Various extensions based on Munich Chain Ladder method

Topic 3: Liability Risk - Reserve models

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Abstract.
In that paper we present some extension and possible generalisation of Munich Chain ladder reserving method. Discussed themes are method of estimate of regression parameters and its impact on the value of the reserves, calculation of mean square error that enables to specify a safety margin of the reserve and also multivariate generalisation of Munich Chain ladder method that is based on recent derivation of multivariate standard chain ladder models.

Keywords: Reserve variability, multivariate models, gap between Paid and Incurred projections

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Introduction

Some new methods how to extend the standard chain ladder techniques of reserving, presented in Mack (1993), were achieved recently. They dealt separately with multivariate extension for correlated portfolio (see Schmidt, Prohl (2005) and Schmidt, Hess, Zossner (2006) and also Kremer (2005)) and also with the problem of a gap between projection of Paid and Incurred data in one dimensional case was described in the paper of Quarg and Mack (2004). Its solution was called the Munich Chain ladder method.

Aim of that paper is to derive properties of Munich Chain ladder that were not shown in original paper and moreover to extend both of the generalisation (more portfolio, both type of data) in one model that could be named as Multivariate Chain ladder model.

We will notify $Y^P_{i,j}, Y^I_{i,j}$ $i = 0,\dots,n,\ j = 0,\dots,n-i$ the cumulative data of paid or incurred claims occurred in period i and reported to insurer after j period after its occurrence.

Munich Chain Ladder - basic recalls and remarks

Firstly we will recall some basic principles of Munich Chain ladder method. Dependencies between Paid and Incurred data are modelled by ratios of paid and incurred values

$$Q_{i,j} = (P/I)_{i,j} = \frac{Y^P_{i,j}}{Y^I_{i,j}}.$$

Average ratio for development period j is defined as

$$q_j = (P/I)_j = \frac{\sum_{i=0}^n Y^P_{i,j}}{\sum_{i=0}^n Y^I_{i,j}}.$$

MCL provides us very nice solution for reducing the gap between Paid and Incurred data projection. This adjustment is based on an idea that if current paid incurred ratio is low (i.e. below average) it means that it is not paid enough or reserved more than enough comparing to another accident years. So it is expected that the amount of payments will be increased in future period which implies that the corresponding paid development factor should be increased and corresponding incurred factor should be lower than usual. If oppositely paid and incurred ratio is above average it may be interpreted that the future payment will be lower or increase of incurred will be substantially higher.

These types of dependencies are modelled for all development period together after standardisation. Thus residual values with mean 0 and standard deviation 1 are used since $\mathrm{Res}(X|C) = \frac{X - \mathrm{E}(X|C)}{\sigma(X|C)}$. In MCL two regression models which finally produce following estimates of development factors are proposed

$$\mathrm{E}\left(\mathrm{Res}\left(\frac{Y^P_{i,s+1}}{Y^P_{i,s}}\right)|\mathbf{B}_i(s)\right) = \lambda^P \cdot \mathrm{Res}(Q^{-1}_{i,s}|Y^P_i(j))$$

and for incurred data

$$\mathrm{E}\left(\mathrm{Res}\left(\frac{Y^I_{i,s+1}}{Y^I_{i,s}}\right)|\mathbf{B}_i(s)\right) = \lambda^I \cdot \mathrm{Res}(Q_{i,s}|Y^I_i(j)).$$

It was switched from paid incurred ratio $Q_{i,s}$ to incurred paid ratio $Q^{-1}_{i,s}$ to obtain positive correlation in both cases. $\mathbf{B}_i(s)$ notifies two dimensional process $(Y_i(s)^P, Y_i(s)^I)$ of both data types in the time of reserve estimates.

$$\mathrm{E}\left(\frac{Y^P_{i,s+1}}{Y^P_{i,s}}|\mathbf{B}_i(s)\right) = f^P_s + \lambda^P \frac{\sigma\left(\frac{Y^P_{i,s+1}}{Y^P_{i,s}}|Y_i(s)^P\right)}{\sigma(Q^{-1}_{i,s}|Y_i(s)^P)} \cdot (Q^{-1}_{i,s} - E(Q^{-1}_{i,s}|Y_i(s)^P)) \tag{1}$$

resp.

$$\mathrm{E}\left(\frac{Y^I_{i,s+1}}{Y^I_{i,s}}|\mathbf{B}_i(s)\right) = f^I_s + \lambda^I \frac{\sigma\left(\frac{Y^I_{i,s+1}}{Y^I_{i,s}}|Y_i(s)^I\right)}{\sigma(Q^{-1}_{i,s}|Y_i(s)^I)} \cdot (Q_{i,s} - E(Q_{i,s}|Y_i(s)^I)).$$

Moreover we assume that vectors $\mathbf{B}_{i_1}(s)$ and $\mathbf{B}_{i_2}(s)$ are stochastically independent if $i_1 \neq i_2$. Let us assume that $Q_{i,j}$ is defined as $\frac{Y^P_{i,j}}{Y^I_{i,j}}$. Parameters λ^P a λ^I determine then the adjustment of SCL development factors.

For practical implementation it was important to obtain further estimates of $\sigma(Q^{-1}_{i,s}|Y_i(s)^I)$, $\sigma(Q_{i,s}|Y_i(s)^I)$ and $\sigma(Q^{-1}_{i,s}|Y_i(s)^P)$. Estimate of $E(Q_{i,s}|Y_i(s)^I)$ was formulated as $\widehat{q_s} = \sum_{i=0}^{n-s} Y^P_{i,s} / \sum_{i=0}^{n-s} Y^I_{i,s}$ Estimate of variability of paid incurred ratio $\sigma(Q_{i,s}|Y_i(s)^I)$ is suggested as $\widehat{\rho^I_s}/\sqrt{Y^I_{i,s}}$ using $\left(\widehat{\rho^I_s}\right)^2 = \frac{1}{n-s}\sum_{i=0}^{n-s} Y^I_{i,s} \cdot (Q_{i,s} - \widehat{q_s})^2$.

Analogously $\widehat{q_s}^{-1} = \sum_{i=0}^{n-s} Y^I_{i,s} / \sum_{i=0}^{n-s} Y^P_{i,s}$ estimates $E(Q^{-1}_{i,s}|Y_i(s)^P)$ and also $\widehat{\rho^P_s}/\sqrt{Y^P_{i,s}}$ is estimate of $\sigma(Q^{-1}_{i,s}|Y^P_{i,s})$ using $\left(\widehat{\rho^P_s}\right)^2 = \frac{1}{n-s}\sum_{i=0}^{n-s} Y^P_{i,s} \cdot (Q^{-1}_{i,s} - \widehat{q_s}^{-1})^2$.

Estimate of regression parameters λ^P and λ^I was originally in the article Quarg and Mack (2004) obtained by ordinary least square method (OLS). If one changes theoretical values by above presented estimates the final projection could be easily obtained.

Despite the undoubtable benefits of MCL there are some open questions in that field. Some of them will be suggested to solve later in that paper:

1. The underlying regression models for Paid (see formula 1) and Incurred data are regarded in practice as rather volatile. It could imply the question if the OLS method is appropriate for the data or even formulated model based on the Paid to Incurred ratios is the most proper one.

2. From practical point of view the information regarding the known value of reserves is useful for amount of payments in future periods but it does not have to be valid that so far paid amounts are useful to predict future development of incurred. That idea was mentioned by Verdier and Klinger (2005). Moreover it could be more more appropriate to use the value of reserve only as relevant information for Paid projection instead of whole incurred since in fact already paid amount, that is part of incurred amount, gives us no more information beyond standard chain lader model.

3. The consequences of the problem if the run-off is not ended after n period after claims' occurrence was mentioned in Quarg and Mack (2004). If we assume that outstanding reserve is set up adequately after n periods of development one could increase Paid value in upper right cell of triangle to match the paid and incurred data in that position and transformed value of $Y^P_{0,n}$ is to be interpreted as final payment for accident year 0. However in some examples of data with significant reserve development the run-off reserve model should be also mentioned.

Methods how to estimate the slope parameters λ in MCL

In our opinion the proposed OLS method for estimating slope parameters λ^P and λ^P for all data is not the most suitable as was mentioned previously in Verdier and Klinger (2005) who suggested implementation of different mean and slope parameters of the model depending on development periods what on the other hand contradict the parsimony of the model stressed by Qurag and Mack (2004). In our approach we will try not to change the general construction of the model 1 but we will adjust the value of the slope parameters by omitting the outliers which may occur in this kind of situation generally across all development periods, see also Jedlicka (2006).

We try to compare original ordinary least squares estimates of λ parameters with estimates obtained by some robust methods. We decided to use Huber's robust regression approach, bisquare methods and Least trimmed squares (LTS) methods. Generally speaking the first two methods evaluate each observation and the outliers "receive" lower weight. Apart from this approach LTS method directly cut off the outlying observation which does not correspond with probabilistic model. Differences between LTS1 and LTS2 are based on numbers of observations that are assumed not to contradict the model. It is about 60% in first situation and 75% approximately in the latter case.

LTS estimator or regression model parameters (see Cizek (2001) for more details) is generally defined as

$$\hat{\beta}^{LTS} = \arg \min_{\beta \in R^{p+1}} \sum_{i=1}^{h} r^2_{[i]}(\beta),$$

where $r^2_{[i]}(\beta)$ represents i-th smallest value among $r^2_1(\beta), \ldots, r^2_n(\beta)$ and $r_i(\beta) = y_i - x_i^{'}\beta$, represents thus OLS residuals. It is important to specify how to select the value of trimming constant h. Generally holds $\frac{n}{2} < h \leq n$ that agrees with our assumption that 75% and 60% data does not contradict the model.

Parameter estimates of three different portfolio including original data used in the article Quarg and Mack (2004) and two another portfolios are presented in Jedlicka(2006).

Elasticity of reserve

The differences in the ultimate projection depending on applied regression estimate lead us to further sensitivity study of relationship between final projection and parameter estimate values. The derivation will be performed only for Paid data as the principles for Incurred are analogous.

We started from formula (1) to define estimate of development factor used in reserve calculation as

$$\widehat{f^P_{i,k}} = \widehat{f^P_k} + \widehat{\lambda^P} \cdot \frac{\widehat{\sigma^P_k}}{\widehat{\rho^P_k}} \left(\widehat{Q_{i,k}}^{-1} - \widehat{q_k}^{-1} \right).$$

It is straightforward that ultimate value of paid amount due to claims occurred in accident period i is calculated as $\widehat{Y^P_{i,n}} = Y^P_{i,a(i)} \cdot \prod_{j=a(i)}^{n-1} \widehat{f_{i,j}}^P$ using notation $a(i) = n - i$.

If we inspect the value of paid ultimate estimate $\widehat{Y_{i,n}}$ as a function of $\widehat{\lambda^P}$ we can derive how strongly the ultimate values (and thus also reserve since reserve differs only by a known diagonal value) are affected by the choice of appropriate estimate of λ. We can write (all derivative are understood with respect to $\widehat{\lambda^P}$):

$$(\widehat{Y^P_{i,n}})' = \sum_{j=a(i)}^{n-1} \frac{Y^P_{i,a(i)}}{\widehat{f^P_{i,j}}} \cdot (\widehat{f^P_{i,j}})' \cdot \widehat{f^P_{i,a(i)}} \cdots \widehat{f^P_{i,n-1}} = \widehat{Y^P_{i,n}} \sum_{j=a(i)}^{n-1} \frac{\widehat{f^P_{i,j}}'}{\widehat{f^P_{i,j}}}.$$

Using formula $\widehat{f^P_{i,k}} = \widehat{f^P_k} + \widehat{\lambda^P} \cdot (\widehat{f^P_{i,k}})'$ we can make final adjustment of the above mentioned formula

$$\frac{(\widehat{Y^P_{i,n}})'}{\widehat{Y^P_{i,n}}} = \frac{1}{\widehat{\lambda^P}} \cdot \left[\sum_{j=a(i)}^{n-1} (1 - \frac{\widehat{f^P_j}}{\widehat{f^P_{i,j}}}) \right].$$

:

We further derived rather surprising result that $\mathrm{E}\left(\frac{(\widehat{Y^P_{i,n}})'}{\widehat{Y^P_{i,n}}}|\mathbf{B}_i(a(i))\right) = 0$ if the expectation exists. That could be interpreted there is no systematical influence of varying the regression estimates onto the ultimates values. It is rational that we do not see regression estimates as random variable since we are interested in the sensitivity only. It is easy to prove that $E((\widehat{f^P_{i,s}})'|\mathbf{B}_i(s),\widehat{\lambda^P}) = 0$ since the model assumptions imply that $E(Q_{i,s}|\mathbf{B}_i(s),\widehat{\lambda^P}) = q_s$ independently on accident period i.

Using again formula $\widehat{f^P_{i,k}} = \widehat{f^P_k} + \widehat{\lambda^P}\cdot(\widehat{f^P_{i,k}})'$ we get $\mathrm{E}(\widehat{f^P_{i,k}}|\mathbf{B}_i(k),\widehat{\lambda^P}) = E(\widehat{f^P_k})$ Provided that both expectations exist we later obtain

$$\mathrm{E}\left(\frac{\widehat{f^P_{i,k}}}{\widehat{f^P_k}}|\mathbf{B}_i(k),\widehat{\lambda^P}\right) = \mathrm{E}\left(\frac{\widehat{f^P_k}+\widehat{\lambda^P}(\widehat{f^P_{i,k}})'}{\widehat{f^P_k}}|\mathbf{B}_i(k),\widehat{\lambda^P}\right) = 1+\widehat{\lambda^P}\mathrm{E}\left(\frac{(\widehat{f^P_{i,k}})'}{\widehat{f^P_k}}|\mathbf{B}_i(k),\widehat{\lambda^P}\right) = 1.$$

This proves the formula $\mathrm{E}\left(\frac{(\widehat{Y^P_{i,n}})'}{\widehat{Y^P_{i,n}}}|\mathbf{B}_i(a(i))\right) = 0$.

Variability and MSE calculation

Munich Chain Ladder gave us so far only formula for $\mathrm{E}\left(\frac{Y^P_{i,s+1}}{Y^P_{i,s}}|\mathbf{B}_i(s)\right)$ or $\mathrm{E}\left(\frac{Y^I_{i,s+1}}{Y^I_{i,s}}|\mathbf{B}_i(s)\right)$ and no information about the variability of development factors. We will drive this starting from regression model of residual data. It is again sufficient to perform the derivation for paid triangle only.

The standard linear model theory implies that

$$\mathrm{var}\left(\mathrm{Res}\left(\frac{Y^P_{i,s+1}}{Y^P_{i,s}}|Y^P_i(s)\right)|\mathbf{B}_i(s)\right) = \frac{\sigma^2_R\cdot\mathrm{Res}^2\left(\frac{Y^I_{i,s}}{Y^P_{i,s}}|Y^P_i(s)\right)}{\sum_i\sum_{j,i+j\leq n}\mathrm{Res}^2\left(\frac{Y^I_{i,j}}{Y^P_{i,j}}|Y^P_i(s)\right)} = \mathrm{var}(\widehat{\lambda^P})\cdot\mathrm{Res}^2\left(\frac{Y^I_{i,s}}{Y^P_{i,s}}|Y^P_i(s)\right).$$

Rearranging this formula we obtain

$$\mathrm{var}\left(\frac{Y^P_{i,s+1}}{Y^P_{i,s}}|\mathbf{B}_i(s)\right) = \mathrm{var}(\widehat{\lambda^P})\cdot\sigma^2\left(\frac{Y^P_{i,s+1}}{Y^P_{i,s}}|Y^P_i(s)\right)\cdot\mathrm{Res}^2(Y^I_{i,s}/Y^P_{i,s}|Y_i(s)).$$

It is straightforward to substitute the theoretical parameters by their estimates similarly as in formula for expectation

$$\sigma^{\widehat{P,MCL}2}_{i,s} = \mathrm{var}(\widehat{\lambda^P})\cdot\sigma_s^{\widehat{P,SCL}2}\cdot\mathrm{Res}^2\left(\frac{\widehat{Y^I_{i,s}}}{Y^P_{i,s}}|Y_i(s))\right)$$

This potentially enables us to calculate the mean square error for Munich Chain Ladder similarly as for Standard Chain Ladder where holds, see Mack (1993)

$$\mathrm{mse}(\hat{R}_i) = \mathrm{E}(R_i-\widehat{R_i}|\mathbf{Y}_i(j))^2 = \widehat{Y^2_{i,n}}\sum_{k=n-i}^{n}\frac{\widehat{\sigma^2_k}}{\hat{f_k}^2}\left(\frac{1}{\hat{Y}_{i,k}}+\frac{1}{\sum_{j=1}^{n-k}Y_{i,j}}\right)$$

if we substitue factors of SCL by corresponding factors of MCL we will obtain following formula for mean squre error of Paid data

$$\mathrm{mse}(\hat{R}_i) = \mathrm{E}(R_i-\widehat{R_i}|\mathbf{B}_i(j))^2 = \widehat{Y^{P\,2}_{i,n}}\sum_{k=n-i}^{n}\frac{\widehat{\sigma^{P\,2}_{i,k}}}{\widehat{f^P_{i,k}}^2}\left(\frac{1}{\hat{Y}^P_{i,k}}+\frac{1}{\sum_{j=1}^{n-k}Y^P_{i,j}}\right)$$

Multivariate methods of Chain Ladder

Recall of approach suggested by Schmidt

Multivariate analogy of Chain Ladder model is again based on stochastic assumption of original Mack's model. Column vector

$$\mathbf{Y}_{i,j} = (Y^1_{i,j}, \ldots, Y^K_{i,j})'$$

represents cumulative amount of claims occurred in period i and developed after j period after occurrence for all K simultaneously analysed insurance portfolios. Moreover following notation was also used

$$\Upsilon_{i,j} = \mathrm{diag}(\mathbf{Y_{i,j}})$$

Obviously $\mathbf{Y_{i,j}} = \Upsilon_{i,j}\mathbf{1}$, where $\mathbf{1}$ marks union vector of dimension K. Generalisation of one-dimensional formula $Y_{i,j+1} = Y_{i,j} \cdot F_{i,j}$ is then obviously

$$\mathbf{Y}_{i,j+1} = \Upsilon_{i,j} \cdot \mathbf{F}_{i,j}$$

where $\mathbf{F}_{i,j} = (F^1_{i,j}, \ldots, F^K_{i,j})'$ represents multivariate version of individual development factor.

3 basic stochastic assumption proposed by Mack (1993) had to be also extended to multivariate case

(a) conditional expectation
(b) conditional variance
(c) developments of different rows of triangles are independent

If $\mathbf{Y}_i(j)$ represent available information based on j period of development that is based generalisation of the assumption was suggested by Schmidt in the following way.

1. There exists K-dimensional development factor independent on year of occurrence that holds

$$\mathrm{E}\left(\mathbf{Y}_{i,j+1} | \mathbf{Y}_i(j)\right) = \Upsilon_{i,j} \cdot \mathbf{f}_j$$

2. There exists matrix Σ_j so that

$$\mathrm{Cov}(\mathbf{Y}_{i1,j+1}, \mathbf{Y}_{i2,j+1} | \mathbf{Y}_{i1}(j), \mathbf{Y}_{i2}(j)) = \Upsilon^{1/2}_{i,j} \Sigma_j \Upsilon^{1/2}_{i,j}$$

if $i = i_1 = i_2$ and also

$$\mathrm{Cov}(\mathbf{Y}_{i1,j+1}, \mathbf{Y}_{i2,j+1} | \mathbf{Y}_{i1}(j), \mathbf{Y}_{i2}(j)) = 0$$

otherwise.

These assumption imply that

$$\mathrm{E}\left(\mathbf{F}_{i,j} | \mathbf{Y}_i(j)\right) = \mathbf{f}_j$$

and

$$\mathrm{Cov}(\mathbf{F}_{i1,j+1}, \mathbf{F}_{i2,j+1} | \mathbf{Y}_{i1}(j), \mathbf{Y}_{i2}(j)) = \Upsilon^{-1/2}_{i,j} \Sigma_j \Upsilon^{-1/2}_{i,j},$$

that is obvious analogy of one-dimensional formulae

$$\mathrm{E}(F_{i,j} | \mathbf{Y}_i(j)) = f_j$$

and

$$\mathrm{var}(F_{i,j} | \mathbf{Y}_i(j)) = \sigma^2_j / Y_{i,j} \qquad i = 0, \ldots, n \qquad j = 0, \ldots n-1$$

:

We recall that in one-dimensional case of Mack's model estimate of f_j is to be found as

$$\widehat{f_j} = \sum_{i=0}^{n-j-1} w_i F_{i,j}$$

This estimate is unbiased if $\sum_{i=0}^{n-j-1} w_i = 1$. Linear model theory implies that OLS estimate is achieved if

$$w_i = \frac{Y_{i,j}}{\sum_{i=0}^{n-j-1} Y_{i,j}}$$

That gives us univariate Chain ladder estimator.

In multivariate case Schmidt suggested estimator $\mathbf{f}_j$ as

$$\widehat{\mathbf{f}_j} = \sum_{i=0}^{n-j-1} W_i \widehat{F}_{i,j}$$

Conditionally unbiased estimate is achieved if $\sum_{i=0}^{n-j-1} W_i = I$

Estimator that minimalize mean square error is derived form linear model theory as

$$\widehat{\mathbf{f}_j} = \left(\sum_{i=0}^{n-j-1} \Upsilon_{i,j}^{1/2} \Sigma_j^{-1} \Upsilon_{i,j}^{1/2} \right) \sum_{i=0}^{n-j-1} \Upsilon_{i,j}^{1/2} \Sigma_j^{-1} \Upsilon_{i,j}^{1/2} \mathbf{F}_{i,j}$$

We suppose that estimator of Σ_j is important for practical purposes as well. However its specification is not included in the mentioned paper of Schmidt and Prohl (2004).

We could use classical estimator as

$$\widehat{\Sigma_j} = \frac{1}{n-j-1} \sum_{i=0}^{n-j-1} \left(\Upsilon_{i,j}^{1/2} \left(\widehat{\mathbf{F}_{i,j}} - \widehat{\mathbf{f}_j} \right) \right) \cdot \left(\Upsilon_{i,j}^{1/2} \left(\widehat{\mathbf{F}_{i,j}} - \widehat{\mathbf{f}_j} \right) \right)'$$

Drawback of that approach might be seen that $\widehat{\Sigma_j}$ is not well defined if $j \geq n-k$ what implies limited benefit of that method.

Recall of approach suggested by Kremer

Multivariate model in the paper of Kremer (2005) is suggested as follows

$$\begin{aligned} Y_{i,j+1} &= Y_{i,j}.f_j + \varepsilon_{i,j} \qquad i = 0, \ldots, n \\ \mathrm{E}(\varepsilon_{i,j}|\cdot) &= 0 \qquad \mathrm{var}(\varepsilon_{i,j}|\cdot) = \sigma_j^2.Y_{i,j}. \end{aligned}$$

Thus it is assumed that $\forall j$ holds

$$Y_{i,j+1}^k = Y_{i,j}^k.f_j^k + \varepsilon_{i,j}^k \qquad i = 0, \ldots, n \qquad k = 1, \ldots, K$$

So original linear model is assumed for all of K analysed run-off triangles. Moreover it is assumed

$$\mathrm{cov}(\varepsilon_{i,j}^{k1}, \varepsilon_{i,j}^{k2}|\cdot) = C_i^{k1,k2} \cdot \sqrt{Y_{i,j}^{k1}} \cdot \sqrt{Y_{i,j}^{k2}}$$

and

$$\mathrm{var}(\varepsilon_{i,j}^k|\cdot) = \sigma_j^{k,2}.$$

:

If $i_1 \neq i_2$ or $j_1 \neq j_2$ then residuals are assumed to be uncorrelated, that is

$$\mathrm{cov}(\varepsilon_{i1,j1}^{k1}, \varepsilon_{i2,j2}^{k2}|\cdot) = 0$$

Not only the estimate of development factor but also the estimator of variance is stressed in that approach. Estimate of $\mathbf{f_j}$ is suggested as Aitken's estimator since it corresponds to regression estimate with nonconstant variance of residuals. However as is stated in Schmidt (2006) this approach could be seen as not effective enough since computation of large-dimensional inverse matrix $\widehat{\Psi}^{-1}$ might be time consuming.

In the proposed model, estimators of f_j^k are firstly calculated for each triangle separately. These estimators would be the optimal ones if $C_{i,j}^{k1,k2} = 0 \forall i, j, k_1, k_2$ For each run-off triangle k variability estimator corresponding above mentioned estimates of development factor is derived through standard formulae

$$\widehat{\sigma_j^{2,k}} = \frac{\sum_{i=1}^{n-j-1}(Y_{i,j+1}^k - \widehat{f}_j^k Y_{i,j}^k)^2}{\sum_{i=1}^{n-j-1} Y_{i,j}}$$

and also covariance estimator as

$$\widehat{C}_i^{k1,k2} = \frac{\sum_{i=1}^{n-j-1}(Y_{i,j+1}^{k1} - \widehat{f}_j^{k1} Y_{i,j}^{k1})(Y_{i,j+1}^{k2} - \widehat{f}_j^{k2} Y_{i,j}^{k2})}{\sum_{i=1}^{n-j-1} \sqrt{Y_{i,j}^{k1}} \sqrt{Y_{i,j}^{k2}}}$$

In lth step the calculated estimators are used for updating a correlation structure that implies new estimator of development factors $\mathbf{f_j}^{l+1}$ based on inverse matrix $\widehat{\sigma_j^{2,k}}^{l}$ and $\widehat{C}_i^{k1,k2l}$. This **iterative procedure** is repeated until the parameters estimates do not converge.

Proposal of Multivariate Munich Chain ladder model

In our opinion it is more convenient to use Kremer's approach for generalisation of Munich Chain ladder model in the multivariate case. Similar idea as presented in Kremer (2005) is applied for linear model that with slope parameters λ^P a λ^I as in MCL. Thus the vector of parameters of $(\lambda^{P,1}, \ldots, \lambda^{P,K})$ is to be estimated simultaneously if MCL model assumption holds for all triangles $k = 1, \ldots, K$

$$\mathrm{Res}\left(\frac{Y_{i,s+1}^{P,k}}{Y_{i,s}^{P,k}}|Y_i(s)^{P,k}\right)|B_i(s)^k = \lambda^{P,k} \cdot \mathrm{Res}((Q_{i,s}^k)^{-1}|Y_i(s)^P) + (\varepsilon_{i,j}^k|Y_i(s)^{P,k})$$

In univariate case it is assumed

$$\mathrm{E}(\varepsilon_{i,j}|\cdot) = 0$$

and

$$\mathrm{var}(\varepsilon_{i,j}|\cdot) = \sigma^2$$

This could be extended into multivariate model as follows

$$\mathrm{cov}(\varepsilon_{i1,j1}^{k1}, \varepsilon_{i2,j2}^{k2}|\cdot) = 0$$

if $i_1 \neq i_2$ and

$$\mathrm{cov}(\varepsilon_{i,j1}^{k1}, \varepsilon_{i,j2}^{k2}|\cdot) = 0$$

:

if $j_1 \neq j_2$ and for equal occurrence and development periods

$$\mathrm{cov}(\varepsilon_{i,j}^{k1}, \varepsilon_{i,j}^{k2}|\cdot) = \sigma_{k1,k2}$$

and moreover we will mark

$$\sigma_{k,k} = \sigma_k^2$$

In more details we could specify multivariate version of MCL via following linear model of regression equations.

$$\begin{pmatrix} \mathbf{Y}^{P,1} \\ \mathbf{Y}^{P,2} \\ \vdots \\ \mathbf{Y}^{P,K} \end{pmatrix} = \begin{pmatrix} \mathbf{X}^{P,1} & & & \\ & \mathbf{X}^{P,2} & & \\ & & \ddots & \\ & & & \mathbf{X}^{P,K} \end{pmatrix} \cdot \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_K \end{pmatrix} + \begin{pmatrix} \varepsilon^{P,1} \\ \varepsilon^{P,2} \\ \vdots \\ \varepsilon^{P,K} \end{pmatrix}$$

we use obvious notation

$$\mathbf{Y}^{P,k} = \begin{pmatrix} \mathrm{Res}\left(\frac{Y_{0,1}^{P,k}}{Y_{0,0}^{I,k}}|\cdot\right) \\ \mathrm{Res}\left(\frac{Y_{0,2}^{P,k}}{Y_{0,0}^{I,k}}|\cdot\right) \\ \vdots \\ \mathrm{Res}\left(\frac{Y_{n-1,1}^{P,k}}{Y_{n-1,0}^{I,k}}|\cdot\right) \end{pmatrix}$$

for response variable of the k-th model of development factors MCL of Paid data where corresponding explanatory variable is

$$\mathbf{X}^{P,k} = \begin{pmatrix} \mathrm{Res}\left(\frac{Y_{0,0}^{I,k}}{Y_{0,0}^{P,k}}|\cdot\right) \\ \mathrm{Res}\left(\frac{Y_{0,1}^{I,k}}{Y_{0,1}^{P,k}}|\cdot\right) \\ \vdots \\ \mathrm{Res}\left(\frac{Y_{n-1,0}^{I,k}}{Y_{n-1,0}^{P,k}}|\cdot\right) \end{pmatrix}$$

and also $\beta_k = \lambda^{P,k}$.

Based on above mentioned assumption of uncorrelated residuals in different periods we get

$$\mathrm{var}\begin{pmatrix} \varepsilon^{P,1} \\ \varepsilon^{P,2} \\ \vdots \\ \varepsilon^{P,K} \end{pmatrix} = \Sigma \bigotimes I$$

Multivariate model is thus specified via set of linear regression equations and proposed procedure for practical implementation is then as follows

1. We get standard OLS estimator likewise in univariate case

$$\widehat{\lambda^{P,k}} = b_k = (\mathbf{X}^{P,k'} \cdot \mathbf{X}^{P,k})^{-1}\mathbf{X}^{P,k'}\mathbf{Y}^{P,k}$$

:

2. Matrix Σ is estimated using following formula

$$\widehat{\sigma_{k1,k2}} = \frac{\widehat{\varepsilon_{.,k1}}\widehat{\varepsilon_{.,k2}}}{n \cdot (n-1)/2}$$

where $\widehat{\varepsilon_{.,k1}}$ represent the vector of OLS calculated residuals of $k1$th model.
3. Estimator with non constant variance $\beta = \lambda^{\mathbf{P}}$ is derived as

$$\beta = (Z^{'}\widehat{\Psi}^{-1}Z)^{-1}Z^{'}\widehat{\Psi}^{-1}\mathbf{Y}^{P}$$

where $\widehat{\Psi} = \widehat{\Sigma} \bigotimes I$ a Z is block-diagobal matrix $\mathbf{X}^{P,k}$, thus $Z = \mathrm{diag}(\mathbf{X}^{P,1}, \dots, \mathbf{X}^{P,K})$.

This process could be performed repeatedly similarly as in Kremer (2005) if initial estimator is replaced by that one calculated in the 3th step. This is repeated until the estimated do not converge

Possible alternative of modelling dependencies between Reserve and Paid amount

Following idea might help to predict future payments and Incurred values (eventually with a tail factor too) based on data of both triangles. For simplicity of notation we define $P_{i,j} \equiv Y^{P}_{i,j}$ a $I_{i,j} \equiv Y^{I}_{i,j}$ and incremental value of Paid amount in calendar period $i+j$ is to be signed as $P^{d}_{i,j} = P_{i,j} - P_{i,j-1}$.

It is convenient to assume that paid amount in the next development period could be explained by the value of reserve in the present $R_{i,j} = I_{i,j} - P_{i,j}$. We can suggest following model for prediction of future payments

$$P^{d}_{i,j+1} = \alpha_j R_{i,j} + \varepsilon^{A}_{i,j}, \qquad \mathrm{var}(\varepsilon^{A}_{i,j}) = \sigma^2_A R_{i,j}$$

that respect the key idea of Munich Chain Ladder that one might expect higher future amount of paid compensation in case of higher reserve and vice versa. If we want to calculate the estimators $\widehat{P^{d}_{i,j}}$ if $i+j > n$ the estimator $\widehat{R}_{i,j}$ of amount of reserve in unknown part of triangle is also important.

One might propose for example quite simple model for reserve development

$$R_{i,j+1} = \beta_j R_{i,j} + \varepsilon^{B}_{i,j}, \qquad \mathrm{var}(\varepsilon^{B}_{i,j}) = \sigma^2_B R_{i,j}$$

that is similar to standard chain ladder evolution.

This model could be later generalise to consider run-off reserve as well that is important if we want to model reserve evolution as well.

We can assume following equation for reserve evolution

$$R_{i,j+1} = R_{i,j} - P^{d}_{i,j+1} + R^{T}_{i,j+1} - R^{R}_{i,j+1}$$

where $R^{T}_{i,j+1}$ shows increase of reserve (if new claims are detected) a $R^{R}_{i,j+1}$ represents decrease of reserve without following payments if some previously reserved claims are detected as irrelevant. Run-off of reserve could be modelled as

$$R^{T}_{i,j} - R^{R}_{i,j} = \gamma_j R_{i,j} + \varepsilon^{C}_{i,j}, \qquad \mathrm{var}(\varepsilon^{C}_{i,j}) = \sigma^2_C R_{i,j}$$

as following equality holds

$$R_{i,j+1} = R_{i,j} - P^{d}_{i,j+1} + R^{T}_{i,j+1} - R^{R}_{i,j+1} = R_{i,j} - \alpha_j R_{i,j} + R^{T}_{i,j+1} - R^{R}_{i,j+1} + \varepsilon^{A}_{i,j} = \beta_j R_{i,j} + \varepsilon^{B}_{i,j}$$

that implies $\beta_j + \alpha_j - 1 = \gamma_j$ and $\varepsilon^{C}_{i,j} = \varepsilon^{A}_{i,j} + \varepsilon^{B}_{i,j}$.

Conclusion and tasks for further research

The recent developments of the most popular method of actuarial reserving in non life insurance were described and widely discussed in this paper. It has been shown that standard method of estimates are not the best solution in its recently published generalisation. We succeeded in deriving some more properties of that Munich Chain Ladder method that looks as useful especially related to various parameter estimates. Moreover we discussed formula for variability of development factors that could be used for means square error calculation similarly as in Mack's model and possible generalisation of MCL to multivariate case and some alternative approach for incorporating the idea of MCL.

In the future research we would like to perform and present numerical study of that method and derive other properties of presented models (MSE for Multivariate Chain ladders and for models based on reserve values).

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