A GAMMA IBNR CLAIMS RESERVING MODEL WITH DEPENDENT DEVELOPMENT PERIODS

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AGENDA

1. Homogenous allocation in collective risk theory
2. Model with independent development periods
3. Model with dependent development periods
4. Numerical example and comparison

1. Homogenous allocation in collective risk theory

Problem:

allocation of the ultimate claims of a line of business (LoB) to the ultimate claims of the underwriting periods

Quantities:

V : premium volume LoB
U : ultimate claims of LoB
$\mu = E[U]$: mean ultimate claims of LoB
$k = CoV[U]$: coefficient of variation of LoB
V_i : premium volume of i-th period
U_i : ultimate claims of i-th period
$\mu_i = E[U_i]$: mean ultimate claims of i-th period
$k_i = CoV[U_i]$: coefficient of variation i-th period

Homogeneous allocation principle of collective risk theory (Hürlimann(2002))

In the standard collective model of risk theory for the ultimate claims, suppose that the **claim sizes** of the underwriting periods are **identically distributed**. Assume the **premium volumes** are calculated according to the **expected value principle** such that

$V = (1+\theta)\cdot\mu$, $V_i = (1+\theta)\cdot\mu_i$, $i = 1,\ldots,n$, with θ the loading factor. Then the means and coefficients of variation $(\mu_i, k_i), i = 1,\ldots,n$, relate to (μ, k) according to the following rules:

(R1) $$\frac{\mu_i}{V_i} = \frac{\mu}{V}, \quad i = 1,\ldots,n$$

(**invariance of the mean ultimate claims per unit of premium volume**)

(R2) $$k_i = \sqrt{\frac{V}{V_i}}\cdot k, \quad i = 1,\ldots,n$$

(**coefficients of variation inverse proportional to the square-root premium volumes**)

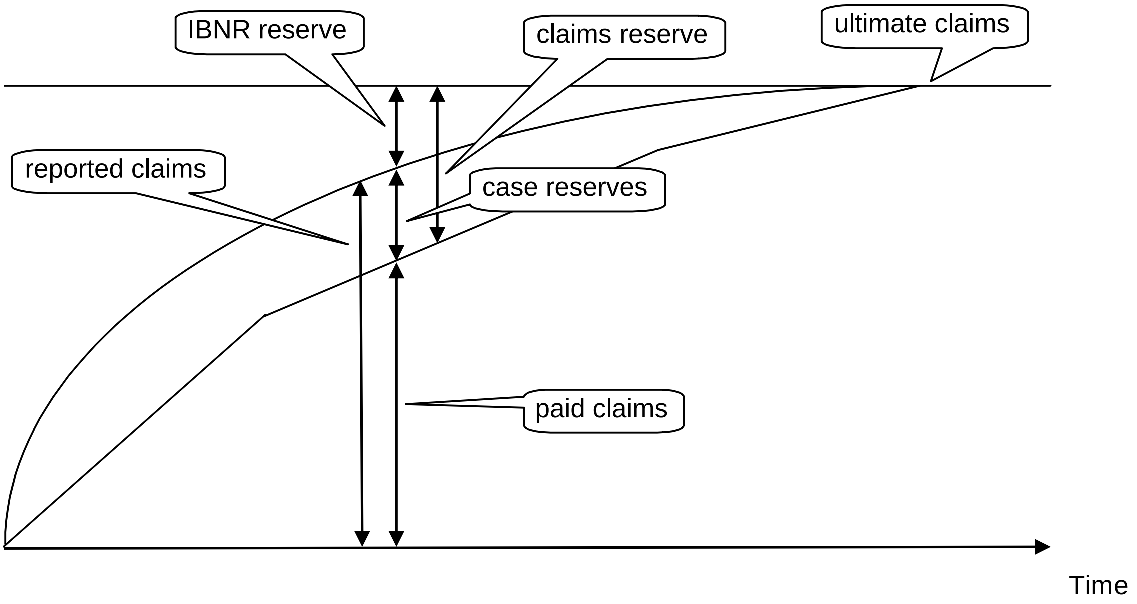
2. Model with independent development periods.

ultimate claims

= paid claims + outstanding case reserves

+ IBNR claims reserve

Figure 1: IBNR, paid claims, case reserves, reported claims, claims reserve and ultimate claims



Gamma IBNR claims reserve model (I)

Under the assumptions of the homogeneous allocation principle in collective risk theory, assume that the paid claims S_{ik}, $1 \le i,k \le n$, satisfy the following model assumptions:

(I.1) $E[S_{ik}] = \left(\frac{V_i}{V}\right) \cdot x_i y_k$, for some parameters x_i, y_k, $1 \le i,k \le n$,

(I.2) $CoV[S_{ik}] = CoV[U_i] = \sqrt{\frac{V}{V_i}} \cdot CoV[U] = \sqrt{\frac{V}{V_i}} \cdot \sqrt{\frac{1}{\alpha}}$, for some parameter $\alpha > 0$,

(I.3) The paid claims S_{ik}, $1 \le i,k \le n$, follow independent gamma distributions

- Probability density of the paid claims:

$$f_{ik}(s)=\Gamma(\alpha_i)^{-1}\left(\frac{s}{\beta_{ik}}\right)^{\alpha_i}\cdot\frac{\exp\left(-\frac{s}{\beta_{ik}}\right)}{s},\quad 1\le i,k\le n,$$

$$\alpha_i=\left(\frac{V_i}{V}\right)\cdot\alpha,\quad \beta_{ik}=\frac{1}{\alpha}\cdot x_i y_k.$$

- Apply **maximum likelihood method** on the loss triangle of paid claims to get estimates of the model parameters $x_i, y_k,\ 1\le i,k\le n,$ and α

- Estimate the **mean** and **variance** of the **IBNR claims reserves**:

$$\hat{R}_i=\sum_{k=n-i+2}^{n}\hat{E}[S_{ik}]=\left(\frac{V_i}{V}\right)\cdot\hat{x}_i\cdot\sum_{k=n-i+2}^{n}\hat{y}_k,\quad i=2,\dots,n,$$

$$\hat{Var}[R_i]=\sum_{k=n-i+2}^{n}\hat{Var}[S_{ik}]$$

$$=\left(\frac{V_i}{V}\right)\cdot\frac{1}{\alpha}\cdot\hat{x}_i^2\cdot\sum_{k=n-i+2}^{n}\hat{y}_k^2,\quad i=2,\dots,n.$$

- **Distribution** of the **IBNR claims reserve** $R_i,\ i=2,\dots,n,$ for the Gamma model (I) has **approximate Gamma** distribution $\Gamma(\beta_i x,\alpha_i)$ with parameters

$$\alpha_i = \frac{1}{k_i^2}, \quad \beta_i = \frac{1}{k_i^2 \mu_i},$$

$$\mu_i = \frac{V_i}{V} x_i \cdot \sum_{k=n-i+2}^{n} y_k, \quad k_i^2 = \frac{\frac{V_i}{V\alpha} x_i^2 \cdot \sum_{k=n-i+2}^{n} y_k^2}{\mu_i^2}.$$

3. Model with dependent development periods.

Assume that the bivariate margins of the paid claims random vector $S^{(i)} = \left(S_{i1}, S_{i2}, \ldots, S_{in-i+2}\right)$ belong to the same parametric family of linear Spearman copulas

$$(1) \quad F_{i,k,\ell}(x,y) = \left(1-\theta_{k\ell}\right) \cdot F_{i,k,\ell}^{\perp}\left(x,y\right) + \theta_{k\ell} \cdot F_{i,k,\ell}^{+}(x,y), \quad 0 \le \theta_{k\ell} \le 1,$$

$\theta_{k\ell}$: Spearman's grade correlation coefficient

$F_{i,k,\ell}^{\perp}(x,y)$: distribution of an independent version $\left(S_{ik}^{\perp}, S_{i\ell}^{\perp}\right)$ of $\left(S_{ik}, S_{i\ell}\right)$

$F_{i,k,\ell}^{+}(x,y)$: distribution of a comonotone version $\left(S_{ik}^{+}, S_{i\ell}^{+}\right)$ of $\left(S_{ik}, S_{i\ell}\right)$

Consider Fréchet like multivariate distribution

(2) $$F_i^*(x) = (1-\theta)\cdot F_i^{\perp}(x) + \theta\cdot F_i^{+}(x), \\ 0 \le \theta \le 1, \quad x = \left(x_1,\dots,x_{n-i+2}\right)$$

$F_i^{\perp}(x)$: distribution of an independent version of $S^{(i)}$

$F_i^{+}(x)$: distribution of a comonotone version of $S^{(i)}$

Motivation from "prudent" IBNR claims reserving:

The bivariate margins of (2) are at least as positively dependent as those obtained from the bivariate model (1) in the concordance ordering.

The bivariate distribution of the bivariate margin of (2) are given by

$$F_{i,k,\ell}^*(x,y) = (1-\theta)\cdot F_{i,k,\ell}^{\perp}(x,y) + \theta\cdot F_{i,k,\ell}^{+}(x,y)$$

The stated condition means that $F_{i,k,\ell}(x,y) \le F_{i,k,\ell}^*(x,y)$, and this is equivalent to

$$(\theta-\theta_{k\ell})\cdot F_{i,k,\ell}^{\perp}(x,y) \le (\theta-\theta_{k\ell})\cdot F_{i,k,\ell}^{+}(x,y)$$

Since $F^{\perp}_{i,k,\ell}(x,y) \le F^{+}_{i,k,\ell}(x,y)$ this means that $\theta \ge \theta_{k\ell}$, for all (k,ℓ).

Therefore, for prudent IBNR evaluation set

$$\theta = \max_{(k,\ell)}\{\theta_{k\ell}\}$$

(most conservative model for IBNR claims reserving with respect to the concordance order for the bivariate margins of this model)

Gamma IBNR claims reserve model (II)

Under the assumptions of the homogeneous allocation principle in collective risk theory, assume that the paid claims S_{ik}, $1 \le i,k \le n$, satisfy the following model assumptions:

(II.1) $E[S_{ik}] = \left(\frac{V_i}{V}\right) \cdot x_i y_k$, for some parameters x_i, y_k, $1 \le i,k \le n$,

(II.2) $CoV[S_{ik}] = CoV[U_i] = \sqrt{\frac{V}{V_i}} \cdot CoV[U] = \sqrt{\frac{V}{V_i}} \cdot \sqrt{\frac{1}{\alpha}}$, for some parameter $\alpha > 0$, (II.3) The random vector of paid claims $S^{(i)} = \left(S_{i1}, S_{i2}, \ldots, S_{in-i+2}\right), \quad i = 2, \ldots, n,$ follows a Fréchet like multivariate distribution $F_i^*(x) = (1-\theta)\cdot F_i^{\perp}(x) + \theta \cdot F_i^{+}(x)$ with gamma distributed margins

The distribution of the sum $R_i = \sum_{k=n-i+2}^{n} S_{ik}, \quad i = 2, \ldots, n,$ can be approximated by the analytical expression

$$F_{R_i}(x) = (1-\theta)\cdot \Gamma\left(\frac{1}{k_i^2 \mu_i} x; \frac{1}{k_i^2}\right) + \theta \cdot \Gamma\left(\frac{V_i \alpha}{V \mu_i} x; \frac{V_i}{V}\alpha\right),$$

$$\theta \in [0,1].$$

For this approximation, the **mean IBNR reserve**

$$E[R_i] = \int_0^\infty \left[1 - F_{R_i}(x)\right] dx = (1-\theta)\cdot \int_0^\infty \left[1 - \Gamma\left(\tfrac{1}{k_i^2 \mu_i} x; \tfrac{1}{k_i^2}\right)\right] dx$$

$$+\, \theta \cdot \int_0^\infty \left[1 - \Gamma\left(\tfrac{V_i \alpha}{V \mu_i} x; \tfrac{V_i \alpha}{V}\right)\right] dx = (1-\theta)\cdot \mu_i + \theta \cdot \mu_i = \mu_i .$$

coincides with the expected value in model (I). The **variance** is given by

$$Var[R_i] = \left[(1-\theta)\cdot k_i^2 + \theta \cdot \frac{V}{V_i \alpha}\right] \cdot \mu_i^2 .$$

In model (II), the two **extreme cases** are the **independent case** $\theta = 0$ and the **comonotone case** $\theta = 1$. One has **inequalities**, for the **variance**

$$
\begin{aligned}
& Var\left[R_i^{\theta=0}\right] = \left(k_i \mu_i\right)^2 \\
& \leq Var\left[R_i^{\theta}\right] \\
& \leq Var\left[R_i^{\theta=1}\right] = \left(\frac{V}{V_i \alpha}\right) \mu_i^2
\end{aligned}
$$

and for the **percentile values** of the IBNR reserve

$$
\begin{aligned}
& Q_{R_i}^{\theta=0}(u) = \Gamma^{-1}\left(u; \frac{1}{k_i^2}\right) k_i^2 \mu_i \\
& \leq Q_{R_i}^{\theta}(u) \\
& \leq Q_{R_i}^{\theta=1}(u) = \Gamma^{-1}\left(u; \frac{V_i}{V} \alpha\right) \frac{V}{V_i \alpha} \mu_i, \quad \theta \in (0,1)
\end{aligned}
$$

4. Numerical example and comparison.

Comparison of gamma model (II) with **optimal** credible loss ratio IBNR claims reserve:

$$R_i^c = Z_i \cdot R_i^{ind} + (1 - Z_i) \cdot R_i^{coll}, \quad i = 1, \ldots, n$$

Theorem (Hürlimann(2005/06))

Under specific assumptions, the optimal credibility weights Z_i^* which minimize the mean squared error $mse(R_i^c) = E\left[(R_i^c - R_i)^2\right]$ and the variance $Var[R_i^c]$ are given by

$$Z_i^* = \frac{p_i}{p_i + t_i^*}, \text{ with } t_i^* = \sqrt{p_i}, \quad i = 1, \ldots, n.$$

Table: Comparison of model (II) and optimal credible method

	IBNR method		
sum over all periods	**ind gamma**	**com gamma**	**optimal credible**
mean IBNR	27'037	27'037	25'648
st.dev. IBNR	2'940	5'051	2'000
80% percentiles	29'473	31'170	27'314
90% percentiles	30'868	33'678	28'242
95% percentiles	32'053	35'846	29'024
99% percentiles	34'359	40'156	30'530