

# **Predictive Distributions for Reserves which Separate True IBNR and IBNER Claims**

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## **Abstract**

This paper considers the model suggested by Schnieper (1991), which separates the true IBNR claims from the IBNER. Stochastic models are defined, using both recursive and non-recursive procedures, within the framework of the models described in England and Verrall (2002). Expressions for the prediction errors of the reserves are derived analytically. A bootstrapping procedure is also described which allows the prediction errors to be estimated straightforwardly. The full predictive distribution of reserves is estimated using the bootstrapping method.

Some extensions to the original Schnieper model are also discussed, together with other possible applications of this type of model.

## **Keywords**

Bootstrapping; Bornhuetter-Ferguson; chain-ladder; claims reserving; predictive distribution

## 1 Introduction

Schnieper (1991) proposed a method of claims reserving, which explicitly separated the incurred data into new claims amounts and changes in incurred amounts for existing claims. The method proposed by Schnieper has not received much attention since then, although Mack (1993) used some of the ideas, indirectly, to derive a method to calculate estimates of the prediction error for the chain-ladder technique. As far as we are aware, there has been no further work in the literature following up the specific modelling structure proposed by Schnieper (1991). We believe that the ideas in Schnieper (1991) should be considered again, and so we present in this paper a number of additions to the work of Schnieper, taking into account the advances in stochastic claims reserves models which have taken place since the paper was published. In particular, we derive recursive formulae for the prediction errors, and show how these, and the full predictive distribution, can also be obtained using bootstrapping. We also make some proposals for extending the Schnieper models, and discuss ideas for extending the application of this type of approach to other types of data.

A useful summary of stochastic claims reserving models can be found in England and Verrall (2002). That paper, and the references given in it, supply a background to the development of the stochastic framework used in this paper. It is assumed that the data have been aggregated by development year and accident year (noting that the methods can be extended straightforwardly to quarterly or monthly data). England and Verrall (2002) considers both recursive and non-recursive models for claims run-off triangles, and both approaches will be used in this paper.

The paper is set out as follows. Section 2 contains a brief description of the model proposed in Schnieper (1991), including the estimates of the parameters and of outstanding claims. In Section 3 we describe stochastic models for the Schnieper model, using the approach described in England and Verrall (2002). Section 4 shows how to derive the prediction errors analytically, using the stochastic models outlined in Section 3. In Section 5, we show how to obtain the prediction errors and the full predictive distribution using bootstrapping. This is built on the methodology described in England and Verrall (2006). In Section 6, the prediction errors are estimated for the data used in Schnieper (1991), using both the analytical approach and bootstrapping. Section 6 also estimates the predictive distributions for this set of data. Finally, in Section 7, a discussion is given of the model proposed by Schnieper, suggesting some possible extensions to the model. We also examine in this section whether a similar approach could usefully be applied to other data sets. Thus, as well as investigating stochastic models for the Schnieper method, another purpose of this paper is to show how the basic ideas of stochastic claims reserving can now be extended in order to provide further useful practical approaches to reserving.

## 2. The Schnieper Model

This section describes the model suggested by Schnieper (1991), and also gives details of the estimates of the parameters and of outstanding claims derived by Schnieper. The

method was specifically designed with reinsurance data in mind, but it is possible that it could be useful for other types of data as well.

## 2.1 The data

The Schnieper model deals with incurred data, and the idea is to separate the data into two more detailed, separate parts. These two parts are the new claims that arise at each development period, and the change in the incurred amounts for claims that arose at previous development periods. Clearly, whether or not this is possible depends on the information available, and the original application in Schnieper was to reinsurance data. While recognising that there will be many cases where data is not available at this level of detail, we also believe that the approach is useful when the data are available. It is also our opinion that the general approach may also be adapted to other situations when other types of data are available, also split into two parts.

Without loss of generality, we assume that the data are available in triangular form, indexed by accident year,  $i$ , and development year,  $j$ . The cumulative incurred data are denoted by  $\{X_{ij} : i = 1, 2, \dots, n; j = 1, 2, \dots, n - i + 1\}$ :

$$\begin{array}{cccc} X_{11} & X_{12} & \dots & X_{1n} \\ X_{21} & \dots & X_{2,n-1} & \\ \vdots & & & \\ X_{n1} & & & \end{array}$$

It is assumed that the incremental incurred claims ( $X_{ij} - X_{i,j-1}$ ) are the sum of incremental incurred from the old claims ( $-D_{ij}$ ) and the new claims ( $N_{ij}$ ). In other words,  $-D_{ij}$  represents the change in the cumulative incurred claims for claims reported in previous development periods, and  $N_{ij}$  is the new claims reported in development period  $j$ . Thus,

$$X_{ij} - X_{i,j-1} = -D_{ij} + N_{ij} \quad (2.1)$$

and for cumulative claims:

$$X_{ij} = X_{i,j-1} - D_{ij} + N_{ij} \quad (2.2)$$

As was discussed in Verrall (2000) and England and Verrall (2002), the stochastic models for claims reserving can be formulated either for incremental or cumulative data, with no difference in the results. It is therefore a matter of convenience which is used, and there are advantages to each in different circumstances. For example, when deriving expressions for the estimation error, it is usually easier to use the cumulative claims.

Schnieper also assumes that a measure of the exposure,  $E_i$ , is also available for each accident year  $i$ , and it will be seen that this leads to estimation that has some similarities with the Bornhuetter-Ferguson method (Bornhuetter and Ferguson, 1972). We continue with this assumption, in order to be consistent with Schnieper (1991), but also discuss how the approach may be adapted when exposure data are not available.

In common with Schnieper (1991), we do not attempt to forecast beyond development year  $n$ . We refer to cumulative claims at development year  $n$  as “Ultimate Claims”.

## 2.2 The model assumptions

The general model assumptions concern the independence between accident years, and the mean and variance of the incremental incurred claims amounts from old claims and from new claims. These are given as follows:

Assumption 1: Mean of  $N_{ij}$  and  $D_{ij}$

$$E\left[N_{ij} \mid X_{i,j-1}\right] = E_i \lambda_j \quad (2.3)$$

$$E\left[D_{ij} \mid X_{i,j-1}\right] = X_{i,j-1} \delta_j \quad (2.4)$$

where  $E_i$  is the exposure measure of accident year  $i$ .

According to Assumption 1, the model structure for the incremental incurred claims from new claims,  $N_{ij}$ , is non-recursive in format, with column parameters,  $\lambda_j$ , which have to be estimated from the data and row parameters,  $E_i$ , which are assumed known. Therefore, it also can be seen to be similar to the Bornhuetter-Ferguson method with the row parameters assumed to be known *a priori*. A discussion of the Bornhuetter-Ferguson method in a Bayesian context can be found in Verrall (2005).

The model structure for the incremental incurred claims amounts from existing claims has some similarities with the chain-ladder model, where  $\delta_j$  is similar to a development factor. The difference here is that  $X_{i,j-1}$  is taken from a different triangle, rather than being the previous cumulative claims. For this reason, Mack (1993) describes this method as “a mixture of the Bornhuetter-Ferguson technique and the chain-ladder method”.

Assumption 2: Variance of  $N_{ij}$  and  $D_{ij}$

$$\text{Var}\left[N_{ij} \mid X_{i,j-1}\right] = E_i \sigma_j^2 \quad (2.5)$$

$$\text{Var}\left[D_{ij} \mid X_{i,j-1}\right] = X_{i,j-1} \tau_j^2 \quad (2.6)$$

Assumption 2 defines only the variances of the random variables, and not the full distribution. This approach was also used by Mack (1993) in his stochastic model for the chain-ladder technique. It is possible to use this approach to derive second moments of estimates and prediction errors, but some distributional assumptions are needed in order to derive full predictive distributions. For this reason, we use the natural assumption in this context, which is that the data have a normal distribution, noting that we could also work in a non-parametric context making only the assumptions concerning the mean and variance, and obtain the same results.

Assumption 3: Independence between accident years

It is assumed that the incurred data,  $\{N_{ij}, D_{ij} | i=1, \dots, n; j=1, \dots, n\}$ , are independent between accident years.

### 2.3 Unbiased estimates of the parameters

Following the assumptions in section 2.2, the unbiased estimates of the parameters were provided by Schnieper as follows.

$$\hat{\lambda}_j = \frac{\sum_{i=1}^{n+1-j} N_{ij}}{\sum_{i=1}^{n+1-j} E_i}, \quad j=1, 2, \dots, n. \quad (2.7)$$

and

$$\hat{\delta}_j = \frac{\sum_{i=1}^{n+1-j} D_{ij}}{\sum_{i=1}^{n+1-j} X_{i,j-1}}, \quad j=2, 3, \dots, n \quad (2.8)$$

Also

$$\hat{\sigma}_j^2 = \frac{1}{n-j} \sum_{i=1}^{n+1-j} \frac{1}{E_i} (N_{ij} - \hat{\lambda}_j E_i)^2, \quad j=1, 2, \dots, n-1. \quad (2.9)$$

and

$$\hat{\tau}_j^2 = \frac{1}{n-j} \sum_{i=1}^{n+1-j} \frac{1}{X_{i,j-1}} (D_{ij} - \hat{\delta}_j X_{i,j-1})^2, \quad j=2, 3, \dots, n-1 \quad (2.10)$$

Schnieper (1991) also derived estimates of the variances of  $\hat{\lambda}_j$  and  $\hat{\delta}_j$ :

$$\text{Var}(\hat{\lambda}_j) = \frac{\sigma_j^2}{\sum_{i=1}^{n+1-j} E_i}, \quad j=1, 2, \dots, n.$$

and 
$$\text{Var}(\hat{\delta}_j) = \frac{\tau_j^2}{\sum_{i=1}^{n+1-j} X_{i,j-1}}, \quad j = 2, 3, \dots, n$$

## 2.4 Ultimate claims

The main purpose of claims reserving is to estimate outstanding claims. In order to do this, we require an estimate of ultimate claims for each accident year. We are also interested in the prediction errors for these reserves. When using recursive models, the prediction errors for the reserves are the same as the prediction errors for the ultimate claims, since we always condition on the latest cumulative claims. In other words, the variance of the estimate of the reserve (outstanding claims) is the same as the variance of the estimate of ultimate claims when the latest cumulative claims are assumed to be fixed. Thus, we may consider whichever is easier to deal with. In most cases, this means using ultimate claims, especially when the models are set up in the form of recursive models.

Using a recursive derivation method, Schnieper (1991) provided a formula for the loss reserve for accident year  $i$ ,  $L_i$ . Since we derive prediction errors using ultimate claims, we rewrite Schnieper's formula in terms of the prediction of ultimate claims.

Let  $\hat{X}_{ij} = E[X_{ij} | X_{i,n-i+1}]$  for  $j > n - i + 1$  and note that  $\hat{X}_{in} = L_i + X_{i,n-i+1}$ .

Then 
$$\begin{aligned} \hat{X}_{in} &= E[X_{in} | X_{i,n-i+1}] = X_{i,n-i+1} (1 - \delta_{n-i+2}) \dots (1 - \delta_n) \\ &\quad + E_i[\lambda_{n-i+2} (1 - \delta_{n-i+3}) \dots (1 - \delta_n) + \lambda_{n-i+3} (1 - \delta_{n-i+4}) \dots (1 - \delta_n) + \dots + \lambda_n] \end{aligned} \quad (2.11)$$

It is clear that we require the “ $t$ -steps-ahead” forecast of cumulative claims, in the terminology of time series. Thus, for row 2 we require the 1-step-ahead forecast, for row 3 we require the 2-steps-ahead forecast, and so on. The number of steps ahead from the latest observed time (the leading diagonal) to ultimate claims only depends on which accident year we are considering.

For example, for accident year 2, the 1-step-ahead forecast of cumulative claims, which is ultimate claims, is

$$\begin{aligned} \hat{X}_{2n} &= E[X_{2n} | X_{2,n-1}] = E[X_{2,n-1} - D_{2n} + N_{2n} | X_{2,n-1}] = X_{2,n-1} - \delta_n X_{2,n-1} + \lambda_n E_2 \\ &= X_{2,n-1} (1 - \delta_n) + \lambda_n E_2. \end{aligned}$$

For accident year 3, the 2-steps-ahead forecast of cumulative claims gives predicted value of ultimate claims as

$$\begin{aligned}\hat{X}_{3n} &= E[X_{3n} | X_{3,n-2}] = E[E(X_{3n} | X_{3,n-1}) | X_{3,n-2}] = E[X_{3,n-1}(1-\delta_n) + E_{2n} | X_{3,n-2}] \\ &= X_{3,n-2}(1-\delta_{n-1})(1-\delta_n) + E_2(1-\delta_n)\lambda_{n-1} + E_2\lambda_n.\end{aligned}$$

It is easy to see that the general formula, the  $(i-1)$ -steps-ahead forecast for row  $i$ , gives the prediction of ultimate claims as given above. It is straightforward to verify that this is equivalent to the loss reserve for accident year  $i$ ,  $L_i$ , as derived in Schnieper (1991). (Note that Schnieper called this the reserve, whereas it is, in fact, the estimate of ultimate claims.)

### 3. Stochastic models

In this section, we express the model used by Schnieper in the format of the approach of England and Verrall (2002). The reason for doing this is to show how the properties of the predictive distribution of the loss reserve may be derived and how bootstrapping may be applied.

The formulation in Schnieper (1991) is in the form of a distribution-free approach, with the assumptions being limited to the means and variances of the incremental claims amount from both new and existing claims. Prediction errors and the full predictive distribution were not considered. The alternative to the distribution-free approach is to define the process distributions in full, and the most obvious choice, staying as close as possible to Schnieper's distribution-free model, is to use normal distributions. Thus, we begin by assuming that both  $N_{ij} | X_{i,j-1}$  and  $D_{ij} | X_{i,j-1}$  have normal distributions. For consistency with Section 5, where bootstrap estimates are derived, we define the distributions of  $\frac{N_{ij}}{E_i}$  and  $\frac{D_{ij}}{X_{i,j-1}}$  instead of  $N_{ij}$  and  $D_{ij}$ . Using the same mean and variance as Schnieper (1991), the models are

$$\frac{N_{ij}}{E_i} | X_{i,j-1} \sim \text{Normal}(\lambda_j, \frac{\sigma_j^2}{E_i}) \quad (3.1)$$

and

$$\frac{D_{ij}}{X_{i,j-1}} | X_{i,j-1} \sim \text{Normal}(\delta_j, \frac{\tau_j^2}{X_{i,j-1}}) \quad (3.2)$$

respectively.

The maximum likelihood estimates of the parameters  $\lambda_j$  and  $\delta_j$  in these models are the same as the estimates given in equations (2.7) and (2.8), derived by Schnieper (1991).

For the estimates of the parameters in the variances,  $\sigma_j^2$  and  $\tau_j^2$  it is usual to use the asymptotic distribution of the sum of the squares of the residuals. The Pearson residuals are defined as

$$\frac{\frac{N_{ij} - \hat{\lambda}_j}{E_i}}{\sqrt{\frac{\sigma_j^2}{E_i}}} \quad \text{and} \quad \frac{\frac{D_{ij} - \hat{\delta}_j}{X_{i,j-1}}}{\sqrt{\frac{\tau_j^2}{X_{i,j-1}}}}$$

respectively. The asymptotic distributions of the sums of the squares of the residuals are

$$\sum_{i=1}^{n-j+1} \frac{\left(\frac{N_{ij} - \hat{\lambda}_j}{E_i}\right)^2}{\frac{\sigma_j^2}{E_i}} \sim \chi_{(n-j+1)-1}^2 \quad \text{and} \quad \sum_{i=1}^{n-j+1} \frac{\left(\frac{D_{ij} - \hat{\delta}_j}{X_{i,j-1}}\right)^2}{\frac{\tau_j^2}{X_{i,j-1}}} \sim \chi_{(n-j+1)-1}^2.$$

Equating these to their expected values gives:

$$\sum_{i=1}^{n-j+1} \frac{\left(\frac{N_{ij} - \hat{\lambda}_j}{E_i}\right)^2}{\frac{\hat{\sigma}_j^2}{E_i}} = n - j \quad \text{and} \quad \sum_{i=1}^{n-j+1} \frac{\left(\frac{D_{ij} - \hat{\delta}_j}{X_{i,j-1}}\right)^2}{\frac{\hat{\tau}_j^2}{X_{i,j-1}}} = n - j.$$

This gives the following estimates of  $\sigma_j^2$  and  $\tau_j^2$ :

$$\hat{\sigma}_j^2 = \frac{1}{n-j} \sum_{i=1}^{n-j+1} \frac{(N_{ij} - E_i \hat{\lambda}_j)^2}{E_i} \quad (3.3)$$

and

$$\hat{\tau}_j^2 = \frac{1}{n-j} \sum_{i=1}^{n-j+1} \frac{(D_{ij} - X_{i,j-1} \hat{\delta}_j)^2}{X_{i,j-1}} \quad (3.4)$$

which are exactly the same as the estimates proposed by Schnieper (1991), given in equations (2.9) and (2.10).

It would also be possible to use an over-dispersed Poisson model for the observed incremental incurred amounts from new claims here, i.e.  $\frac{N_{ij}}{E_i} \sim ODP(\lambda_j)$ . In this case,

it should be noted that there would be no extra parameters to estimate in the variance. It can be shown that the maximum likelihood estimates of the parameters  $\lambda_j$  are still the same as Schnieper (1991).

#### 4. Prediction errors

One of the principle reasons for using stochastic models is so that prediction errors and predictive distributions can be estimated. These are useful for solvency requirements, and for capital modelling and risk measurement. We begin by showing how prediction errors may be calculated, and use bootstrapping in the following section for the predictive distribution.

In general, we require the mean square error of prediction. Consider first the simple situation where we have a random variable,  $Y$ , and let  $\hat{Y}$  be the predicted value of  $Y$ . The mean square error of prediction can be written as  $MSEP(Y) = E\left[\left(Y - \hat{Y}\right)^2\right]$ , and this may be approximated by  $MSEP(Y) \approx Var(Y) + Var(\hat{Y})$ . Here the component  $Var(Y)$  is the process variance and  $Var(\hat{Y})$  is the estimation variance. Therefore, in general the prediction variance is (approximately) the sum of process variance and estimation variance. The square root of the mean square error of prediction is known as the prediction error.

There are two different approaches to calculate the prediction error, depending on whether a recursive or non-recursive model is used. England and Verrall (2002) provides detailed derivations and explanations for both, with the expressions for recursive models being in the appendices. In this section we use the recursive approach to derive the prediction errors for the Schnieper model.

As was pointed out in section 2.4, it is equivalent to consider either the ultimate claims or the loss reserve, when recursive models are used, since we always condition on the latest values for cumulative claims. This means that the mean square errors of prediction are the same for the reserve and ultimate claims. In this case it is easier to consider ultimate claims, and derive recursive formulae for these.

##### 4.1 Process variance for accident year (row) $i$

The process variance is derived recursively, which makes it particularly straightforward to implement in a spreadsheet. The  $t$ -steps-ahead formula for the expectation and variance of ultimate claims may be obtained recursively as follows.

Let  $k = n - i + 1$ . We continue to use the notation for the forecast of a future cumulative claims amount given in section 2.4,  $\hat{X}_{ij} = E\left[X_{ij} \mid X_{ik}\right]$  for  $j > k$ . Then the  $t$ -steps-ahead forecast is

$$\hat{X}_{i,k+t} = E\left[X_{i,k+t} \mid X_{ik}\right] = E\left[E\left[X_{i,k+t} \mid X_{i,k+t+1}\right] \mid X_{ik}\right]$$

$$\begin{aligned}
&= (1 - \delta_{k+t}) E[X_{i,k+t-1} | X_{ik}] + E_i \lambda_{k+t} \\
&= (1 - \delta_{k+t}) \hat{X}_{i,k+t-1} + E_i \lambda_{k+t}
\end{aligned} \tag{4.1}$$

and

$$\text{Var}[X_{i,k+t} | X_{ik}] = (1 - \delta_{k+t}^2) \text{Var}[X_{i,k+t-1} | X_{ik}] + \tau_{k+t}^2 E[X_{i,k+t-1} | X_{ik}] + E_i \sigma_{k+t}^2 \tag{4.2}$$

Proof

From the model assumptions, it is easy to get the formulae for *one-step-ahead*:

$$\hat{X}_{i,k+1} = E[X_{i,k+1} | X_{ik}] = E[X_{ik} - D_{i,k+1} + N_{i,k+1} | X_{ik}] = X_{ik} (1 - \delta_{k+1}) + E_i \lambda_{k+1}$$

$$\text{and } \text{Var}[X_{i,k+1} | X_{ik}] = \text{Var}[X_{ik} - D_{i,k+1} + N_{i,k+1} | X_{ik}] = \tau_{k+1}^2 X_{ik} + E_i \sigma_{k+1}^2.$$

This shows that the recursive formulae are (trivially) correct for  $t = 1$ , using the fact that the mean and variance of  $X_{ik} | X_{ik}$  are  $X_{ik}$  and 0.

We assume that the above formulae are true for  $t$  and show that they are correct for  $t+1$ .

$$\begin{aligned}
\hat{X}_{i,k+t+1} &= E[X_{i,k+t+1} | X_{ik}] = E[E(X_{i,k+t+1} | X_{i,k+t}) | X_{ik}] \\
&= E[E(X_{i,k+t} - D_{i,k+t+1} + N_{i,k+t+1} | X_{i,k+t}) | X_{ik}] \\
&= (1 - \delta_{k+t+1}) E[X_{i,k+t} | X_{ik}] + E_i \lambda_{k+t+1} \\
&= (1 - \delta_{k+t+1}) \hat{X}_{i,k+t} + E_i \lambda_{k+t+1}
\end{aligned}$$

as required.

$$\begin{aligned}
\text{Var}[X_{i,k+t+1} | X_{ik}] &= E[\text{Var}(X_{i,k+t+1} | X_{i,k+t}) | X_{ik}] + \text{Var}[E(X_{i,k+t+1} | X_{i,k+t}) | X_{ik}] \\
&= E[\text{Var}(X_{i,k+t} - D_{i,k+t+1} + N_{i,k+t+1} | X_{i,k+t}) | X_{ik}] + \text{Var}[E(X_{i,k+t} - D_{i,k+t+1} + N_{i,k+t+1} | X_{i,k+t}) | X_{ik}] \\
&= E[\tau_{k+t+1}^2 X_{i,k+t} + E_i \sigma_{k+t+1}^2 | X_{ik}] + \text{Var}[(1 - \delta_{k+t+1}) X_{i,k+t} + E_i \lambda_{k+t+1} | X_{ik}] \\
&= \tau_{k+t+1}^2 E[X_{i,k+t} | X_{ik}] + E_i \sigma_{k+t+1}^2 + (1 - \delta_{k+t+1})^2 \text{Var}[X_{i,k+t} | X_{ik}]
\end{aligned}$$

which concludes the proof.

Using this recursive formula, it is possible to derive recursive expressions for  $E[X_{in} | X_{ik}] = E[X_{in} | X_{i,n-i+1}]$  and  $\text{Var}[X_{in} | X_{ik}] = \text{Var}[X_{in} | X_{i,n-i+1}]$ . The formula for  $E[X_{in} | X_{i,n-i+1}]$  can be used to derive the same expression for the loss reserve as was derived by Schnieper. The formula for  $\text{Var}[X_{in} | X_{i,n-i+1}]$ , gives the process variance for

the row total for accident year  $i$  (which is the same as the process variance of the loss reserve).

#### 4.2 Estimation variance for accident year (row) $i$

The estimation variance can also be calculated in a recursive format. Again, it is simpler to consider ultimate claims rather than the reserve. From section 4.1, the recursive formula for the  $t$ -steps-ahead forecast is

$$\hat{X}_{i,k+t} = \hat{X}_{i,k+t-1} (1 - \hat{\delta}_{k+t}) + E_i \hat{\lambda}_{k+t} \quad (4.3)$$

The variance of this estimate can also be obtained recursively using the following formula:

$$\begin{aligned} \text{Var} \left[ \hat{X}_{i,k+t} \right] &= \hat{X}_{i,k+t-1}^2 \text{Var} \left[ \hat{\delta}_{k+t} \right] + (1 - \hat{\delta}_{k+t})^2 \text{Var} \left[ \hat{X}_{i,k+t-1} \right] \\ &\quad + \text{Var} \left[ \hat{\delta}_{k+t} \right] \text{Var} \left[ \hat{X}_{i,k+t-1} \right] + E_i^2 \text{Var} \left[ \hat{\lambda}_{k+t} \right] \end{aligned} \quad (4.4)$$

Proof

From the model assumptions in section 2.2, we have the following recursive structure for the variance of the *one*-step-ahead prediction:

$$\text{Var} \left[ \hat{X}_{i,k+1} \right] = \text{Var} \left[ X_{ik} (1 - \hat{\delta}_{k+1}) + E_i \hat{\lambda}_{k+1} \mid X_{ik} \right] = X_{ik}^2 \text{Var}(\hat{\delta}_{k+1}) + E_i^2 \text{Var}(\hat{\lambda}_{k+1}).$$

Thus, equation (4.4) is correct for  $t = 1$ .

Again we assume that the above recursive formula for the estimation variance, (4.4), is true for  $t$  and show that it is correct for  $t+1$ .

$$\begin{aligned} \text{Var} \left[ \hat{X}_{i,k+t+1} \right] &= \text{Var} \left[ \hat{X}_{i,k+t} (1 - \hat{\delta}_{k+t+1}) + E_i \hat{\lambda}_{k+t+1} \right] \\ &= \text{Var} \left[ \hat{X}_{i,k+t} (1 - \hat{\delta}_{k+t+1}) \right] + \text{Var} \left[ E_i \hat{\lambda}_{k+t+1} \right] \\ &= \hat{X}_{i,k+t}^2 \text{Var} \left[ \hat{\delta}_{k+t+1} \right] + (1 - \hat{\delta}_{k+t+1})^2 \text{Var} \left[ \hat{X}_{i,k+t} \right] + \text{Var} \left[ \hat{X}_{i,k+t} \right] \text{Var} \left[ \hat{\delta}_{k+t+1} \right] + E_i^2 \text{Var} \left[ \hat{\lambda}_{k+t+1} \right] \end{aligned}$$

which completes the proof.

This proof uses the independence between  $\hat{\lambda}_j$  and  $\hat{\delta}_j$ , which is easy to see from model assumption 3. Note that this independence is similar to the independence between the estimates of the parameters in the chain-ladder technique noted by Mack (1993).

### 4.3 Prediction variance for the Total Loss Reserve

The final step for the prediction errors is to obtain the prediction variance for the overall total. Let  $R$  denote the overall reserve,  $\hat{R}$  be the estimate of  $R$ , and  $H_m$  denote the set of those variables in the  $N$  and  $D$  triangles which are observed up to calendar year  $m$

$$H_m = \{N_{ij}, D_{ij} \mid i + j \leq m + 1\}.$$

$H_n$  is the set of all variables which have been observed so far.  $H_{i+j-2}$  is the history of the process up to the calendar year immediately preceding the emergence of  $N_{ij}$  and  $D_{ij}$ .

The mean squared error of prediction of  $\hat{R}|H_n$  can be written as:

$$MSEP(\hat{R}|H_n) \approx Var(R|H_n) + Var(\hat{R}|H_n) \quad (4.5)$$

Under the independence assumptions of the model, the process variance of the overall reserve is the sum of the process variances of the row reserves. However, it becomes more complicated for the estimation variance of the overall reserve due to the covariance between the accident years. It is easy to see that the estimates of the row reserves are not independent, as they use the same column parameters. Therefore, we need to consider the estimation covariance between row reserves, and a recursive procedure is again used in this context. We first note that the covariance of the reserves for rows  $t$  and  $s$  ( $2 \leq t \leq n$  and  $t < s$ ) is the same as  $Cov[\hat{X}_m, \hat{X}_{sn}]$ . Thus, given the latest observed data, the covariance between the estimates of ultimate claims for any two rows has to be calculated. In order to do this, we make use of the following result.

$$\begin{aligned} Cov[\hat{X}_{ij}, \hat{X}_{sj}] &= Var[\hat{\delta}_j] \left( Cov[\hat{X}_{t,j-1}, \hat{X}_{s,j-1}] + \hat{X}_{t,j-1} \hat{X}_{s,j-1} \right) \\ &\quad + \left( E[1 - \hat{\delta}_j] \right)^2 Cov[\hat{X}_{t,j-1}, \hat{X}_{s,j-1}] + E_t E_s Var(\hat{\lambda}_j) \end{aligned} \quad (4.6)$$

where  $n + t - 1 < j \leq n$ .

Proof

This proof is also in a recursive format. We start from the *one*-step-ahead estimation covariance, i.e. when  $j = n - t + 2$

$$\begin{aligned}
Cov\left[\hat{X}_{ij}, \hat{X}_{sj}\right] &= Cov\left[\hat{X}_{t, n-t+2}, \hat{X}_{s, n-t+2}\right] \\
&= Cov\left[X_{t, n-t+1} \left(1 - \hat{\delta}_{n-t+2}\right) + \hat{N}_{t, n-t+2}, X_{s, n-t+1} \left(1 - \hat{\delta}_{n-t+2}\right) + \hat{N}_{s, n-t+2}\right] \\
&= Var\left[\hat{\delta}_{n-t+2}\right] X_{t, n-t+1} E\left[X_{s, n-t+1} \mid X_{s, n-s+1}\right] + Cov\left[\hat{N}_{t, n-t+2}, \hat{N}_{s, n-t+2}\right] \\
&= Var\left[\hat{\delta}_{n-t+2}\right] X_{t, n-t+1} \hat{X}_{s, n-t+1} + E_t E_s Var\left[\hat{\lambda}_{n-t+2}\right]
\end{aligned}$$

Again, this shows that the recursive formula (4.6) is (trivially) correct for the one-step ahead case.

In the same way as for the proof of the recursive formulae for the process or estimation variance, we assume that the above formula is true for  $j$  and show that it is correct for  $j+1$ .

Firstly, from equation (4.3), the *one*-step-ahead prediction is  $\hat{X}_{ij} = \hat{X}_{i, j-1} (1 - \hat{\delta}_j) + E_i \hat{\lambda}_j$ . We now consider the covariance for  $j+1$ , using this *one*-step-ahead prediction formula:

$$\begin{aligned}
Cov\left[\hat{X}_{t, j+1}, \hat{X}_{s, j+1}\right] &= Cov\left[\hat{X}_{tj} \left(1 - \hat{\delta}_{j+1}\right) + E_t \hat{\lambda}_{j+1}, \hat{X}_{sj} \left(1 - \hat{\delta}_{j+1}\right) + E_s \hat{\lambda}_{j+1}\right] \\
&= Cov\left[\hat{X}_{tj} \left(1 - \hat{\delta}_{j+1}\right), \hat{X}_{sj} \left(1 - \hat{\delta}_{j+1}\right)\right] + E_t E_s Var\left[\hat{\lambda}_{j+1}\right] \\
&= Var\left[\hat{\delta}_{j+1}\right] Cov\left[\hat{X}_{tj}, \hat{X}_{sj}\right] + Var\left[\hat{\delta}_{j+1}\right] \hat{X}_{tj} \hat{X}_{sj} \\
&\quad + \left(E\left[1 - \hat{\delta}_{j+1}\right]\right)^2 Cov\left[\hat{X}_{tj}, \hat{X}_{sj}\right] + E_t E_s Var\left[\hat{\lambda}_{j+1}\right] \\
&= Var\left[\hat{\delta}_{j+1}\right] \left(Cov\left[\hat{X}_{tj}, \hat{X}_{sj}\right] + \hat{X}_{tj} \hat{X}_{sj}\right) \\
&\quad + \left(E\left[1 - \hat{\delta}_{j+1}\right]\right)^2 Cov\left[\hat{X}_{tj}, \hat{X}_{sj}\right] + E_t E_s Var\left[\hat{\lambda}_{j+1}\right]
\end{aligned}$$

which completes the proof.

For ultimate claims, we put  $j = n$ , and derive the required covariances using this recursive formula. In order to see how this result is used, consider first the case when  $t = 2$ . Putting  $t = 2$  and  $j = n$ , the result (4.6) becomes

$$Cov\left[\hat{X}_{2n}, \hat{X}_{sn}\right] = Var\left[\hat{\delta}_n\right] X_{2, n-1} \hat{X}_{s, n-1} + E_2 E_s Var\left[\hat{\lambda}_n\right]$$

So, for row 2 we use the iterative formula once, and in general for row  $t$ , we use formula (4.6)  $t-1$  times. This enables us to derive the covariances between the estimates of any of the row totals, as required when calculating the estimation variance for the overall

total. Thus, we can go on and derive the prediction variance for the overall reserves using equations (4.2), (4.4), (4.5) and (4.6).

We write equation (4.5) as

$$\begin{aligned} MSEP(\hat{R}|H_n) &\approx Var(R|H_n) + Var(\hat{R}|H_n) \\ &= Var\left(\sum_{i=1}^n X_{in} | X_{ik}\right) + Var\left(\sum_{i=1}^n \hat{X}_{in}\right) \\ &= \sum_{i=1}^n Var(X_{in} | X_{ik}) + \sum_{i=1}^n Var(\hat{X}_{in}) + 2 \sum_{t=1}^{n-1} \sum_{s=t}^n Cov(\hat{X}_m, \hat{X}_{sn}) \end{aligned}$$

Applying equations (4.2) (4.4) and (4.6), it can be seen that the prediction variance has the following form:

$$\begin{aligned} MSEP(\hat{R}|H_n) &= \sum_{i=1}^n \left\{ (1 - \delta_n^2) Var[X_{i,n-1} | X_{ik}] + \tau_n^2 E[X_{i,n-1} | X_{ik}] + E_i \sigma_n^2 \right\} \\ &\quad + \sum_{i=1}^n \left\{ \hat{X}_{in}^2 Var[\hat{\delta}_n] + (1 - \hat{\delta}_n)^2 Var[\hat{X}_{i,n-1}] + Var[\hat{\delta}_n] Var[\hat{X}_{i,n-1}] + E_i^2 Var[\hat{\lambda}_n] \right\} \\ &\quad + 2 \sum_{t=1}^{n-1} \sum_{s=t+1}^n \left\{ Var(\hat{\delta}_n) [Cov[\hat{X}_{t,n-1}, \hat{X}_{s,n-1} | X_{t,n-t+1}, X_{s,n-s+1}] + (\hat{X}_{t,n-1} \hat{X}_{s,n-1})] + (E[(1 - \hat{\delta}_n)])^2 Cov[\hat{X}_{t,n-1}, \hat{X}_{s,n-1}] + E_t E_s Var[\hat{\lambda}_n] \right\} \end{aligned} \tag{4.7}$$

Although this equation for the prediction variance of the overall reserve looks very complicated, it is in fact straightforward to implement in a spreadsheet, using recursive formulae. This spreadsheet is available on request from the corresponding author.

## 5. Bootstrapping

Bootstrapping is a very simple but powerful approach for the calculation of the estimation error. It has gained popularity in a practical context partly because of the ease with which it can be implemented in a spreadsheet. In a statistical context, bootstrapping is used to obtain simulated values of the estimation error, but it is straightforward to extend it to the prediction error by including simulations from a process distribution. Bootstrapping was first suggested by England and Verrall (1999), and was also discussed by Pinheiro et al (2003). Bootstrapping for recursive models was first considered by England and Verrall (2006).

Since the Schnieper method includes a recursive model, we follow the bootstrapping procedure in England and Verrall (2006). This means considering the residuals of the ratios  $\left(\frac{N_{ij}}{E_i} \text{ and } \frac{D_{ij}}{X_{i,j-1}}\right)$  rather than the observed data  $(N_{ij} \text{ and } D_{ij})$ .

From section 3, the mean and variance assumptions for the Schnieper model are:

$$E\left[\frac{N_{ij}}{E_i} \middle| X_{i,j-1}\right] = \lambda_j \text{ and } E\left[\frac{D_{ij}}{X_{i,j-1}} \middle| X_{i,j-1}\right] = \delta_j,$$

and

$$\text{Var}\left[\frac{N_{ij}}{E_i} \middle| X_{i,j-1}\right] = \frac{\sigma_j^2}{E_i} \text{ and } \text{Var}\left[\frac{D_{ij}}{X_{i,j-1}} \middle| X_{i,j-1}\right] = \frac{\tau_j^2}{X_{i,j-1}}.$$

The idea of bootstrapping is to generate new triangles of data (“bootstrap samples”) which are representative of the underlying distributions. When this has been done a reasonable number of times and the required results saved, the sampling properties may be estimated by simply looking at the properties of the bootstrap samples. So, for example, to obtain a bootstrap estimate of the estimation error of the overall reserve, we generate a reasonable number (in this case we used 1000) of new sets of data from the original data and estimate the reserve for each of these. The standard deviation of these reserve estimates is the bootstrap estimate of the estimation error. To include the process error, we add an extra simulation at the end of each bootstrap, using the appropriate process distribution. For a more detailed discussion, we refer the reader to one of the papers mentioned above.

In the context of claims reserving, the bootstrap samples are generated from the residuals rather than the raw data. We define  $f_{ij} = \frac{N_{ij}}{E_i}$  and  $g_{ij} = \frac{D_{ij}}{X_{i,j-1}}$ . The scaled

Pearson residuals are

$$r_{ij} = r_{PS}\left(f_{ij}, \hat{\lambda}_j, X_{i,j-1}, \hat{\sigma}_j\right) = \frac{\sqrt{E_i}\left(f_{ij} - \hat{\lambda}_j\right)}{\hat{\sigma}_j} \quad (5.1)$$

and

$$s_{ij} = r_{PS}\left(g_{ij}, \hat{\delta}_j, X_{i,j-1}, \hat{\tau}_j\right) = \frac{\sqrt{X_{i,j-1}}\left(g_{ij} - \hat{\delta}_j\right)}{\hat{\tau}_j}. \quad (5.2)$$

These residuals are sampled, with replacement, to generate bootstrap samples of residuals,  $r_{ij}^B$  and  $s_{ij}^B$ , for  $i = 1, 2, \dots, n; j = 1, 2, \dots, n - i + 1$ . The triangles of pseudo data are then calculated by inverting the residual definition:

$$f_{ij}^B = r_{ij}^B \frac{\hat{\sigma}_j}{\sqrt{E_i}} + \hat{\lambda}_j \quad (5.3)$$

and

$$g_{ij}^B = s_{ij}^B \frac{\hat{\tau}_j}{\sqrt{X_{i,j-1}}} + \hat{\delta}_j. \quad (5.4)$$

The appealing aspect of bootstrapping is that the calculations now only involve the simple spreadsheet operations used in the original method to calculate the loss reserves. Thus, for each bootstrap sample, the bootstrap estimates of the parameters in the mean,  $\tilde{\lambda}_j$  and  $\tilde{\delta}_j$ , are obtained using the usual weighted average of the individual development factors:

$$\tilde{\lambda}_j = \frac{\sum_{i=1}^{n-j+1} f_{ij}^B N_{ij}}{\sum_{i=1}^{n-j+1} E_i} \quad (5.5)$$

and.

$$\tilde{\delta}_j = \frac{\sum_{i=1}^{n-j+1} f_{ij}^B D_{ij}}{\sum_{i=1}^{n-j+1} X_{i,j-1}} \quad (5.6)$$

Note that the observed data,  $N_{ij}$ ,  $D_{ij}$  and the exposure  $E_i$  act as the weights here: it is not correct to use bootstrapped data for the weights. The bootstrap estimates of the reserves for each row and the total reserves are obtained by applying the bootstrap values of the parameters,  $\tilde{\lambda}_j$  and  $\tilde{\delta}_j$ , in the original formula, (2.11).

Bootstrapping only addresses the estimation error for the model. In section 6 we apply bootstrapping to the data from Schnieper (1991) and show that the results are very close to the results for the analytical estimation error derived in section 4. To obtain the prediction error and the full predictive distribution of the reserves, it is necessary to take account of the process distributions to complete the simulation. The stochastic model outlined in section 3 uses normal distributions for both  $N_{ij}$  and  $D_{ij}$ , although other models are also possible, such as the over-dispersed Poisson distribution for  $N_{ij}$ . The final step in the process to obtain simulations of the loss reserves suitable for calculating prediction errors and the predictive distribution is to simulate from these process distributions, using the bootstrap sample values for the means. i.e. we simulate from the following distributions:

$$\frac{N_{ij}}{E_i} \Big| X_{i,j-1} \sim \text{Normal}(\tilde{\lambda}_j, \frac{\sigma_j^2}{E_i}) \quad (5.8)$$

and

$$\frac{D_{ij}}{X_{i,j-1}} \Big| X_{i,j-1} \sim \text{Normal}(\tilde{\delta}_j, \frac{\tau_j^2}{X_{i,j-1}}) \quad (5.7)$$

respectively.

This gives simulated values,  $\tilde{D}_{ij}$  and  $\tilde{N}_{ij}$ , which are then inserted into equation (2.2), to obtain  $\tilde{X}_{ij}$  for each simulation.

The bootstrapping method is illustrated in Section 6, to show that the bootstrap prediction errors are very close to the analytical results.

## 6. Example

In this section we use the data from Schnieper (1991) and calculate the prediction errors and the predictive distribution using analytical and bootstrapping methods. The data used by Schnieper consisted of an IBNR triangle,  $X_{ij}$ , and exposure,  $E_i$ , which are shown in Table 1. Tables 2 and 3 show the more detailed data, consisting of the new claims,  $N_{ij}$ , and the changes in the existing claims,  $-D_{ij}$ . These data were taken from a practical motor third party liability excess-of-loss pricing problem.

**Table 1.** Cumulative IBNR ( $X_{ij}$ ) and Exposure ( $E_i$ ) for both new and existing claims

Dev year \ Accident year	1	2	3	4	5	6	7	Exposure
1	7.5	28.9	52.6	84.5	80.1	76.9	79.5	10224
2	1.6	14.8	32.1	39.6	55	60		12752
3	13.8	42.4	36.3	53.3	96.5			14875
4	2.9	14	32.5	46.9				17365
5	2.9	9.8	52.7					19410
6	1.9	29.4						17617
7	19.1							18129

**Table 2.** Incremental incurred claims from new claims ( $N_{ij}$ )

Dev year \ Accident year	1	2	3	4	5	6	7
1	7.5	18.3	28.5	23.4	18.6	0.7	5.1
2	1.6	12.6	18.2	16.1	14	10.6	
3	13.8	22.7	4	12.4	12.1		
4	2.9	9.7	16.4	11.6			
5	2.9	6.9	37.1				
6	1.9	27.5					
7	19.1						

**Table 3.** Incremental incurred claims from existing claims ( $D_{ij}$ )

Dev year \ Accident year	2	3	4	5	6	7
2	-3.1	4.8	-8.5	23	3.9	2.5
3	-0.6	0.9	8.6	-1.4	5.6	
4	-5.9	10.1	-4.6	-31.1		
5	-1.4	-2.1	-2.8			
6	0	-5.8				
7	0					

Tables 4 and 5 show the parameters estimates for each development year for the models. These are obtained by maximum likelihood estimation, or using equations (2.7) and (2.8), as derived in Schnieper (1991).

**Table 4.** Estimates of the parameters in the N triangle obtained from the Normal model, using maximum likelihood estimation.

$\lambda_1$	$\lambda_2$	$\lambda_3$	$\lambda_4$	$\lambda_5$	$\lambda_6$	$\lambda_7$
0.0005	0.0011	0.0014	0.0012	0.0012	0.0005	0.0005

**Table 5.** Estimates of the parameters in the D triangle obtained from the Normal model, using maximum likelihood estimation.

$\delta_2$	$\delta_3$	$\delta_4$	$\delta_5$	$\delta_6$	$\delta_7$
-0.3595	0.0719	-0.0476	-0.0536	0.0703	0.0325

The reserve estimates, estimation errors and prediction errors, obtained using the analytical results from section 4 and using the bootstrap method described in section 5 are shown in Table 6.

**Table 6.** Overall reserve estimates, estimation errors and prediction errors

Accident year	Reserves Estimates		Estimation errors		Prediction errors		Prediction errors %	
	Analytical	Bootstrap	Analytical	Bootstrap	Analytical	Bootstrap	Analytical	Bootstrap
<b>2</b>	4.4	4.4						
<b>3</b>	4.8	5.2	6.0	6.0	9.5	9.8	196%	187%
<b>4</b>	32.5	32.1	13.6	13.2	27.2	30.3	84%	95%
<b>5</b>	61.6	60.0	21.8	20.9	39.0	41.5	63%	69%
<b>6</b>	78.6	77.2	22.3	21.3	41.7	45.8	53%	59%
<b>7</b>	105.4	104.4	26.7	25.5	47.6	50.3	45%	48%
<b>Total</b>	287.3	283.3	77.1	80.3	110.9	112.4	39%	40%

It can be seen that there is a good agreement between the analytical results and those obtained using bootstrapping (allowing for the fact that bootstrapping is a simulation-based method).

## 7. Discussion

This paper has extended the analysis of the Schnieper model to include prediction errors and predictive distributions, using both analytical and bootstrapping methods. We believe that the model deserves to be reconsidered in the context of its original, using the new stochastic framework. One limitation of the application of the Schnieper model is that it requires a relatively detailed data set. i.e. the exposure of every accident year, the time when the claims occur and how they develop in every calendar year. For this reason, the model can not be used in every application. The model was also originally used in a specific context, and it is likely that this is a further reason why it has not been considered any further since it was published. However, the ideas from Schnieper (1991) of modelling two sets of data have some similarities with other slightly different problems, for example, the consideration of paid and incurred run-off triangles.

We believe that this paper may pave the way for new approaches to paid and incurred data, which may use the results derived here for the predictive distributions and the prediction errors. For example, we could suggest a straightforward extension to paid and incurred data, which may have some practical appeal. This is to use the model for incremental incurred amounts from new claims,  $N_{ij}$ , for incremental paid data for all development years apart from the first. In fact, the claims amount from first development years can always be ignored as they represent, in effect, non-random elements of the model. Similarly, incremental case reserves can be fitted by the model

for the incremental decreases in incurred amount from existing claims,  $D_{ij}$ , proposed by Schnieper (1991). The underlying requirement of the consistent ultimate losses projection between paid and incurred can be reflected by the exposure assumption.

Further, a prior distribution could be applied to model parameters, such as  $\lambda_j$  and  $\delta_j$ , and a Bayesian approach adopted. This would have the advantages that the flexibilities of the stochastic models are improved by introducing expert opinion: the prediction errors for the ultimate losses parameters could also be calculated, even when the model parameters are intuitively adjusted by experts under certain circumstances. For example, an appropriate prior distribution for the development factor parameters could be used in order to reflect significant changes observed in the data, which may be caused by changes in the management of claims.

As discussed before, the Schnieper model is a mixture of a chain-ladder model and the Bornheutter-Ferguson method. Another possible application of the Schnieper model is to change the Bornheutter-Ferguson model for the losses from new claims to a chain-ladder model type, so that we can drop the exposure requirement. This could be done as following,

$$\frac{N_{ij}}{N_{i,j-1}} \Big| N_{i,j-1} \sim Normal(\lambda_j, \frac{\sigma_j^2}{N_{i,j-1}}) \quad (7.1)$$

and

$$\frac{D_{ij}}{X_{i,j-1}} \Big| X_{i,j-1} \sim Normal(\delta_j, \frac{\tau_j^2}{X_{i,j-1}}) \quad (7.2)$$

Notice that in the above model, the equation (7.1) is the exactly chain ladder stochastic model and replaces equation (3.1) used by Schnieper (1991).

In summary, we suggest that the Schnieper model, in the stochastic framework given in this paper, deserves further consideration, now that these types of stochastic model are gaining practical acceptance. We would also suggest that this provides a useful avenue for further research in models that have useful practical applications.

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