RUIN THEORY WITH K LINES OF BUSINESS

Stéphane Loisel[§]

[§]*Université Claude Bernard Lyon I, Ecole ISFA.*
50 avenue Tony Garnier, 69007 Lyon, France
Email: stephane.loisel@univ-lyon1.fr
Tel : +33 4 37 28 74 38, Fax : +33 4 37 28 76 32.

Abstract

This paper deals with the evolution of the reserves of an insurance company with $K \geq 1$ lines of business facing dependent risks. We consider risk measures based on the behavior of the multivariate risk process from an academic point of view. To deal with multivariate risk processes, we propose a multi-risks model. We then explain how to determine the optimal reserve allocation of the global reserve to the lines of business in order to minimize those risk measures. The impact of dependence on the risk perception and on the optimal allocation is studied and used to test the consistency of the risk measures. This paper is mainly based on the two following papers Loisel (2004 2005b).

Topic 1 : Risk management of an insurance entreprise.
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1. INTRODUCTION

This paper is based on the two following papers Loisel (2004 2005b), which the interested reader is encouraged to consult for further details.

We consider here the case of an insurance company with $K \geq 1$ lines of business. Some authors, like Cossette and Marceau (2000), and many others, considered multi-risk models. However, in most cases, they focus on the unidimensional risk process representing the total wealth of the company.

From now on, we consider a fixed accounting time horizon T, which may be infinite, and study the evolution of the wealths of the lines of business of the company between times 0 and T. With three lines of business (three kinds of activities), for example liability, disablement and driving insurance, it is not the same situation to have $(1M, 2M, -2.8M)$ (*id est* 1 million euros for the first branch, 2 million euros for the second one and to be short of 2.8 million euros for the last branch), or to have $(0, 0.1M, 0.1M)$. Considering only the total wealth (0.2 million euros) does not reflect the situation of the company very well. A few years ago, a holding company mainly had two large airlines companies. The first one was doing well, say its wealth was 10M dollars. The second one was undergoing a bad period, with a debt of, say, 2M dollars. Even if the subcompanies were collateralized, the holding company was estimated 4M dollars, instead of 8M dollars, by the market. The reason was that analysts expected the healthy line of business to be penalized by the other one, which was in the red. To be able to detect such penalties, and to compute probabilities of such unfavorable events, the multi-dimensional process has to be studied as an additional indicator.

The solvency II project is an additional motivation for these considerations. For an international insurance group, the required capital, formerly determined at the group level, which took into account possible mutualization of risks supported by the different subcompanies, will probably be determined subcompany by subcompany. This would breed for all subcompanies a need for a much higher capital.

We consider in section 2 risk measures based on the behavior of the multivariate risk process from an academic point of view. To deal with multivariate risk processes, we propose in section 3 a multi-risks model. We then explain in section 4 how to determine the optimal reserve allocation of the global reserve to the lines of business in order to minimize those risk measures. The impact of dependence on the risk perception and on the optimal allocation is studied and used to test the consistency of the risk measures in section 5.

2. RISK MEASURES WITH K LINES OF BUSINESS

For a unidimensional risk process, one classical goal is to determine the minimal initial reserve u_ϵ needed for the probability of ruin to be less than ϵ.

In a multidimensional framework, modelling the evolution of the different lines of business of an insurance company by a multirisk process $(u_1 + X_t^1, \dots, u_K + X_t^K)$ ($u_k + X_t^k$ corresponds to the wealth of the k^{th} line of business at time t), one could look for the global initial reserve u which ensures that the probability of ruin ψ satisfies

$$\psi(u_1, \dots, u_K) \leq \epsilon$$

for the optimal allocation $(u_1, \dots, u_K)$ such that

$$\psi(u_1, \dots, u_K) = \inf_{v_1 + \cdots + v_K = u} \psi(v_1, \dots, v_K).$$

There exist different ruin concepts for multivariate processes. Most of them may be represented by the multivariate, time-aggregated claim process to enter some domain of $\mathbb{R}^K$, called insolvency region (see Picard et al. (2003), Loisel (2004)). Considering insolvency regions of the kind

$$\{x \in \mathbb{R}^K, \quad x_1 + \cdots + x_K > u + ct\},$$

corresponds to the classical unidimensional ruin problem for the global company. However, one could consider that ruin occurs when at least one line of business gets ruined :

$$\psi(u_1, \dots, u_K) = P(\exists k \in [1, K], \exists t > 0, u_k + X_t^k < 0).$$

To measure the severity of ruin, one may consider penalty functions which quantify the penalty undergone by the company due to insolvency of some of its lines of business.

Instead of the probability of crossing some barriers, it may thus be more interesting to minimize the sum of the expected cost of the ruin for each line of business until time T, which may be represented by the expectation of the sum of integrals over time of the negative part of the process. In both cases, finding the global reserve needed requires determination of the optimal allocation.

The multidimensional risk measure A, which does not depend on the structure of dependence between lines of business, is one example of what can be considered :

$$A(u_1, \dots, u_K) = \sum_{i=k}^{K} EI_T^k$$

where

$$EI_T^k = E\left[\int_0^T |R_t^k| 1_{\{R_t^k < 0\}} dt\right]$$

with $R_t^k = u_k + X_t^k$ under the constraint $u_1 + \cdots + u_K = u$.

Another possibility would be to minimize the sum

$$B = \sum_{k=1}^{K} E\tau_k'(u)$$

where

$$E\tau_k'(u) = E\left(\int_0^T 1_{\{R_t^k < 0\}} 1_{\{\sum_{j=1}^K R_t^j > 0\}} dt\right).$$

To determine the total initial reserve u needed to have an acceptable risk level requires to find the optimal allocation. Before tackling this problem in section 4, we propose a dependence structure in section 3 to be able to work on risk measures like B or ψ which take dependence into account.

3. A MULTIDIMENSIONAL RISK MODEL

We consider the process modelling the wealth of the K lines of business of an insurance company. Typical lines of business are driving insurance, house insurance, health, incapacity, death, liability,... Two main kinds of phenomena may generate dependence between the aggregated claim amounts of these lines.

- Firstly, in some cases, claims for different lines of business may come from a common event: for example, a car accident may cause a claim for driving insurance, liability and disablement insurance. Hurricanes might cause losses in different countries. This should correspond to simultaneous jumps for the multivariate process. The most common tool to take this into account is the Poisson common shock model.

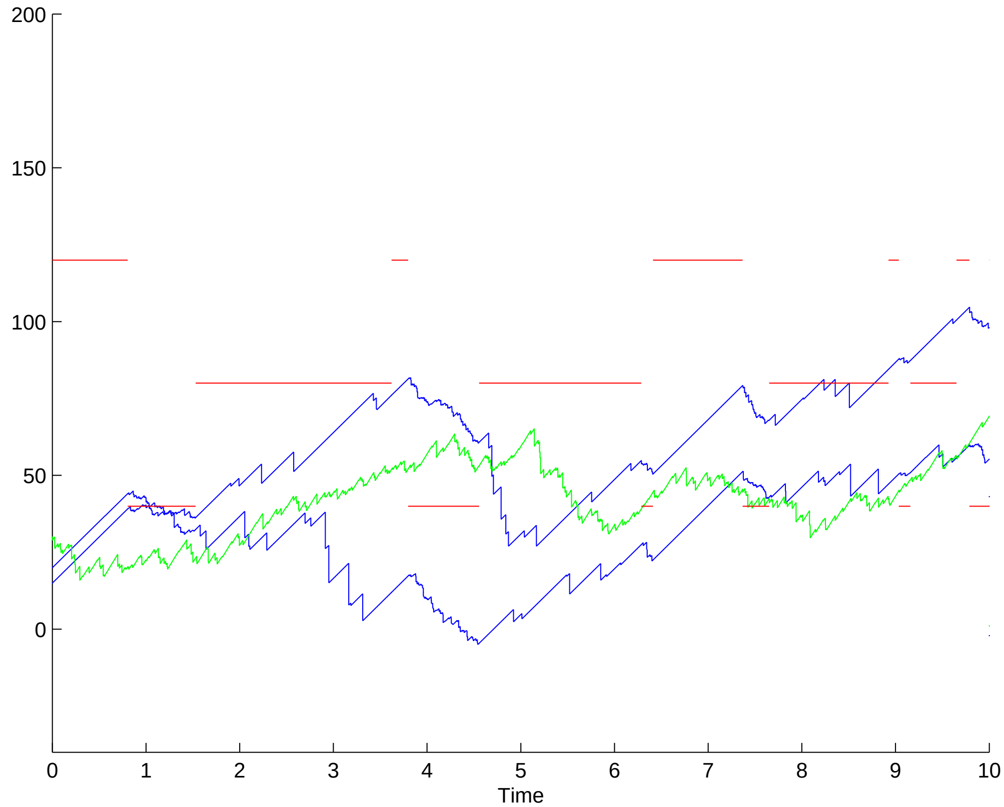

Figure 1: Sample path for three lines of business: The green one does not depend on the state of the environment. The two blue lines of business have identical parameters, and are independent conditionally on the environmental state. Occupation periods for environment states in red.

- Secondly, there exist other sources of dependence, for example the influence of the weather on health insurance and on agriculture insurance. In this case, claims seem to outcome independently for each branch, depending on the weather. This seems to correspond rather to models with modulation by a Markov process which describes the evolution of the state of the environment.

Common Poisson shock models are quite easy to understand. To fix ideas, figure 1 shows a sample path of the surpluses of the 3 lines of business of the insurance company, under a Markovian environment, but without common shock. The set of states of the environment has cardinality three. State 3 is the most favorable for the company, almost no claim occurs for lines 1 and 2 (in blue) in this state. State 1 is the least favorable state for the company, claim frequencies and severities are higher for lines 1 and 2. The state of the environment is represented in red. Events for the third line of business (in green) are independent from the state of the environment. One can see the strong positive dependence between lines 1 and 2 (in blue), but also their independence conditionally to the environment state. At some moment, the two blue curves separate each other because of this conditional independence.

Let us define more precisely the model we propose, which takes into account these two different sources of dependence: the Markov-modulated, Common Shock, Multivariate Compound Poisson Process model ((MM,CS)-MCPP), Conditionally on the state of the environment, the multivariate claim process is modelled by a Compound Poisson Process with Common Shocks. The intensity, the claim size distribution and even the common shock parameters may vary in function of the state of the environment, which is modelled by a Markov process.

The environment state process, denoted by $J(t)$, is a Markov process with state space $S = \{1, \ldots, N\}$, initial distribution μ and intensity matrix A.
For example, state 1 might correspond to periods of frequent heavy rains, or very hot weather, to hurricane seasons. It might also indicate frequent speed controls, law modifications,...

There are $m \geq 1$ different types of shock. If $J(t) = i$, then shocks of type e $(1 \leq e \leq m)$ occur according to a Poisson process with intensity $\lambda_{e,i}$. These shock counting processes are independent conditionally on $J(.)$.
For example, a shock may be a big car accident, a particular hurricane, an explosion, a medical mistake with consequences in a hospital,...

If $J(t) = i$, at the r^{th} occurrence of type e $(1 \leq e \leq m)$, the Bernouilli vector $I_i^{e,r} = (I_{1,i}^{e,r}, \ldots, I_{Ki}^{e,r})$ indicates whether a loss occurs for branch $k \in [1, K]$, and the potential losses are represented by $X_i^{e,r} = (X_{1,i}^{e,r}, \ldots, X_{K,i}^{e,r})$.
For example, a car accident may cause claims in driving insurance, liability, incapacity, death.

For a fixed state i and fixed shock type e, the successive $I_i^{e,r}$ are i.i.d.,
the successive $X_i^{e,r}$ are i.i.d.

Besides, the $I_i^{e,r}$ are independent from the $X_i^{e,r}$. However, for a fixed event i, e, r, the loss triggers $(I_{1,i}^{e,r}, \ldots, I_{K,i}^{e,r})$ and the potential losses generated by this event $X_i^{e,r} = (X_{1,i}^{e,r}, \ldots, X_{K,,i}^{e,r})$ may be dependent. In most real-world cases, the by-claims amounts seem to be positively correlated. This is the reason why we allow this kind of dependence. Between time 0 and time t, denote by $N_i^e(t)$ the number of shocks of type e that occurred while J was in state i. Then the aggregate claim amount vector up to time t is $S(t) = (S_1(t), \ldots, S_K(t))$ where for a branch $k \in [1, K]$,

$$S_k(t) = \sum_{i=1}^{N} \sum_{e=1}^{m} \sum_{r=1}^{N_i^e(t)} I_{k,i}^{e,r}.X_{k,i}^{e,r}$$

In case of no common shock, $S_k(t)$ (the aggregate claim amount vector up to time t for branch k) is a compound Cox process with intensity $\lambda_{k,J(t)}$ and claim size distribution $F_{k,J(t)}$.

$(S(t), J(t))$ is a Markov process. (and of course $S(t)$ is not!) This has an impact on computation times, because we have to keep track of the environment state during the computations.

In this model, the use of Monte Carlo methods is necessary if the number of states of the environment or the number of lines of business is too large.

For small values of these parameters, we explain in Loisel (2004) how to compute finite-time ruin probabilities using an algorithm generalizing the algorithm of Picard et al. (2003). In case of financial interactions between some lines of business, we can also use martingale methods based on results of Asmussen and Kella (2000) and Frostig (2004) to provide a theoretical way to compute the expected time to ruin of the main line of business, and the impact of the other lines on the time to ruin and on the dividends paid to the shareholders until ruin of the main line (see Loisel (2005a)).

We will only recall here results from Loisel (2004) on the impact of dependence on multidimensional, finite-time ruin probabilities in section 5.

4. OPTIMAL RESERVE ALLOCATION

Using differentiation theorems from Loisel (2005b), it is possible to determine a very intuitive optimal reserve allocation strategy to minimize the functional A defined in section 2. The problem is to minimze

$$A(u_1,\dots,u_K)=\sum_{i=k}^{K}EI_T^k$$

where

$$EI_T^k=E\left[\int_0^T |R_t^k|1_{\{R_t^k<0\}}dt\right]$$

with $R_t^k=u_k+X_t^k$ under the constraint $u_1+\cdots+u_K=u$. This does not depend on the dependence structure between the lines of business because of the linearity of the expectation. Denote $v_k(u_k)$ the differentiate of EI_T^k with respect to u_k. Using Lagrange multipliers implies that if $(u_1,\dots,u_K)$ minimizes A, then $v_k(u_k)=v_1(u_1)$ for all $1\le k\le K$. Compute $v_k(u_k)$:

$$v_k(u_k)=\left(E\left[\int_0^T |R_t^k|1_{\{R_t^k<0\}}dt\right]\right)'=-E\tau^k=-\int_0^T P\left[\{R_t^k<0\}\right]dt$$

where τ^k represents the time spent in the red between 0 and T for line of business k.
The differentiation theorem of the previous section justifies the previous derivation. The sum of the average times spent under 0 is a decreasing function of the u_k. So A is strictly convex. On the compact space

$$\mathcal{S}=\{(u_1,\dots,u_K)\in(\mathbb{R}^+)^K,\quad u_1+\cdots+u_K=u\},$$

A admits a unique minimum.

Theorem 4.1 *The optimal allocation is thus the following: there is a subset $J\subset[1,K]$ such that for $k\notin J$, $u_k=0$, and for $k,j\in J$, $E\tau_k=E\tau_j$.*

The interpretation is quite intuitive: the safest lines of business do not require any reserve, and the other ones share the global reserve in order to get equal average times in the red for those lines of business.

Relaxing nonnegativity, on $\{u_1+\cdots+u_K=u\}$, if $(u_1,\dots,u_K)$ is an extremum point for A, then for the K lines of business, the average times spent under 0 are equal to one another. If it is a minimum for the sum of the times spent below 0 for each line of business, then the average number of visits is proportional to the c_k, and in infinite time the ruin probabilities are in fixed proportions. However the existence of a minimum is not guaranteed, because $(u_1,\dots,u_K)$ is no longer compact. The problem would be more tractable with the average time in the red or with minimization on the c_k, because some factors penalize very negative u_k in these cases.

The multidimensional risk measure A, which does not depend on the structure of dependence between lines of business, is one example of what can be considered. Another possibility was to minimize the sum

$$B = \sum_{k=1}^{K} E\tau'_k(u)$$

where

$$E\tau'_k(u) = E\left(\int_0^T 1_{\{R_t^k<0\}} 1_{\{\sum_{j=1}^K R_t^j>0\}} dt\right).$$

Here B takes dependence into account, and the following proposition shows what can be done:

Proposition 4.2 *Let $X_t = ct - S_t$, where S_t satisfies hypothesis (H1) of Theorem* **??**. *Define B by $B(u_1, \ldots, u_K) = \sum_{k=1}^K E(\tau'_k(u))$ for $u \in \mathbb{R}^K$. B is differentiable on $(\mathbb{R}_*^+)^K$, and for $u_1, \ldots, u_K > 0$,*

$$\frac{\partial B}{\partial u_k} = -\frac{1}{c_k} EN_k^0(u, T),$$

where $N_k^0(u,T) = Card\left(\{t \in [0,T], \quad (R_t^k = 0) \cap \left(\sum_{j=1}^K R_t^j > 0\right)\}\right)$.

All these results are drawn from Loisel (2005b). The proofs and some examples may be found there.

5. IMPACT OF DEPENDENCE

The fact that the optimal reserve allocation strategy for the functional A does not depend on the dependence structure between risks makes it a good benchmark to compare with other risk measures, for example the probability that at least one line of business gets ruined, which we call from now on multidimensional ruin probability (m.r.p.).

Suppose that your lines of business face dependent risks, with fixed marginals. An incontestable fact is that any actuary would prefer negative dependence between risks, in order to profit from their mutualization. This is in agreement with the univariate risk model, in which positive dependence between risks increase the probability of ruin for the global company. However, this is in total contradiction with the results obtained obtained in Loisel (2004) for the m.r.p.. It is shown that positive dependence between risks decrease the probability that at least one line of business gets ruined. This is quite intuitive since for m.r.p. you do not care if only one or all your lines are ruined.
In particular we have the following result. Assume that the transition rate matrix of the environment process is stochastically monotone, and that one can order the states from the least to the most favorable state for the company for all lines of business at the same time. Then, the m.r.p. in the multidimensional model in which all lines of business are impacted by the same environment is less than the m.r.p. in the similar model in which the K lines of business are impacted by independent copies of the environment process. Picard et al. (2003) had also proved for processes with independent increments, if the risks were $PUOD$, then the times to ruin were $PLOD$.

This shows that the m.p.r. is interesting as a complementary information about the solvency of the lines of business, and may be used with other criteria to determine the reserve allocation once the risk have been selected, but that to select risks and to determine the capital requirements, one should use as before risk measures on the aggregated process first. Using only m.r.p. would be a poor idea.

6. CONCLUSION

We proposed and studied risk measures and models for multidimensional risk models. The impact of dependence on the risk perception and on the optimal reserve allocation strategy gives an idea on how to use them. Interesting problems to consider are the problems of parameter estimation and the link with credit risk theory. It would also be interesting to consider more general risk processes to take investment into account.

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