate them from the simulated event catalog, rather we will assume both marginal distributions to be of the Weibull type, with mean 1000 and standard deviation 2700, i.e.

$$F(x) = G(x) = \left(1 - \exp\left(-\left(\frac{x}{b}\right)^a\right)\right) \Theta(x), \quad a = 0.4397, b = 382.7$$

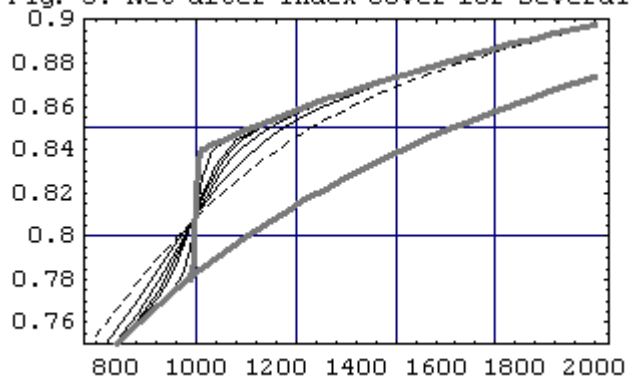
Now a simple quadrature rule suffices to obtain the values of the cumulated distribution $H(z)$ of the net severity. Using the same conventions as in section 1 (bold for the gross severity, light broken line for the indemnity cover, light continuous line for Gumbel with parameter 7.3) we obtain the following plot [fig. 4]:

Fig. 4: Distributions of gross (bold), indemnitynet (broken), index net (light solid)



Now it's easy to analyse the dependence of the result on the value of the Gumbel parameter. The graph below [fig. 5] shows the gross and indemnity net distribution with a bold line, the index cover for the estimated value 7.3 of the Gumbel copula with a broken line, and light solid lines for parameters 15, 25, 35, 45, 100. We note how the index cover slowly approaches the indemnity case as the Gumbel copula becomes more and more comonotonicity-like as the parameter θ grows.

Fig. 5: Net after index cover for several θ 's



We would be able to capture the behaviour of the intermediate cases with some indemnity programme composed of several layers, starting at smaller loss sizes and extending to larger ones, but covering only a smaller share of the loss.

If we were to define the risk content of the net portfolio through application of an appropriate risk measure to its distribution we can study the dependence of this risk measure on the copula parameter. The notion of appropriateness here essentially means that we are interested in the tail of the distribution, describing the large losses – which the risk measure should reflect. One such risk measure (which we will use in our example) is the average loss, given that the loss exceeds the 99%-percentile:

$$\rho(H) = \frac{1}{1 - 0.99} \int_{0.99}^1 H^{-1}(p) dp$$

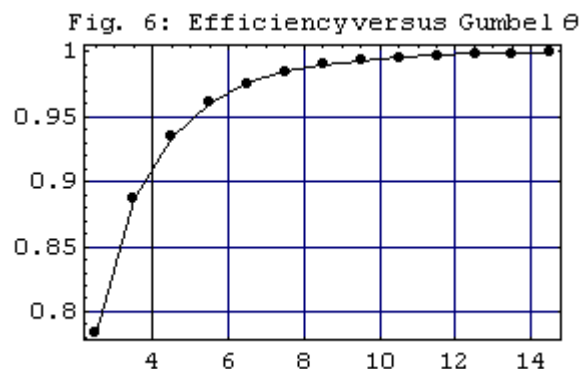
Once we have settled for a risk measure we can define the efficiency of the index protection as the ratio of the risk reduction it effects to the reduction an indemnity cover would produce.

So far, we have based our example on a layer with a very low attachment point, which had the advantage of clear illustrative graphics. As these graphics show the tail of the distribution is hardly affected at all. We wouldn't see any relevant differences in efficiency, regardless of the quality of the index.

However, the situation will change if we look at a layer with a reasonably high attachment point. To become concrete, let us consider a layer of 100% of 500 xs 10000. Its expectation value is 7.17.

N.B.: In „reality“ we would perhaps rather look at the annual distribution of aggregated losses, than at the event distribution. However, our basic line of argument stands without this complication.

So let's conclude the analysis of our example with a look at a plot showing the efficiency defined above as a function of the Gumbel parameter θ [fig. 6]:



Reference

There are quite a few books out now about copulas. For our purposes the following item offers more than enough:

Roger B. Nelsen, An Introduction to Copulas, Springer-Verlag 1998.