A Multiline Risk Factor Model

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Abstract: For a risk model to provide a realistic picture of the risks of an insurance or reinsurance enterprise, it is essential that the model capture systemic risks and parameter risks that are often neglected, including market risks (e.g. pricing and underwriting), parameter risks related to embedded projections (e.g. trend and development), and risks related to processes in the external environment (e.g. inflation and economic performance). These risks are typically non-diversifying and create interdependencies (correlations) among different lines of business and between current underwriting and risk accumulated from prior underwriting (reserves). This paper presents a modeling structure that provides for the reflection of various types of risks in a form structurally related to the operation of such risks. The structure is first presented for a single accident year and then extended to reflect the accumulation of risk over time. The various risk factors provide a basis for understanding the source of correlations and a mechanism for reflecting them. The resulting model can provide a basis for Enterprise Risk Management, capital allocation and risk-adjusted performance measures, and for evaluating the impact and effectiveness of reinsurance.

I. INTRODUCTION

As Enterprise Risk Management increasingly becomes an essential aspect of insurance and reinsurance company management, there are increasing needs for actuaries and others to build realistic whole-company risk models. Other important applications include capital allocation or other risk-adjusted performance measurement, and evaluating the impact of reinsurance on capital needs and ROE. This paper addresses liability-side risks such as underwriting and reserve risks, but not asset risks or other operational risks.

There are many existing models of process risk, notably including the collective risk model and catastrophe risk models developed by various vendors. When parameters of the process risk distributions have been estimated from data samples using statistical methods, modern statistics provides measurements of the accuracy of those parameters, and provisions for sample-size related parameter risk can be included. Catastrophe model vendors include estimates for the uncertainty of the parameters in their models. This paper assumes that process risk distributions and estimates of sample-size related parameter risk are available.

There is far less literature and practice regarding the many other contributors to insurance risk. These include systemic risks, such as pricing and underwriting risks from insurance markets, changes in inflation and other economic conditions, and changes in insurance trends. There are also parameter risks that are frequently neglected, such as the parameters of the underlying trend and development projection models.

For a realistic risk model, it is essential to include reasonable provisions for these many other risks. For a large insurance enterprise, they may in fact constitute the most important sources of risk. Furthermore, these risk factors create interdependency (i.e. correlation). Depending on how the model is constructed, there will be correlations among various lines of business, portfolios, sub-portfolios or components. (Without loss of generality, all such subdivisions will be referred to hereinafter as lines of business, or "LOB's"). Unless dependency among the LOB's is modeled, the model of the overall enterprise will be unrealistically stable. Consideration of specific risk factors creates a basis for understanding the causes of correlations and reflecting them in the model.

Reserve risk is often treated as separate from underwriting risk, but it is of course merely underwriting risk from previous periods that has not yet been discharged, and current underwriting will create reserve risk in the future. To properly measure the risk assumed by writing insurance or ceded by purchasing reinsurance, it is necessary to consider the time frame until the risk exposure is retired, the accumulation of exposure that occurs as a result, and the correlation between the current underwriting and the accumulated exposure.

II. THE ACCIDENT YEAR RISK FACTOR MODEL

The period of future exposure for which risk is being modeled is frequently, but not always, an accident year. To avoid cumbersome language, we will refer to the future exposure period as an accident year, without loss of generality.

2.1 Sources of Risk

There are many risk factors that affect future insurance loss payments. Many (but undoubtedly not all) of these risk factors are discussed in greater detail in what follows. Some may be modeled based on an individual insurance company's dataset; others arise from the external environment, are less easily measured, and may sometimes be neglected in risk modeling. This section provides a brief review of the elements and structure of insurance risk as modeled herein.

2.1.1 Process Risk

Given a specified mathematical model of the insurance loss process with specified parameters, the process risk refers to the variability of the actual losses that occurs because of the random component of the modeled loss process.

(i) The Collective Risk Model

The loss process is generally modeled using the collective risk model. This model assumes that losses arise from separate and independent frequency and severity processes.

The frequency process is described by a mathematical distribution (Poisson or negative binomial) that generates the number of non-zero claims in the exposure period. The distribution forms are naturally diversifying, i.e. the coefficient of variation ("CV") of the frequency is inversely proportional to the square root of the number of expected claims, approaching zero as the number of expected claims becomes large.

The claim size for each claim is described by sampling from a severity distribution. The sampling from the severity distribution is independent from the sampling from the frequency distribution, and each sampled claim size is independent from the others.

(ii) Other Loss Processes

Some loss processes (most notably catastrophe losses) are not described by the collective risk model. In these cases, the process risk is generated by other models, often catastrophe models provided by various vendors.

2.1.2 Parameter Risk

Parameter risk is often broadly defined as including all risks of inaccuracy in the model of the process risk. It includes not only the risk of misestimation of the parameters of the process risk model, but also the risk that the form of the process risk model does not describe the true loss process. Within this broad definition, we include the risks of error in various projections that are part of the estimation of the frequency and severity parameters (e.g. loss development, trend, on-level premiums, etc.) as well as other processes (e.g. economic processes) that contribute to risk.

The importance of parameter risk is noted, for example, in references [1], [2], and [3].

The following list of sources of parameter risk is undoubtedly not exhaustive:

(i) Limited Sample Size

This describes the risk of misestimation of the process risk model parameters that is related to the size of the available data sample. It is important to note that the "data sample" is often not actual data, but rather projections from data, involving, for example, loss development, trend, etc. "Sample size related parameter risk" as used herein relates only to the parameter risk as if the data sample were actual data -- in other words, as if any underlying projections were perfectly accurate.

(ii) Loss Development

Loss development refers to the estimation of ultimate losses for past periods from incomplete data. The most common form of this estimation is through "loss development factors", but the risk arises more generally from the errors of the estimation process, whether that is described as error in the factors or error in the form and parameters of the estimation model.

(iii) Trend

Trend generally describes the phenomena of changes in distribution parameters over time. These may be related to general inflation, "social inflation", frequency trend, etc.

Specific trend factors are applied in most actuarial analyses. Some of these same effects also impact loss development.

(iv) Changes in Trend

Even given perfect estimation of expected future trends, actual future trends will be different, for reasons including changes in general inflation as well as factors more specific to insurance losses. This is in effect an additional process risk, arising from the external environment, and non-diversifying. We list it here as a separate risk factor to differentiate it from the risk of misestimation of the expected trend.

(v) Market Risks

Insurance markets can be volatile. The accident year is in the future, and the insurer's actual rates in place during the accident year may be different from the rates assumed in the analysis. Furthermore, the insurer may not actually achieve its rates in the market, particularly in commercial insurance where there are substantial judgment elements in insurance pricing. Market pressures affect underwriting as well. In a soft market, an insurer that insists upon achieving its price will typically suffer a decrease in its risk quality (adverse selection) while an insurer that insists on maintaining risk quality will typically have to make concessions on price.

(vi) Exposure/On-Level Premiums

The relative level of exposure to losses for different exposure periods is frequently measured through on-level premiums, and sometimes through other exposure measures. In addition to potential inaccuracy in the on-level analysis, risk arises from the fact that no exposure measure perfectly reflects a portfolio's exposure to loss.

(vii) Frequency Contagion

As previously noted, the diversifying frequency risk models assume that the CV of frequency decreases, ultimately approaching zero, as the size of the portfolio increases. Actual experience of large companies contradicts this prediction of the model, and the non-diversifying frequency risk is described as "contagion." Observed contagion may be partially reflective of previously discussed risk factors (imperfect exposure measurement, uncertain frequency trends). Frequency contagion can also describe phenomena that give rise to multiple claims that are not reflected in catastrophe models.

(viii) Differences Between the Historical Portfolio and the Future Portfolio

Property/casualty insurance portfolios change every year. Actuarial analyses often assume that the historical portfolio and the future portfolio are identical. Alternatively, the analysis may include adjustments (imperfect by definition) for changes in the portfolio. In specific cases, changes in the portfolio may be known to be greater than normal, increasing this risk, but the risk exists in all cases.

(ix) Other Economic Risks

In addition to inflation, the losses in an insurance portfolio may be affected by other economic conditions. Examples include unemployment, general economic performance including the performance of specific sectors/industries, real estate markets, etc. These impacts will vary with the lines of business written.

(x) Payment Timing

Estimates of expected payment timing are subject to misestimation and to the risk that the payment pattern changes in the future.

2.2 Model Structure

This section presents a model structure for a single accident year and a single LOB (the “Model”). We describe the form of the Model and how that form relates to the elements of process and parameter risk previously discussed. In general, the reader may note that the Model structure subdivides risks into those that are time sensitive (i.e. that are related to the time between the experience data and the future loss payments) and those that are not. We present the form as a model of aggregate payments, but it can also be used in conjunction with a frequency/severity model.

In the model form as presented below, "$\boldsymbol{i}$" is the payment year ($\boldsymbol{i}$ = 1, 2, … , $\boldsymbol{n}$ for an $\boldsymbol{n}$ year payment pattern).

Random variables (“RV’s”) are denoted as italicized capital letters, and have been given the following names:

- ***A***: "Process+" Risk
- ***B***: Accident Year Deviation
- ***C***: Payment Timing Risk
- ***D***: Trend/Development Parameter Risk
- $\boldsymbol{E_i}$: Future Trend Process Risk

A has a special status and may be a placeholder for output from another model. RV’s ***B*** through ***E*** are the “Risk Factors” of the “Risk Factor Model.” Each of these will be described and discussed subsequently.

Other inputs:

- $\mathbf{P}_i$: Cumulative expected portion of losses paid at time $\boldsymbol{i}$.
- **G**: Average date of payments in the historical data.
- $\mathbf{H}_i$: Average date of payment for payment year $\boldsymbol{i}$ (usually assumed to be at the midpoint of payment year $\boldsymbol{i}$).

The accident year payment in payment year $\boldsymbol{i}$ is denoted $\boldsymbol{AY_i}$ and is modeled as follows:

$$AY_i = A \times B \times D^{(Hi\text{-}G)} \times E_i \times (P_{i \times (C)} - P_{(i\text{-}1) \times (C)}) \qquad [2.2.1]$$

A. **"Process+" Risk**

"Process+" risk is defined as the combination of process risk (Section 2.1.1) and "sample size related parameter risk" (Section 2.1.2, item (i)).

We have chosen to combine process risk and sample-size related parameter risk for convenience. These are the areas of the greatest existing literature and practice (and the least focus for this paper). In some cases, more elaborate models reflecting these risks may already exist, and RV ***A*** would be the placeholder for the output of such models.

As an aggregate model, ***A*** is sampled once per year from an aggregate annual distribution reflecting Process+ risk.[1] The distribution may enter the model either in parametric form or as an empirical CDF or other output from another model.

B. **Accident Year Deviation**

B reflects that portion of parameter risk that is not specifically related to the time between the experience data and the future accident year. It can be considered as including the risk factors described in Section 2.1.2, items (v) through (ix) (Market Risks, Exposure/On-Level Premiums, Frequency Contagion, Differences Between the Historical Portfolio and the Future Portfolio, and Other Economic Risks)**.**

B is sampled once per year from a distribution with likely mean unity, although some practitioners may believe that an upward bias is appropriate.

In a frequency/severity context, ***B*** would most likely be applied to expected frequency.

C. **Payment Timing Risk**[2]

C reflects the risks described in item (x). ***C*** is sampled once per year from a distribution with mean unity.

The operation of ***C*** is in the last term of equation [2.2.1]. If the value of ***C*** is 1.0, then that term becomes the typical payment pattern increment for payment year ***i***. Values of ***C*** of (for example) 0.9 and 1.1 represent "10% deceleration" and "10% acceleration",

[1] Since ***A*** is sampled only once, AY_i does not reflect the subdivision of the Process+ risk into its individual payment year components. This anticipates that we are most interested in the sum of all AY_i's. If the breakdown of Process+ risk into individual payment year components is important, Section 3 discusses an alternative formulation for AY_i.

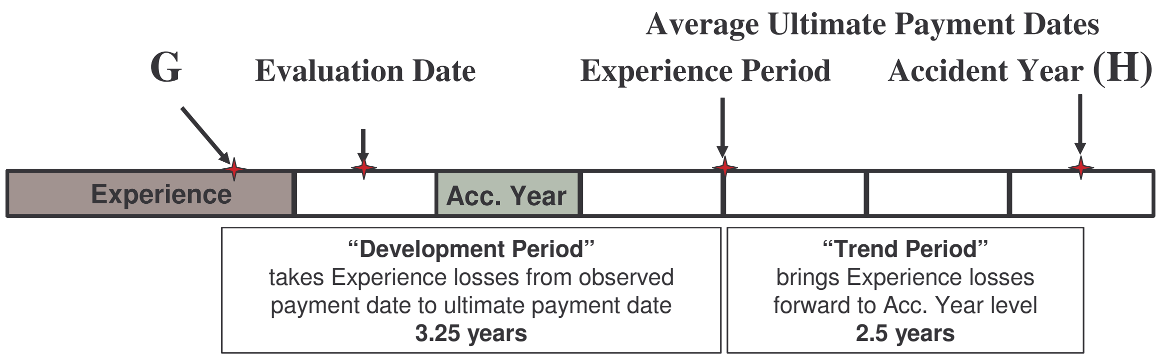
[2] Payment timing risk is important for certain reinsurance structures that are sensitive to variations in payment timing. In most other cases, the impact of payment timing risk appears to be negligible and this term may be ignored.

respectively, and cause projected payments to be made either 10% later or 10% sooner than expected.

D. Trend/Development Parameter Risk

D reflects the risks described in Section (ii) (Loss Development) and (iii) (Trend) but not including (iv) (Changes in Trend). ***D*** is sampled once per year from a distribution with likely mean unity (lognormal with $\mu = 0$, i.e. median unity, is a convenient choice).

The operation of ***D*** is in the third term of equation [2.2.1]. The value of ***D*** represents an *annualized* error which is compounded for the time period from **G** to $\mathbf{H}_i$. The following timeline provides an illustration of the time period:



In the illustrated example, the Experience Period is two years long. The Evaluation Date (i.e. "now") is six months before the beginning of the accident year. The assumed payment pattern has a mean time to payment of four years from the accident date.

G is the average date of the actual loss payments in the experience data. Since all such payments have occurred between the beginning of the Experience Period and the Evaluation Date, **G** is necessarily somewhere between those two dates. The ultimate average date of payment for Experience period losses is four years from the midpoint of the experience period. The development process projects the actual observed losses from the experience period to their ultimate value, and thus 3.25 years (in the example) is the "trend" period for the loss development process.

The ultimate average date of payment for the accident year, labeled **H,** corresponds to the average of the $\mathbf{H}_i$'s for all payment years ***i***. The period of 2.5 years (in the example) is the period that conventional trend factors are often applied. It is also equal to the time from the midpoint of the Experience Period to the midpoint of the accident year.

By compounding ***D*** from time **G** to time **H**, the intended effect of this term is to reflect the parameter risk in both the loss development and trend processes. The decision to combine the parameter risks for trend and development into a single compounded annual error is based on the structural similarity of development and trend processes. In fact, Zehnwirth [4] describes development in the form of trend. While the single distribution of ***D*** will not precisely replicate the effects of several different development parameters with different standard errors, there is substantial structural similarity. Other reserve uncertainty models such as Mack [5] and Murphy [6] also provide development parameters with annual standard errors that are compounded to project payments, creating a structurally similar effect of parameter uncertainty.

Because of the multiple year compounding, ***D*** usually creates an upward bias that the author believes is appropriate. The bias increases with the length of the projection period.

E. Future Trend Process Risk

$\boldsymbol{E_i}$ reflects the risks described in item (ii) (Changes in Trend). As the subscript implies, there are different values of $\boldsymbol{E_i}$ for each payment year. The $\boldsymbol{E_i}$'s are the outputs of another model, and are generated once per year.

The $\boldsymbol{E_i}$'s are generated by a time series model. Models of this type are used for processes involving random changes over time. We have used one of the simplest time series forms, a first order autoregressive process ("AR-1"). This is a mean reverting process, meaning that while the trend may drift away from the long-term mean, it will have a tendency to revert to the mean.

Since we are modeling deviations from the mean trend, the deviations necessarily have a mean of zero. The following model form reverts to a mean of zero:

$\boldsymbol{X_i}$ ($\boldsymbol{i}$ = 1, 2, …, $\boldsymbol{n}$) are independent normal mean zero RV's drawn from the same distribution.

$$\boldsymbol{t_1 = X_1} \qquad [2.2.2]$$

$$t_k = \rho \times t_{k-1} + X_k \qquad [2.2.3]$$

$$E_i = \prod_{k=1}^{i} (1 + t_k) \qquad [2.2.4]$$

or alternatively,

$$E_i = \exp\left(\sum_{k=1}^{i} t_k\right) \qquad [2.2.5]$$

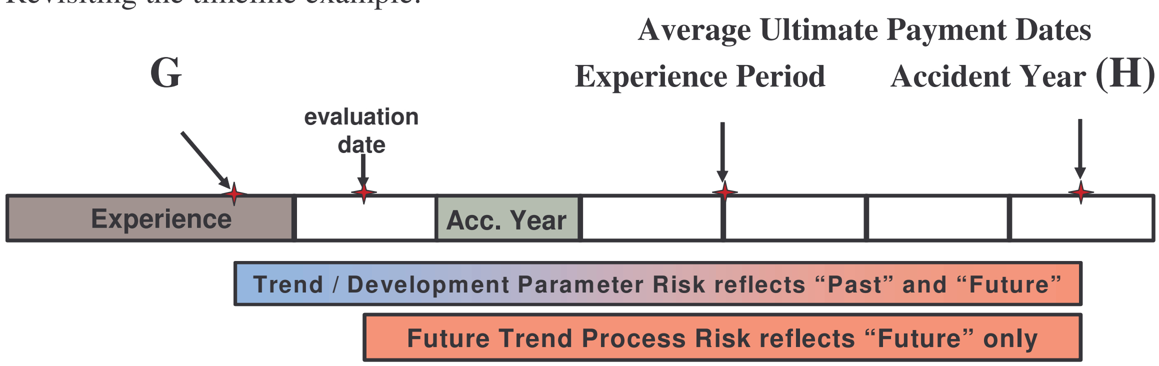
$0 \le \rho \le 1$ is the autocorrelation coefficient. Equation [2.2.5] is the more elegant and mathematically tractable form.

Equations [2.2.2] and [2.2.3] define the AR-1 process. The values $\boldsymbol{t_k}$ represent the deviation from the mean trend in each future calendar year. The value $\boldsymbol{E_i}$ in equation

[2.2.4] or [2.2.5] represents the cumulative multiplicative impact of all of the annual deviations affecting payment year $\boldsymbol{i}$. As with RV $\boldsymbol{D}$, the $\boldsymbol{E_i}$'s will typically create an upward bias.

Briefly revisiting the entire future trend model, it consists of mean future trend (whose true value is not known) and deviations of the actual trend in each future year from the mean. The mean future trend is assumed to be reflected in the base actuarial projection. The risks related to future trend are possible misestimation of the mean trend, a parameter risk reflected in $\boldsymbol{D}$, and possible future deviations from the mean trend, a non-diversifying process risk reflected in $\boldsymbol{E}$, which exists regardless of the accuracy of the projection.

Revisiting the timeline example:



Trend and development projections are from the experience data (in the past) to the projected payment (in the future), and thus the potential errors in the projections, reflected by $\boldsymbol{D}$, apply for this period. Uncertainty as to future deviations from the mean trend, reflected by $\boldsymbol{E}$, applies only to the future period.

The additional risk created by the time series model is an unusual feature, but, in the author's opinion, an essential one. Without it, existing models implicitly assume that we know with certainty that the future is identical to the past – we just don't know the parameters. The time series model provides a mechanism for rejecting that notion. The increasing uncertainty with the length of the projection period more realistically reflects the dynamic aspect of risk.

2.3 Correlation in the Multiline Model

The Model to this point has been for a single LOB, but an important goal of the Model is to provide a basis for reflecting the correlations which may be expected to occur among LOB's. When several LOB's are included, the model allows correlation among the LOB's for the corresponding RV's. For example, if there are three LOB's, three correlated values of $\boldsymbol{B}$ may be sampled for each year. However, the RV's represented by different letters are always mutually independent.

RV $\boldsymbol{A}$ is not usually expected to be correlated, while the risks reflected in the remaining RV's would generally be expected to exhibit some cross-LOB correlation. In the following, the issue of correlation is briefly discussed for each of the components of the Model discussed above.

2.3.1 Process+ Risk (*A*)

Process risk is defined as random and diversifying and is not usually expected to be correlated. The data "samples" analyzed are usually separate for the different LOB's, so that sample size related parameter risk would also not usually be correlated. Therefore, in the great majority of cases, Process+ risk is not expected to be correlated across LOB's.

There are exceptions. For example, if two separate LOB's are exposed to losses from the same catastrophe events, this will create correlation (which is usually analyzed with catastrophe models). Another example of process risk correlation is umbrella losses and primary liability losses, where the umbrella policies are written over the primary policies.

As previously noted, ***A*** can be a placeholder for the output of another model. If so, that other model may reflect some cross-LOB correlation, e.g. the correlation created by exposure to the same catastrophe events.

2.3.2 Accident Year Deviation (*B*)

While this component is intended to represent a number of risk factors, the most significant risk factor is the effect of insurance market volatility. While the markets for different LOB's do not move identically, there is a strong tendency for them to move together. Insurers frequently experience deteriorating or improving underwriting results for many LOB's simultaneously. The strength of the correlation will tend to vary with how closely related the markets are. However, the entire insurance industry tends to experience cycles of capital shortage and capital abundance, affecting market prices for all products, so some correlation would normally be expected in most cases.

2.3.3 Payment Timing Risk (*C*)

This risk has elements of parameter risk (misestimation of the payment pattern) and a non-diversifying process risk (possible systematic future changes in the payment pattern), with probably a larger weight to the former. Refer to discussion of the next item.

2.3.4 Trend/Development Parameter Risk (*D*)

Misestimation of trend and development parameters is believed to be correlated among LOB's and among different insurers/reinsurers. Evidence of this correlation is the frequently observed phenomenon of loss reserve deterioration occurring simultaneously for different LOB's and different companies. Within a company (the applicable case), likely explanations for cross-LOB correlation include: (1) similar judgments regarding trends for different LOB's; (2) similar management philosophy regarding reserves; and (3) similar management philosophy regarding settlement practices.

2.3.5 Future Trend Process Risk (*E*)

This component is significantly influenced by the uncertainty in general inflation, and all claims for all LOB's (within a country) occur in the same inflationary environment. Subcategories of inflation (e.g. medical costs vs. construction costs) move independently to some degree; however, most tend to be significantly correlated. Insurance losses for different LOB's are frequently exposed to the same inflation components. For example, both bodily injury liability and workers compensation losses compensate for medical costs and lost wages, and bodily injury liability pays for similar items whether it arises from an auto or a general liability claim, a personal or a commercial policy. As a result, correlations for this risk component are often expected to be fairly high.

Recall that the E_i's are the result of n different RV's, $X_i \ldots X_n$. For each payment year i, the X_i's for different LOB's can be correlated. However, the X_i's for different payment years are always mutually independent.

III. ACCUMULATION OF RISK OVER TIME

Property-Casualty insurance (or General Insurance) contracts provide coverage for exposures during a limited period of time (usually one year), but the insurer's obligations that arise from that exposure are not typically time-limited. The insurer's loss reserves are a valuation of the total of such obligations accumulated through a point in time. The actual realization of those obligations may differ substantially from their current valuation, and the uncertainty regarding the true value of existing obligations is described as *loss reserve risk*.

Underwriting risk is usually considered a separate risk category, referring to additional risk to which the insurer will become exposed by continuing to accumulate exposures for an additional period of time, usually one year.[3] Underwriting risk relates to uncertainty regarding premiums, losses and expenses, but not to uncertainty relating to investments, other operational risks, etc.

Capital is required to provide adequately for both reserve risk and underwriting risk. Required capital formulas and ratios used by regulators and rating agencies include charges for both categories of risk. More theoretically, the adequacy of capital is often measured in terms of the company's ability to satisfy existing obligations plus those that will be accumulated for another year.

When insurance business is written, it contributes to underwriting risk during the period when premium is earned and loss obligations are incurred. Thereafter, it contributes to reserve risk until all of the obligations are satisfied. When we consider the capital required to write insurance business (for pricing, capital allocation, etc.), we must include

[3] For the purpose of this paper, underwriting risk will refer to premiums earned and losses incurred during an accident year ("AY").

all the capital that will be required over the time period for which it will be required, i.e., its contribution to current underwriting risk and future reserve risk. Similarly, when we measure the impact of (prospective) reinsurance on risk, we must include its impact on future reserve risk.

3.1 A Convenient Mechanism: The "As-If" Reserves

First, some notation:

$\mathbf{P}_i$ **:** cumulative AY payment pattern, $i = 1, \ldots n$; $\mathbf{P_0} = 0$; and $\mathbf{P}_x = 1$ for $x \geq n$.
$\mathbf{p}_i$ **:** incremental AY payment pattern; $\mathbf{p}_i = \mathbf{P}_i - \mathbf{P}_{i-1}$
μ_{AY} **:** mean losses for the accident year.
TF : annual trend (as a factor, e.g. 1.05)

We will use the operators E( ), CV( ), var( ), cov( ), and corr( ) to denote expected value, coefficient of variation, variance, covariance, and correlation.

In the year following the accident year, the expected contribution of the accident year losses to reserves will be $\mu_{AY}(1 - \mathbf{P}_1)$. With annual interest rate ***int***, the discounted sum of all future expected contributions to reserves $= \mu_{AY} \sum_{i=1}^{n-1} (1+\text{int})^{-i}(1-P_i)$.

The "As-If" reserves is a closely related concept. Define the As-If reserves as the reserves the company would have accumulated at the beginning of the accident year, if it had always written the current volume of business (adjusted for trend). Let ***R*** denote the random losses for the As-If reserves, in total. Then,

$$E(R) = \mu_{AY} \sum_{i=1}^{n-1} TF^{-i}(1-P_i)$$

If the trend rate and interest rate were identical, the As-If reserves would equal the discounted sum of the accident year's future contributions to reserves.

The As-If reserves mechanism has other advantages:

1. It can measure the impact of the risk expected to be accumulated at a point in time caused by correlated risk factors.
2. Reinsurance being analyzed or considered can be applied to the accident year and the As-If reserves, providing a more valid measure of the impact of the reinsurance on accumulated risk and on capital absorbed over the full life of the accident year.

We will also need $\mathbf{Q}_i$, the cumulative payment pattern for ***R***:

$$Q_i = \sum_{j=1}^{n-1} TF^{-j}(P_{i+j} - P_j) \bigg/ \sum_{j=1}^{n-1} TF^{-i}(1-P_j)$$

3.2 Applying the Risk Factor Model to As-If Reserves

We previously discussed the expansion of the Model from a single LOB to a multiline model. For each LOB in the model, we now add the effect of accumulated risk by including the corresponding As-If reserves as an additional component using a nearly identical structure.

The reserve risk arises from a series of consecutive accident years. There are significant dependencies among the risks related to the individual accident year components of the reserve. These same dependencies create correlation between the reserve risk and the underwriting risk.

We start by recalling the single accident year aggregate risk factor model (equation [2.2.1]).

$$AY_i = A \text{ x } B \text{ x } D^{(\mathbf{H}i\text{-}\mathbf{G})} \text{ x } E_i \text{ x } (\mathbf{P}_{i \text{ x } (C)} - \mathbf{P}_{(i\text{-}1) \text{ x } (C)}) \qquad [2.2.1]$$

We make the following adaptations and interpretations of equation [2.2.1] for the accumulated risk model:

- RV's C, D and E_i are sampled once per year and apply identically to the accident year and the reserves.
- We require a distribution of A for reserves that corresponds to A for the accident year. Denote the two RV's as A_{AY} and A_R (A_{AY} is the previous A). In the following section, we determine the mean and CV of A_R using a model that subdivides RV A_{AY} into payment period components.
- We require a distribution of B for reserves that corresponds to B for the accident year. Denote the two RV's as B_{AY} and B_R (B_{AY} is the previous B). In the second following section, we determine both the CV for B_R and the correlation between B_{AY} and B_R using a model of the relationship among values of RV B_{AY} for the various accident year components of the reserves.

Thus we rewrite equation [2.2.1] as:

$$AY_i = A_{AY} \text{ x } B_{AY} \text{ x } D^{(\mathbf{H}i\text{-}\mathbf{G})} \text{ x } E_i \text{ x } (\mathbf{P}_{i \text{ x } (C)} - \mathbf{P}_{(i\text{-}1) \text{ x } (C)}) \qquad [3.2.1]$$

and for the reserves:

$$R_i = A_R \text{ x } B_R \text{ x } D^{(\mathbf{H}i\text{-}\mathbf{G})} \text{ x } E_i \text{ x } (\mathbf{Q}_{i \text{ x } (C)} - \mathbf{Q}_{(i\text{-}1) \text{ x } (C)}) \qquad [3.2.2]$$

The subsequent sections further develop the values of the RV's A_R and B_R. As previously noted, the RV's C, D and E_i will be identical for the reserves and the accident year and will not be discussed further.

3.3 Process Risk for the As-If Reserves

To determine the relationship between CV($\boldsymbol{A_{AY}}$) and CV($\boldsymbol{A_R}$), we treat the process+ risk structurally as if it were process risk. Thus $\boldsymbol{A_{AY}}$ and $\boldsymbol{A_R}$ will be considered as independent.[4] We make the additional simplifying assumption that the process risks for the individual payment year components of the accident year are mutually independent.[5]

Denote the CV of the accident year losses paid in year $\boldsymbol{i}$ as $\boldsymbol{CV_i}$. We further assume that the CV is unaffected by trend. Therefore $\boldsymbol{CV_i}$ is the CV corresponding to payment period $\boldsymbol{i}$ for any accident year contributing to the As-If reserves.

The mean and CV[2] for $\boldsymbol{A_R}$ are then calculated in a straightforward manner by combining the independent accident year components.[6] Thus:

$$E(A_R) = E(A_{AY})\sum_{i=1}^{n-1} TF^{-i}(1-P_i) = E(A_{AY})\sum_{i=1}^{n-1} TF^{-i}\sum_{j=i+1}^{n} p_j$$

$$CV^2(A_R) = \sum_{i=1}^{n-1} TF^{-2i}\sum_{j=i+1}^{n} p_j^2 CV_j^2 \bigg/ \left(\sum_{i=1}^{n-1} TF^{-i}(1-P_i)\right)^2$$

Various published stochastic development models lead to a number of different forms for the relationships among the $\boldsymbol{CV_i}$'s. The following example is from Gluck [7], which develops variance relationships based on collective risk theory and a functional relationship between payment lag and mean claim severity.[7]

$$CV_i^2 = CV^2(A_{AY})\left(\frac{2+(P_i+P_{i-1})(e^c-1)}{(p_i)(e^c+1)}\right) \qquad [3.3.1]$$

In equations [3.2.1] and [3.2.2], the RVs $\boldsymbol{A_{AY}}$ and $\boldsymbol{A_R}$ are each sampled only once, and thus the resulting values of $\boldsymbol{AY_i}$ and $\boldsymbol{R_i}$ do not truly reflect the process risk of the individual payment year components, but will be generally be adequate when only the

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[4] Sample-size related parameter risk is undoubtedly not independent between the accident year and the reserves, and may even be identical.
[5] This is a fairly common assumption in stochastic models of the development triangle, although clearly not accurate for LOB's with significant periodic payment of claims.
[6] Note (in the equation that follows) that E($\boldsymbol{A_{AY}}$) is not the same as $\boldsymbol{\mu_{AY}}$. This is because the terms involving risk factors $\boldsymbol{C}$, $\boldsymbol{D}$ and $\boldsymbol{E}$ are not generally unbiased.
[7] The parameter $-\infty < \boldsymbol{c} < \infty$ in equation [3.3.1], defined in [7], controls the relationship between average claim severity and payment lag. At $\boldsymbol{c} = 0$, claim severity and payment lag are unrelated, and the variance of incremental payments is proportional to the mean (a fairly common version, noted in England and Verral [8], Gluck [9], and others). More often, $\boldsymbol{c} > 0$, increasing the variance associated with the tail. Negative values of $\boldsymbol{c}$, decreasing the tail variance, are also possible and may be used to reflect the impact of periodic payments, where the independence assumption is not valid.

sum of the $\boldsymbol{AY_i}$'s and $\boldsymbol{R_i}$'s are used. However, in some applications this distinction may be important. In this case, the values $\boldsymbol{A_{AY}}$ and $\boldsymbol{A_R}$ in equations [3.2.1] and [3.2.2] would be replaced by separate independent RV's for each of the $\boldsymbol{AY_i}$'s and $\boldsymbol{R_i}$'s, denoted $\boldsymbol{A(AY_i)}$ and $\boldsymbol{A(R_i)}$.

Then, $\quad E(\boldsymbol{A(AY_i)}) = E(\boldsymbol{A_{AY}})$ for all $\boldsymbol{i}$,

$E(\boldsymbol{A(R_i)}) = E(\boldsymbol{A_R})$ for all $\boldsymbol{i}$,

$CV(\boldsymbol{A(AY_i)}) = \boldsymbol{CV_i}$,

and

$$CV^2(\boldsymbol{A(R_i)}) = \sum_{j=i+1}^{n} TF^{2(i-j)} p_j^2 CV_j^2 \Bigg/ \left(\sum_{j=i+1}^{n} TF^{i-j} p_j \right)^2$$

3.4 Accident Year Deviation for the As-If Reserves

In equation [3.2.1], $\boldsymbol{B_{AY}}$ reflects the systemic risk falling on the accident year, especially pricing and underwriting risk. Reserve deficiencies often accumulate during extended periods of unfavorable pricing and underwriting. To model these elements of risk in a reserve context, two issues arise:

1. the extent to which these risks are correlated across adjacent accident years; and
2. the rate at which the risks reduce as the accident year matures.

Issue #1:

We reflect the correlation of values of $\boldsymbol{B_{AY}}$ across accident years by considering $\boldsymbol{B_{AY}}$ to be the result of a first order autocorrelated (AR-1) process. Denoting the value of $\boldsymbol{B_{AY}}$ for accident year $\boldsymbol{k}$ as $\boldsymbol{B_{AY}(k)}$,

$$\boldsymbol{B_{AY}(k)} - 1 = \boldsymbol{\omega}\,(\boldsymbol{B_{AY}(k}-1)-1) + \boldsymbol{Y_k} = \sum_{i=0}^{\infty} \omega^i Y_{k-i} \qquad [3.4.1]$$

where the $\boldsymbol{Y_k}$'s are independent, mean zero RV's with common standard deviation $\boldsymbol{\sigma_Y}$, and $0 \le \boldsymbol{\omega} \le 1$ is the autocorrelation coefficient.[8]

It follows that $E(\boldsymbol{B_{AY}}) = 1$ and $\mathrm{var}(\boldsymbol{B_{AY}}) = \boldsymbol{\sigma^2_Y} / (1 - \boldsymbol{\omega^2})$ [3.4.2]

$\boldsymbol{B_R}$ is a weighted average of its accident year components:

[8] Section II includes an AR-1 process underlying the $\boldsymbol{E_i}$'s using autocorrelation coefficient $\boldsymbol{\rho}$ and annual independent RV's $\boldsymbol{X_i}$. We use $\boldsymbol{\omega}$ and $\boldsymbol{Y_i}$ here to distinguish between the two AR-1 processes.

$$B_R(k)-1=\frac{\sum_{i=1}^{n-1} TF^{-i}(1-P_i)B_{AY}(k-i)}{\sum_{i=1}^{n-1} TF^{-i}(1-P_i)}=\frac{\sum_{i=1}^{n-1} TF^{-i}(1-P_i)\sum_{j=0}^{\infty}\omega^j Y_{k-i-j}}{\sum_{i=1}^{n-1} TF^{-i}(1-P_i)}$$

Regrouping to a weighted sum of $\boldsymbol{Y_k}$'s:

$$B_R(k)-1=\frac{\sum_{i=1}^{\infty} Y_{k-i}\sum_{j=1}^{i} TF^{-j}(1-P_j)\omega^{i-j}}{\sum_{i=1}^{n-1} TF^{-i}(1-P_i)}$$

$$E(B_R)=1.0$$

$$\text{var}(B_R)=\sigma_Y^2\,\frac{\sum_{i=1}^{\infty}\left(\sum_{j=1}^{i} TF^{-j}(1-P_j)\omega^{i-j}\right)^2}{\left(\sum_{i=1}^{n-1} TF^{-i}(1-P_i)\right)^2}=\sigma_Y^2\,\text{Z1} \qquad [3.4.3]$$

$$\text{cov}(B_{AY},B_R)=\sigma_Y^2\,\frac{\sum_{i=1}^{\infty}\;\sum_{j=1}^{i} TF^{-j}(1-P_j)\omega^{2i-j}}{\sum_{i=1}^{n-1} TF^{-i}(1-P_i)}=\sigma_Y^2\,\text{Z2}$$

$$corr(B_{AY},B_R)=\frac{\text{Z2}}{\sqrt{\text{Z1}/(1-\omega^2)}} \qquad [3.4.4]$$

Equations [3.4.2], [3.4.3], and [3.4.4] provide the information necessary to model $\boldsymbol{B_{AY}}$ and $\boldsymbol{B_R}$ as correlated RV's.

Issue #2:
The AR-1 process defined in equation [3.4.1] can be considered to reasonably address issue #2 as well, with suitable choice of the parameter $\boldsymbol{\omega}$.

Consider the following alternative structure for reducing the uncertainty in the values $\boldsymbol{B_{AY}(k)}$ for older accident years. Equation [3.4.1] continues to apply, but the $\boldsymbol{Y}$'s no longer have constant variance $\boldsymbol{\sigma^2_Y}$. Instead, var($\boldsymbol{Y_{k-i}}$) = $(\boldsymbol{\tau^2})^{\boldsymbol{i}}\,\boldsymbol{\sigma^2_Y}$, for $\boldsymbol{i}$ = 0, 1, … ,

$0 \leq \tau \leq 1$. Equations [3.4.2], [3.4.3], and [3.4.4] are unchanged except for replacing ω with $\omega\tau$. Thus, the additional parameter τ is unnecessary; it can be subsumed in the parameter ω.

3.5 Correlations in the Expanded Model

In section 3.2 we recommended, without elaboration, that RV's ***C***, ***D*** and ***E*** be identical for the accident year and As-If reserves for a given LOB. The underlying trend and development projections made at a point in time are typically based on the same analysis for the accident year and reserves, and thus the parameter risk reflected in ***D*** should be identical. RV ***E*** reflects additional time-related uncertainty associated with future payments in a given LOB, regardless of the accident year that gave rise to those payments.

Both ***D*** and ***E*** have substantially greater impact as the length of the payment tail increases, and a longer payment tail also gives rise to larger As-If reserves. The 100% correlations of ***D*** and ***E*** between the accident year and the reserves reflect the non-diversifying risks arising from accumulated exposure to long tail business. Cross-lines correlation is likely to be significant as well, particularly for ***E***. In combination, the risk associated with long payment tails as modeled herein will be greater than in many existing models, an important differentiating feature.

The expansion of the Model to incorporate As-If reserves creates new RV's A_R and B_R for each LOB. If RV's A_{AY} and A_R are modeled as independent, then presumably A_R will be independent of all other RV's as well. However, for the B_R's, additional cross-LOB correlations will be needed, e.g. between B_R for one LOB and B_{AY} for another.

For LOB's α and β, we have used the following correlations:

$$\text{corr}(B_{AY}(\alpha), B_R(\beta)) = \text{corr}(B_{AY}(\alpha), B_{AY}(\beta)) \times \text{corr}(B_{AY}(\beta), B_R(\beta)) \quad [3.5.1]$$

and

$$\text{corr}(B_R(\alpha), B_R(\beta)) = \text{corr}(B_{AY}(\alpha), B_{AY}(\beta)) \times \text{ the lesser of :}$$
$$\text{corr}(B_{AY}(\alpha), B_R(\alpha)) \div \text{corr}(B_{AY}(\beta), B_R(\beta)) \quad \text{or}$$
$$\text{corr}(B_{AY}(\beta), B_R(\beta)) \div \text{corr}(B_{AY}(\alpha), B_R(\alpha)) \quad [3.5.2]$$

The logic of equation [3.5.2] vs. [3.5.1] is that the correlation between $B_R(\alpha)$ and $B_R(\beta)$ is higher than the correlation between $B_{AY}(\alpha)$ and $B_R(\beta)$ and more similar to the correlation between $B_{AY}(\alpha)$ and $B_{AY}(\beta)$, since $B_R(\alpha)$ and $B_R(\beta)$ arise from more similar accident years.

IV. THE RISK DISTRIBUTION AND REQUIRED CAPITAL

Typically, theoretical required capital is related to some risk measure, or perhaps a co-measure to allocate capital. The question we address briefly here is not what risk measure or co-measure to use, but to what distribution shall we apply it? We discuss a few possibilities.

4.1 An Accounting View

Total Gain = E(U/W) + Expected Nominal Losses - Actual Nominal Losses,

where U/W is the underwriting gain. Note that U/W refers only to the accident year, while expected and actual losses refer to the accident year plus As-If reserves.

It may be simpler and more convenient to use:

Gain vs. Mean = Expected Nominal Losses - Actual Nominal Losses.

Pros: If the accounting basis is nominal, then nominal results drive the balance sheet. Impairment and insolvency will be determined on this basis. If the company loses its rating, it will lose its business. It won't matter whether the assets plus investment income are enough to pay the claims.

Cons: Nominal results are economically unrealistic. When time-related risk factors ($\boldsymbol{D}$ and $\boldsymbol{E_i}$) are included, risk can be far more severe for long-dated payments. The increased risk is realistic, but considering the long-dated payment equally as important as the short-dated payment is not. The outlying scenarios for time-related risk are scenarios in which the losses would take many years to be recognized.

4.2 An Economic View

Total Gain = E(U/W) + Expected Nominal Losses - Actual Discounted Losses

or more simply:

Gain vs. Mean = Expected Nominal Losses – Actual Discounted Losses

Pros: Given nominal accounting, available assets will equal expected nominal losses. Thus, this is a true economic view of the sufficiency of assets to pay losses.

Cons: This approach effectively reverses (or ignores) the accounting requirement for undiscounted reserves, and the related solvency requirements. Required capital under this view would need to be set at very remote probabilities, causing us to rely more heavily on the tails of the distributions where the model is least likely to be accurate.

4.3 Discounted Deviation from the Mean

Total Gain = E(U/W) + Expected Discounted Losses - Actual Discounted Losses or more simply:

Gain vs. Mean = Expected Discounted Losses – Actual Discounted Losses

Pros: This is a compromise position. By looking at deviations from the mean on a discounted basis, we give appropriately smaller weight to long-dated payments. However, we don't reverse the undiscounted reserve requirement.

Cons: It doesn't correspond exactly to the accounting view or the economic view.

V. CONCLUSION

The modeling framework developed herein can be used to incorporate measures of process risks, parameter risks, and systemic risks, and can include the impact of accumulated risk and cross-LOB correlation. Reasonable reflection of these many risk factors and their interdependencies is necessary to produce a meaningful model of a company's aggregate risk exposure. Meaningful companywide models are necessary for measuring overall capital adequacy, and for Enterprise Risk Management. Meaningful measures of LOB contributions to corporate risk are necessary for capital allocation or other risk-adjusted price and/or performance measures.

The remaining task is to develop parameters for the various elements of the risk model. The selection of parameters may be based on studies of individual company experience, insurance industry experience, and economic data.

References

[1] American Academy of Actuaries, Committee on Property Liability Financial Reporting, "Risk Transfer Testing Practice Note," November, 2005, p. 15.

[2] Casualty Actuarial Society, "Report of the Working Party on Risk Transfer Testing," August 1, 2005, p. 21.

[3] Kreps, Rodney E., "Parameter Uncertainty in (Log) Normal Distributions," PCAS etc.

[4] Zehnwirth

[5] Mack

[6] Murphy

[7] Gluck, Spencer M., "Process Variance and Development Time," unpublished.

[8] England and Verrall

[9] Gluck, Spencer M., "Balancing Development and Trend in Loss Reserve Analysis," PCAS LXXXIV, 1997, p.515.