Various Extensions Based on Munich Chain Ladder Method

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20th June 2007, 50th Anniversary ASTIN Colloquium
1. Introduction

2. Robust Regression

3. Addition of MSE calculation to MCL model

4. Multivariate Extensions to Chain Ladder

5. Multivariate MCL

6. Other Approaches to model Paid and Incurred data
Scope of presentation

1. Introduction
2. Robust Regression
3. Addition of MSE calculation to MCL model
4. Multivariate Extensions to Chain Ladder
5. Multivariate MCL
6. Other Approaches to model Paid and Incurred data
Munich Chain Ladder - Introduction

- Derived and presented by Munich Re → its name (MCL) (see Paper of Quarg 2004)

Used Notation

- $a(i) = n - i$ level of development
- Data of Paid to Incurred Ratio: $Q_{i,j} = (P/I)_{i,j}$
- $Y_{i,j} = Y_{P_{i,j}}$ and $I_{i,j}$
- $\hat{Y}_{I_{i,n}}$ and $\hat{Y}_{P_{i,n}}$ for $i \geq 1$
- It does not hold for SCL

Various Extensions Based on MCL Method

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Munich Chain Ladder - Introduction

- Derived and presented by Munich Re ⇒ its name (MCL) (see Paper of Quarg 2004)
- Analysis of both Paid $Y_{i,j}^P$ a Incurred $Y_{i,j}^I$ schemes

Used Notation

- $a(i) = n - i$ level of development
- Data of Paid to Incurred Ratio $Q_{i,j} ≡ \left(\frac{P}{I}\right)_{i,j}$
- $Y_{i,j} = 0, \ldots, n_i + j \leq n$
Munich Chain Ladder - Introduction

- Derived and presented by Munich Re ⇒ its name (MCL) (see Paper of Quarg 2004)
- Analysis of both Paid $Y_{i,j}^P$ and Incurred $Y_{i,j}^I$ schemes
- Extension of model of Mack (SCL)

Used Notation

- $a(i) = n - i$ level of development
- Data of Paid to Incurred Ratio $Q_{i,j} \equiv (P/I)_{i,j}$
- $\hat{Y}_{i,j}^P$ and $\hat{Y}_{i,j}^I$
- Joint available information $B_i(s) = (Y_i(s); Y_i(s))$
Munich Chain Ladder - Introduction

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- Analysis of both Paid $Y_{i,j}^P$ a Incurred $Y_{i,j}^I$ schemes
- Extension of model of Mack (SCL)
- Significant improvement: If $Y_{0,n}^I \sim Y_{0,n}^P$ MCL reduces gap between $\hat{Y}_{i,n}^I$ and $\hat{Y}_{i,n}^P$ for $i \geq 1$

Used Notation

- $\beta(i, n) = n - i$ level of development
- Data of Paid to Incurred Ratio $Q_{i,j} \equiv (P/I)_{i,j}$
- $Y_{i,j} \equiv Y_{i,j}^I + Y_{i,j}^P$ for $i = 0, \ldots, n, j = 0, \ldots, n$
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Used Notation

- $a(i) = n - i$ level of development
- $Q_{i,j} \equiv (P/I)_{i,j} \equiv Y_{i,j}^P / Y_{i,j}^I$
- $i = 0, \ldots, n \leq j$ joint available information
- $B_i(s) = (Y_i(s), Y_i(s))$
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Used Notation

- $a(i) = n - i$ level of development $i$
- Data of Paid to Incurred Ratio

$$Q_{i,j} \equiv (P/I)_{i,j} \equiv \frac{Y_{i,j}^P}{Y_{i,j}^I} \quad i = 0, \ldots, n \quad i + j \leq n$$
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- Joint available information $B_i(s) = (Y_i(s)^P; Y_i(s)^I)$
Average PI Estimate $j \ P/I_j = \frac{\sum_{i=0}^{n} Y_{i,j}^P}{\sum_{i=0}^{n} Y_{i,j}^I}$

It holds true that $\frac{P}{I_{i,j}} = \frac{P}{I_{i,j}}_{a(i)}$

See Quarg 2004 for proof

Interpretation explains drawback of SCL method:

1. Low $(P/I_{i,j})$ for known data in diagonal $\Rightarrow$ low $(P/I_{i,j})$ for prediction
2. Disparity between both projection
3. Systematic weakness of SCL
Average PI Estimate \( j \frac{P}{I_j} = \frac{\sum_{i=0}^{n} Y_{i,j}^P}{\sum_{i=0}^{n} Y_{i,j}^I} \)

If \( i + j > n \) PI is defined as \( (P/I)_{i,j} = \frac{\hat{Y}_{i,j}^P}{Y_{i,j}^I} \)
Munich Chain Ladder - Paid to Incurred Ratio (PI)

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  \]
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  1. Low \((P/I)_{i,j}\) for known data in diagonal \(\Rightarrow\) low \((P/I)_{i,j}\) for prediction
Munich Chain Ladder - Paid to Incurred Ratio (PI)

- Average PI Estimate
  \[ P/I_j = \frac{\sum_{i=0}^{n} Y_{i,j}^P}{\sum_{i=0}^{n} Y_{i,j}^I} \]

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  3. Systematic weakness of SCL
Fundamental Principles of MCL

- Adjustment of development factors according to \((P/I)_{i,a(i)}\)
Fundamental Principles of MCL

- Adjustment of development factors according to \((P/I)_{i,a(i)}\)
- If PI Ratio is below average ⇒ for Paid data

[High level of Claims Reserving]

Lower increase of incurred amount is expected ⇒ Below average factor

Good to decrease standard estimate

\[\hat{f}_{P,j}^{corr}(Y_{P,i,j}, Y_{I,i,j}; Y_{P,i,j+1}, Y_{I,i,j+1}) < 0\]

[Low level of claim settlement]

Could be accelerated in future periods ⇒ Above average factor

Good to increase standard estimate

\[\hat{f}_{I,j}^{corr}(Y_{P,i,j}, Y_{I,i,j}; Y_{P,i,j+1}, Y_{I,i,j+1}) > 0\]
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Adjustment of development factors according to $(P/I)_{i,a(i)}$

If PI Ratio is below average $\Rightarrow$ for Paid data

- Low level of claim settlement
- Could be accelerated in future periods
- $\Rightarrow$ Above average factor $Y_{i,j+1}/Y_{i,j}$

If PI Ratio is below average $\Rightarrow$ for Incurred data

- High level of Claims Reserving
- Lower increase of incurred amount is expected
- $\Rightarrow$ Below average factor $Y_{i,j+1}/Y_{i,j}$
Adjustment of development factors according to \((P/I)_{i,a(i)}\)

If PI Ratio is below average ⇒ for Paid data
- Low level of claim settlement
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- ⇒ Above average factor \(Y_{i,j+1}^P / Y_{i,j}^P\)
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If PI Ratio is below average ⇒ for Incurred data
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Introduction

Fundamental Principles of MCL

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  - \(\text{corr} \left( \frac{Y_{i,j}^{P}}{Y_{i,j}^{I}} ; \frac{Y_{i,j+1}^{I}}{Y_{i,j}^{I}} \right) > 0\)
Variables are standardised $\Rightarrow$ conditional residuals

$$Res(X|C) = \frac{X - E(X|C)}{\sigma(X|C)}$$
Variables are standardised ⇒ conditional residuals

\[ \text{Res}(X | C) = \frac{X - \text{E}(X | C)}{\sigma(X | C)} \]

Dependency structure ⇒ Paid data MCL assumption

\[ E \left( \text{Res} \left( \frac{Y_{i,s+1}^P}{Y_{i,s}^P} | Y_i(s)^P \right) | B_i(s) \right) = \lambda^P \cdot \text{Res}(Q_{i,s}^{-1} | Y_i(s)^P) \]
Variables are standardised ⇒ conditional residuals

\[
\text{Res}(X|C) = \frac{X - \text{E}(X|C)}{\sigma(X|C)}
\]

Dependency structure ⇒ Paid data MCL assumption

\[
\text{E} \left( \text{Res} \left( \frac{Y_{i,s+1}^P}{Y_{i,s}^P} \right) | B_i(s) \right) = \lambda_P \cdot \text{Res}(Q_{i,s}^{-1} | Y_i(s)^P)
\]

Could be transformed onto

\[
\text{E} \left( \frac{Y_{i,s+1}^P}{Y_{i,s}^P} | B_i(s) \right) = f_s^P + \lambda_P \frac{\sigma \left( \frac{Y_{i,s+1}^P}{Y_{i,s}^P} \right)}{\sigma(Q_{i,s}^{-1} | Y_i(s)^P)} \cdot (Q_{i,s}^{-1} - \text{E}(Q_{i,s}^{-1} | Y_i(s)^P))
\]
Regression Models of MCL method - Incurred Data

- Analogous as for Paid
Analogous as for Paid

\[
\mathbb{E} \left( \text{Res} \left( \frac{Y_{i,s+1}^l}{Y_{i,s}^l} \middle| Y_i(s)^l \right) \middle| B_i(s) \right) = \lambda^l \cdot \text{Res}(Q_{i,s} \middle| Y_i(s)^l)
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Regression Models of MCL method - Incurred Data

- Analogous as for Paid

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E \left( \text{Res} \left( \frac{Y_{i,s+1}^l}{Y_{i,s}^l} \middle| Y_i(s)^l \right) \middle| B_i(s) \right) = \lambda^l \cdot \text{Res}(Q_{i,s} \middle| Y_i(s)^l)
\]

- Transformation

\[
E \left( \frac{Y_{i,s+1}^l}{Y_{i,s}^l} \middle| B_i(s) \right) = f_s^l + \lambda^l \frac{\sigma \left( \frac{Y_{i,s+1}^l}{Y_{i,s}^l} \middle| Y_i(s)^l \right)}{\sigma(Q_{i,s} \middle| Y_i(s)^l)} \cdot (Q_{i,s} - E(Q_{i,s} \middle| Y_i(s)^l))
\]

Note - differences between models

\[Q_{i,j} \text{ is explanatory variable at Incurred Model} \]

\[Q_{i,j} - 1 \text{ is explanatory variable at Paid Model} \]

⇒ in rational cases should be \[\lambda^P > 0 \text{ and } \lambda^I > 0\]
Regression Models of MCL method - Incurred Data

- Analogous as for Paid

$$
E \left( \text{Res} \left( \frac{Y_{i,s+1}^I}{Y_{i,s}^I} \bigg| Y_i(s)^I \right) \bigg| B_i(s) \right) = \lambda^I \cdot \text{Res}(Q_{i,s} | Y_i(s)^I)
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- Transformation

$$
E \left( \frac{Y_{i,s+1}^I}{Y_{i,s}^I} \bigg| B_i(s) \right) = f_s^I + \lambda^I \frac{\sigma \left( \frac{Y_{i,s+1}^I}{Y_{i,s}^I} \bigg| Y_i(s)^I \right)}{\sigma(Q_{i,s} | Y_i(s)^I)} \cdot (Q_{i,s} - E(Q_{i,s} | Y_i(s)^I))
$$

Note - differences between models

- $Q_{i,j}$ is explanatory variable at Incurred Model
- $Q_{i,j}^{-1}$ is explanatory variable at Paid Model
- $\Rightarrow$ in rational cases should be $\lambda^P > 0$ and $\lambda^I > 0$
Implementation of Regressions

- Originally traditional OLS method
Implementation of Regressions

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- Explanatory Power of the model rather weak (especially for Incurred model)
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- Interpretation of causality relation between Paid and Incurred?
Implementation of Regressions

- Originally traditional OLS method
- Explanatory Power of the model rather weak (especially for Incurred model)
- Interpretation of causality relation between Paid and Incurred?
- The Best achieved results by standard approach not so appropriate
Scope of presentation

1. Introduction
2. Robust Regression
3. Addition of MSE calculation to MCL model
4. Multivariate Extensions to Chain Ladder
5. Multivariate MCL
6. Other Approaches to model Paid and Incurred data
Detection of outliers of the model
Application of Robust Regression

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- Various available method performed
Application of Robust Regression

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- For example Huber, Bi square,
Application of Robust Regression

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- For example Huber, Bi-square,
  - Lower weight given to outlying observation
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- For example Huber, Bi square,
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- Least Trimmed squares
Application of Robust Regression

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- Various available method performed
- For example Huber, Bi square,
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- Least Trimmed squares
  - Selected portion of outliers is directly cut off the model
Application of Robust Regression

- Detection of outliers of the model
- Various available method performed
  - For example Huber, Bi square,
    - Lower weight given to outlying observation
  - Least Trimmed squares
    - Selected portion of outliers is directly cut off the model
- Outliers have strong influence onto model
Least Trimmed Squares

- LTS estimator $\hat{\beta}^{LTS} = \arg \min_{\beta \in \mathbb{R}^{p+1}} \sum_{i=1}^{h} r_{[i]}^{2}(\beta)$

$r_{[i]}(\beta) = y_i - x_i' \beta$

Computational algorithm of LTS

1. Randomly select $h$ observations and perform OLS regression for them.
2. Compute OLS residuals based on the model for all data and choose $h$ with smallest absolute values of residuals.
3. For newly selected $h$ observations compute OLS regression again. Did RSS for selected model decrease?
   - Yes $\Rightarrow$ go to 2
   - No $\Rightarrow$ stop
Least Trimmed Squares

- LTS estimator $\hat{\beta}^{LTS} = \arg\min_{\beta \in \mathbb{R}^{p+1}} \sum_{i=1}^{h} r_{[i]}^2(\beta)$
- $r_{[i]}^2(\beta)$ represents $i$-th smallest value among $r_1^2(\beta), \ldots, r_n^2(\beta)$
Least Trimmed Squares

- LTS estimator \( \hat{\beta}^{LTS} = \arg\min_{\beta \in \mathbb{R}^{p+1}} \sum_{i=1}^{h} r^2_{[i]}(\beta) \)
- \( r^2_{[i]}(\beta) \) represents the \( i \)-th smallest value among \( r^2_1(\beta), \ldots, r^2_n(\beta) \)
- \( r_i(\beta) = y_i - x_i' \beta \) OLS residuals
Least Trimmed Squares

- LTS estimator $\hat{\beta}^{LTS} = \text{arg min}_{\beta \in \mathbb{R}^{p+1}} \sum_{i=1}^{h} r_{[i]}^2(\beta)$
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- $r_i(\beta) = y_i - x_i'\beta$ OLS residuals
- trimming constant $h$

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1. Randomly select $h$ observation and perform OLS regression for them
2. Compute OLS residuals based on the model for all data and choose $h$ with smallest absolute values of residuals
3. For newly selected $h$ observation compute OLS regression again. Did RSS for selected mode decrease?
   - yes $\Rightarrow$ go to 2
   - no $\Rightarrow$ stop
Least Trimmed Squares

- LTS estimator $\hat{\beta}^{\text{LTS}} = \arg \min_{\beta \in \mathbb{R}^{p+1}} \sum_{i=1}^{h} r_{[i]}^2(\beta)$
- $r_{[i]}^2(\beta)$ represents $i$-th smallest value among $r_1^2(\beta), \ldots, r_n^2(\beta)$
- $r_i(\beta) = y_i - x_i'\beta$ OLS residuals
- trimming constant $h$
- $\frac{n}{2} < h \leq n$
Least Trimmed Squares

- LTS estimator $\hat{\beta}^{LTS} = \arg\min_{\beta \in \mathbb{R}^{p+1}} \sum_{i=1}^{h} r_{[i]}^2(\beta)$
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- Our choices $h = 0.6 \cdot n$ and $h = 0.75 \cdot n$
Least Trimmed Squares

- LTS estimator $\hat{\beta}_{LTS} = \arg \min_{\beta \in \mathbb{R}^{p+1}} \sum_{i=1}^{h} r_{[i]}^2(\beta)$

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- $r_i(\beta) = y_i - x_i'\beta$ OLS residuals

- trimming constant $h$

- $\frac{n}{2} < h \leq n$

- Our choices $h = 0.6 \cdot n$ and $h = 0.75 \cdot n$

Computational algorithm of LTS

1. Randomly select $h$ observation and perform OLS regression for them
2. Compute OLS residuals based on the model for all data and choose $h$ with smallest absolute values of residuals
3. For newly selected $h$ observation compute OLS regression again. Did RSS for selected mode decrease?
   - yes $\Rightarrow$ go to 2
   - no $\Rightarrow$ stop
Numerical Results

- Estimates of $\hat{\lambda}^P$ and $\hat{\lambda}^I$ differ across a method relatively a lot.
- No large influence on ultimates and reserves values.
- Numerical Illustration performed.

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<th>Chain Ladder</th>
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<th>Munich</th>
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<td>101%</td>
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Robust Regression

Derivation of theoretical principles

Elasticity of MCL reserve = sensitivity of ultimates with respect to parameters $\hat{\lambda}$.

- Remark of basic formula

$$f_{i,k}^{P,MCL}(\lambda^P) = f_{k}^{P,SCL} + \hat{\lambda}^P \frac{\sigma_k^P}{\rho_k^P} \cdot \left( \frac{Y_{i,k}^I}{Y_{i,k}^P} - q_k^{-1} \right)$$

- Linearity of the function $\Rightarrow$

$$f_{i,k}^{P,MCL}(\lambda^P) = f_{k}^{P,SCL} + \lambda^P \cdot f_{i,k}^{P,MCL'}(\lambda^P)$$

- So the derivative of development factors could be rewritten to

$$f_{i,k}^{P,MCL'}(\lambda^P) = \frac{\sigma_k^P}{\rho_k^P} \cdot \left( \frac{Y_{i,k}^I}{Y_{i,k}^P} - q_k^{-1} \right)$$
Derivation of theoretical principles

Implications to elasticity of projections depending on $\hat{\lambda}$.

- Standard formula: $\hat{Y}_{i,n}^P = Y_{i,a(i)}^P \cdot \prod_{j=a(i)}^{n-1} \hat{f}_{i,j}^P$
- Rearranging the development factors

$$(Y_{i,n}^P)' = \sum_{j=a(i)}^{n-1} \frac{Y_{i,a(i)}^P}{\hat{f}_{i,j}^P} \cdot (f_{i,j}^P)' \cdot f_{i,a(i)}^P \cdots f_{i,n-1}^P = \frac{\sum_{j=a(i)}^{n-1} (f_{i,j}^P)'}{f_{i,k}^P}$$

$$\Rightarrow \frac{(Y_{i,n}^P)'}{Y_{i,n}^P} = \frac{1}{\lambda^P} \cdot \left( \sum_{j=a(i)}^{n-1} (1 - \frac{\hat{f}_{i,j}^P}{f_{i,j}^P}) \right)$$

**Final Result**

$$E(Q_{i,k}^{-1}) = q_k^{-1} \Rightarrow E \left( \hat{f}_{i,k}^P \right) = 0$$
Conclusions to MCL Elasticity

- Using formula $E \left( \hat{f}_{i,k}^P \right) = 0$

- $\Rightarrow \frac{(Y_{i,n}^P)'}{Y_{i,n}^P} = \frac{1}{\lambda^P} \cdot \sum_{j=a(i)}^{n-1} \frac{f_j^P}{f_{i,j}^P}$

- holds $E \left( \frac{(Y_{i,n}^P)'}{Y_{i,n}^P} \right) = 0$

- analogously also $E \left( \frac{(Y_{i,n}^I)'}{Y_{i,n}^I} \right) = 0$

Interpretation

- Systematic influence does not depend on $\lambda$
- Confirming original numerical results
- Hard to say what is "right" point estimate of MCL Loss Reserve
- $\Rightarrow$ Computation of Risk margin also needed
Scope of presentation

1. Introduction

2. Robust Regression

3. Addition of MSE calculation to MCL model

4. Multivariate Extensions to Chain Ladder

5. Multivariate MCL

6. Other Approaches to model Paid and Incurred data
MCL variability

- MCL provides expectation $E \left( \frac{Y_{i,s+1}^P}{Y_{i,s}^P} | B_i(s) \right)$
- Variability formula $\text{Var} \left( \frac{Y_{i,s+1}^P}{Y_{i,s}^P} | B_i(s) \right) = ?$

Process of Derivation

- Start from linear model of MCL
  $\text{Res} \left( \frac{Y_{i,s+1}^P}{Y_{i,s}^P} | Y_i^P(s) \right) = \lambda^P \text{Res} \left( \frac{Y_{i,s}^l}{Y_{i,s}^P} | Y_i^P(s) \right) + \varepsilon_{i,s}$
- Properties of residuals $E(\varepsilon_{i,s} | B_i(s)) = 0$ a $\text{var}(\varepsilon_{i,s} | B_i(s)) = \sigma_R^2$
- Adjustment of formula

$$\text{Var} \left( \text{Res} \left( \frac{Y_{i,s+1}^P}{Y_{i,s}^P} | Y_i^P(s) \right) | B_i(s) \right) = \sigma_R^2 \frac{\sum_i \sum_s \text{Res}^2 \left( \frac{Y_{i,s}^l}{Y_{i,s}^P} | Y_i^P(s) \right)}{\sum_i \sum_s \text{Res}^2 \left( \frac{Y_{i,s}^l}{Y_{i,s}^P} | Y_i^P(s) \right)}$$
Addition of MSE calculation to MCL model

MCL variability - end of derivation

We use also formula

$$\text{var} \left( \text{Res} \left( \frac{Y_{i,s+1}^P}{Y_{i,s}^P} \bigg| Y_{i}^P(s) \right) \bigg| B_i(s) \right) = \frac{\text{var}(Y_{i,s+1}^P/ Y_{i,s}^P)}{\sigma_{s}^P/ Y_{i,s}^P}$$

If we combine both formulae and remind Mack’s model

$$\text{Var} \left( \frac{Y_{i,s+1}^P}{Y_{i,s}^P} \bigg| Y_{i}^P(s) \right) = \frac{(\sigma_i^P)^2}{Y_i^P} \Rightarrow$$

Variability Formula for MCL model

could be seen as generalisation of Mack’s approach

$$\text{Var} \left( \frac{Y_{i,s+1}^P}{Y_{i,s}^P} \bigg| B_i(s) \right) = \text{var}(\hat{\lambda}^P)\sigma^2 \left( \frac{Y_{i,s+1}^P}{Y_{i,s}^P} \bigg| Y_{i}^P(s) \right) \text{Res}^2 \left( \frac{Y_{i,s}^P}{Y_{i,s}^P} \bigg| Y_{i}^P(s) \right)$$
Addition of MSE calculation to MCL model

Application to Mean Square Error Calculation

- for Incurred analogously

\[
\text{Var} \left( \frac{Y_{i,s+1}}{Y_{i,s}} \middle| B_i(s) \right) = \text{var}(\hat{\lambda})\sigma^2 \left( \frac{Y_{i,s+1}}{Y_{i,s}} \middle| Y_i(s) \right) \text{Res}^2 \left( \frac{Y_{i,s}}{Y_{i,s}} \middle| Y_i(s) \right)
\]

- \Rightarrow Application onto MCL

\[
\text{mse}(\hat{R}_i) = \hat{Y}_{i,n}^2 \sum_{k=n-i}^{N} \hat{\sigma}_k^2 \left( \frac{1}{\hat{Y}_{i,k}} + \frac{1}{\sum_{j=1}^{n-k} Y_{i,j}} \right)
\]

- Substitute the theoretical parameters by their estimates

\[
\sigma^2_{i,s} = \text{var}(\hat{\lambda}) \cdot \sigma^2_{i,s} \cdot \text{Res}^2 \left( \frac{Y_{i,s}}{Y_{i,s}} \middle| Y_i(s) \right)
\]

- Joint information leads to decrease of reserve variability

- See the following illustration
Comparison of MSE calculation between SCL and MCL

<table>
<thead>
<tr>
<th>year of origin</th>
<th>diagonal values</th>
<th>ultimate projection</th>
<th>value of reserve</th>
<th>MSE^0,5</th>
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<th>ultimate projection</th>
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<td>76 686</td>
<td>69 265</td>
<td>795</td>
<td>1.1%</td>
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MSE graph

- MSE is significantly lower in MCL model
Scope of presentation

1. Introduction
2. Robust Regression
3. Addition of MSE calculation to MCL model
4. Multivariate Extensions to Chain Ladder
5. Multivariate MCL
6. Other Approaches to model Paid and Incurred data
Recall of approach suggested by Schmidt

- Column vector
  \[ Y_{i,j} = (Y_{i,j}^1, \ldots, Y_{i,j}^K)' \]
  cumulative amount of claims occurred in period \( i \) and developed after \( j \) period after occurrence

- \( K \) insurance portfolios are analysed simultaneously

- Useful notation \( \Upsilon_{i,j} = \text{diag}(Y_{i,j}) \). Thus \( Y_{i,j} = \Upsilon_{i,j} 1 \)

- One dimensional case:
  \[ Y_{i,j+1} = Y_{i,j} \cdot F_{i,j} \]

- Multivariate extension:
  \[ Y_{i,j+1} = \Upsilon_{i,j} \cdot F_{i,j} \]

- \( F_{i,j} = (F_{i,j}^1, \ldots, F_{i,j}^K)' \) generalisation of individual factor
Multivariate Chain Ladder Stochastic assumptions

- **Conditional Expectation**
  - There exists $K$-dimensional development factor independent on year of occurrence that holds
    \[
    E(Y_{i,j+1}|Y_i(j)) = \tau_{i,j} \cdot f_j
    \]

- **Conditional Variance and inter-row dependance**
  - There exists matrix $\Sigma_j$ so that
    \[
    \text{Cov}(Y_{i_1,j+1}, Y_{i_2,j+1}|Y_{i_1}(j), Y_{i_2}(j)) = \tau_{i,j}^{-1/2} \Sigma_j \tau_{i,j}^{-1/2}
    \]
    if $i = i_1 = i_2$ and also
    \[
    \text{Cov}(Y_{i_1,j+1}, Y_{i_2,j+1}|Y_{i_1}(j), Y_{i_2}(j)) = 0
    \]
    otherwise

**Corollary**

- $E(F_{i,j}|Y_i(j)) = f_j$
- $\text{Cov}(F_{i_1,j+1}, F_{i_2,j+1}|Y_{i_1}(j), Y_{i_2}(j)) = \tau_{i,j}^{-1/2} \Sigma_j \tau_{i,j}^{-1/2}$,
Univariate Case

- estimate of $f_j$ was found as $\hat{f}_j = \sum_{i=0}^{n-j-1} w_i F_{i,j}$
- unbiased if $\sum_{i=0}^{n-j-1} w_i = 1$
- OLS if $w_i = \frac{Y_{i,j}}{\sum_{i=0}^{n-j-1} Y_{i,j}}$

Multivariate Case

- estimator $f_j$ as $\hat{f}_j = \sum_{i=0}^{n-j-1} W_i \hat{F}_{i,j}$
- Conditional unbiased if $\sum_{i=0}^{n-j-1} W_i = I$
- MSE is minimised if

$$\hat{f}_j = \left( \sum_{i=0}^{n-j-1} \gamma_{i,j}^{1/2} \sum_{j}^{j-1} \gamma_{i,j}^{1/2} \right)^{n-j-1} \sum_{i=0}^{n-j-1} \gamma_{i,j}^{1/2} \sum_{j}^{j-1} \gamma_{i,j}^{1/2} F_{i,j}$$
How to estimate Covariance matrix?

- It is important for practical purposes
- It might be defined in a standard way like
  \[
  \hat{\Sigma}_j = \frac{1}{n-j-1} \sum_{i=0}^{n-j-1} \left( \gamma_{i,j}^{1/2} (\hat{F}_{i,j} - \hat{f}_j) \right) \cdot \left( \gamma_{i,j}^{1/2} (\hat{F}_{i,j} - \hat{f}_j) \right)'
  \]
- Drawback: \( \hat{\Sigma}_j \) is not well defined if \( j \geq n - k \)
- Benefit of the method might be limited
Recall of approach suggested by Kremer

**Multivariate model**

\[
Y_{i,j+1} = Y_{i,j} \cdot f_j + \varepsilon_{i,j} \quad i = 0, \ldots, n
\]
\[
E(\varepsilon_{i,j}|\cdot) = 0 \quad \text{var}(\varepsilon_{i,j}|\cdot) = \sigma_j^2 \cdot Y_{i,j}.
\]

⇒ ∀ \( j \) holds

\[
Y_{i,j+1}^k = Y_{i,j}^k \cdot f_j^k + \varepsilon_{i,j}^k \quad i = 0, \ldots, n \quad k = 1, \ldots, K
\]

- Original linear model is assumed for all of \( K \) analysed run-off triangles
- In addition \( \text{cov}(\varepsilon_{i,j}^k, \varepsilon_{i,j}^k|\cdot) = C_i^{k1,k2} \cdot \sqrt{Y_{i,j}^{k1}} \cdot \sqrt{Y_{i,j}^{k2}} \) and \( \text{var}(\varepsilon_{i,j}^k|\cdot) = \sigma_{j}^{k,2} \).
- If \( i_1 \neq i_2 \) or \( j_1 \neq j_2 \) then residuals are assumed to be uncorrelated \( \text{cov}(\varepsilon_{i_1,j_1}^{k1}, \varepsilon_{i_2,j_2}^{k2}|\cdot) = 0 \).
Remarks to model

- Not only the estimate of development factor but also the estimator of variance is stressed.
- Aitken’s estimator of $f_j$.
- Possibly time consuming computation of large-dimensional inverse matrix $\hat{\Psi}^{-1}$.
- More useful for multivariate extension of Munich Chain Ladder.
Computation of estimates - Algorithm

1. Calculation of estimators of $f_j^k$ for each triangle separately.

2. Variability estimator corresponding above mentioned estimates of development factor is derived through standard formulae

$$
\hat{\sigma}^2_{jk} = \frac{\sum_{i=1}^{n-j-1}(Y_{ij}^k - \hat{f}_j^k Y_{ij})^2}{\sum_{i=1}^{n-j-1} Y_{ij}}
$$

and also covariance estimator as

$$
\hat{C}_{i}^{k1,k2} = \frac{\sum_{i=1}^{n-j-1}(Y_{ij}^{k1} - \hat{f}_j^{k1} Y_{ij}^{k1})(Y_{ij}^{k2} - \hat{f}_j^{k2} Y_{ij}^{k2})}{\sum_{i=1}^{n-j-1} \sqrt{Y_{ij}^{k1}} \sqrt{Y_{ij}^{k2}}}
$$

3. Application of these estimates to estimate of development factors $f_j^{l+1}$ based on inverse matrix $\hat{\sigma}^2_{jl}$ and $\hat{C}_{i}^{k1,k2l}$.

4. Repeat it until the parameters estimates do not converge.
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Proposal for multivariate Extensions of Munich Chain Ladder

- Kremer’s approach found more suitable for MMCL
- linear model with slope parameters $\lambda^P$ and $\lambda^I$
- vector of parameters of $(\lambda^{P,1}, \ldots, \lambda^{P,K})$ is to be estimated simultaneously
- MCL model assumption holds for all triangles $k = 1, \ldots, K$

$$\text{Res} \left( \frac{Y_{i,s+1}^{P,k}}{Y_{i,s}^{P,k}} \bigg| Y_i(s)^{P,k} | B_i(s)^k \right) = \lambda^{P,k} \cdot \text{Res}((Q_{i,s}^k)^{-1} | Y_i(s)^P) + \epsilon_{i,j}^k | Y_i(s)^P$$

- Recall univariate case $\mathbb{E}(\epsilon_{i,j}|\cdot) = 0$ and $\text{var}(\epsilon_{i,j}|\cdot) = \sigma^2$
Proposal for multivariate Extensions of Munich Chain Ladder

- Multivariate stochastic assumptions

\[ \text{cov}(\varepsilon_{i_1,j_1}^{k_1}, \varepsilon_{i_2,j_2}^{k_2}|\cdot) = 0 \]

if \( i_1 \neq i_2 \) and

\[ \text{cov}(\varepsilon_{i,j_1}^{k_1}, \varepsilon_{i,j_2}^{k_2}|\cdot) = 0 \]

if \( j_1 \neq j_2 \)

- Moreover for equal occurrence and development periods

\[ \text{cov}(\varepsilon_{i,j}^{k_1}, \varepsilon_{i,j}^{k_2}|\cdot) = \sigma_{k_1,k_2} \]

- General model specification

\[
\begin{pmatrix}
Y_{P,1} \\
Y_{P,2} \\
\vdots \\
Y_{P,K}
\end{pmatrix} =
\begin{pmatrix}
X_{P,1} \\
X_{P,2} \\
\vdots \\
X_{P,K}
\end{pmatrix} \cdot
\begin{pmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_K
\end{pmatrix} +
\begin{pmatrix}
\varepsilon_{P,1} \\
\varepsilon_{P,2} \\
\vdots \\
\varepsilon_{P,K}
\end{pmatrix}
\]
Variables of the model in the multivariate case

Response variable and Explanatory variable

\[ Y_{P,k} = \begin{pmatrix} \text{Res} \left( \frac{Y_{P,k}^{0,1}}{Y_{0,0}^{0,k}} \right) \\ \text{Res} \left( \frac{Y_{P,k}^{0,2}}{Y_{0,0}^{0,k}} \right) \\ \vdots \\ \text{Res} \left( \frac{Y_{P,k}^{n-1,1}}{Y_{n-1,0}^{n-1,k}} \right) \end{pmatrix} \]

\[ X_{P,k} = \begin{pmatrix} \text{Res} \left( \frac{Y_{I,k}^{0,0}}{Y_{0,0}^{P,k}} \right) \\ \text{Res} \left( \frac{Y_{I,k}^{0,1}}{Y_{0,1}^{P,k}} \right) \\ \vdots \\ \text{Res} \left( \frac{Y_{I,k}^{n-1,0}}{Y_{n-1,0}^{P,k}} \right) \end{pmatrix} \]
Multivariate MCL - computation

1. get standard OLS estimator likewise in univariate case

\[ \hat{\lambda}_{P,k} = b_k = (X_{P,k}^T \cdot X_{P,k})^{-1} X_{P,k}^T Y_{P,k} \]

2. Matrix \( \Sigma \) is estimated using following formula

\[ \hat{\sigma}_{k1,k2} = \frac{\hat{\varepsilon}_{.,k1} \hat{\varepsilon}_{.,k2}}{n \cdot (n - 1)/2} \]

\( \hat{\varepsilon}_{.,k1} \) vector of OLS calculated residuals of \( k1 \)th model.

3. Estimator with non constant variance \( \beta = \lambda^P \) is derived as

\[ \beta = (Z^T \hat{\Psi}^{-1} Z)^{-1} Z^T \hat{\Psi}^{-1} Y^P \quad \hat{\Psi} = \hat{\Sigma} \otimes I \]
a \( Z \) is block-diagonal matrix \( X_{P,k} \), thus \( Z = \text{diag}(X_{P,1}^T, \ldots, X_{P,K}^T) \).

Notes

- initial estimator is replaced by that one calculated in the 3th step
- repeat process
- stop if parameters converges
Scope of presentation

1. Introduction
2. Robust Regression
3. Addition of MSE calculation to MCL model
4. Multivariate Extensions to Chain Ladder
5. Multivariate MCL
6. Other Approaches to model Paid and Incurred data
Suggestion how to model Paid and Incurred

- Different idea how to predict future payments and Incurred values
- May work for non-finished schemes as well (tail factor)
- Define $P_{i,j} = Y_{i,j}^P$ and $I_{i,j} = Y_{i,j}^I$
- Incremental value of Paid amount in calendar period $i + j$ is signed:
  \[ P_{i,j}^d = P_{i,j} - P_{i,j-1} \]

**Model specification**

- Paid amount in the next development period could be explained by the value of reserve in the present:
  \[ R_{i,j} = I_{i,j} - P_{i,j} \]
- Linear predictor:
  \[ P_{i,j+1}^d = \alpha_j R_{i,j} + \varepsilon_{i,j}^A, \quad \text{var}(\varepsilon_{i,j}^A) = \sigma_A^2 R_{i,j} \]
- Respect the key idea of Munich Chain Ladder that one might expect higher future amount of paid compensation in case of higher reserve
- Estimator $\hat{R}_{i,j}$ necessary for estimators $\hat{P}_{i,j}^d$ where $i + j > n$
Other Approaches to model Paid and Incurred data

Models for reserve development

- quite simple model for reserve development

\[ R_{i,j+1} = \beta_j R_{i,j} + \varepsilon_{i,j}^B, \quad \text{var}(\varepsilon_{i,j}^B) = \sigma_B^2 \]

- reminds standard chain ladder evolution.

- However it holds \( R_{i,j+1} = R_{i,j} - P_{i,j+1}^d + R_{i,j+1}^T - R_{i,j+1}^R \)

- \( R_{i,j+1}^T \) shows increase of reserve (if new claims are detected) a \( R_{i,j+1}^R \) represents decrease of reserve

- Run-off model

\[ R_{i,j}^T - R_{i,j}^R = \gamma_j R_{i,j} + \varepsilon_{i,j}^C, \quad \text{var}(\varepsilon_{i,j}^C) = \sigma_C^2 \]

- derived from \( R_{i,j+1} = R_{i,j} - P_{i,j+1}^d + R_{i,j+1}^T - R_{i,j+1}^R = R_{i,j} - \alpha_j R_{i,j} + R_{i,j+1}^T - R_{i,j+1}^R + \varepsilon_{i,j}^A = \beta_j R_{i,j} + \varepsilon_{i,j}^B \Rightarrow \beta_j + \alpha_j - 1 = \gamma_j \)

and \( \varepsilon_{i,j}^C = \varepsilon_{i,j}^A + \varepsilon_{i,j}^B \)
Numerical Illustration

- Various portfolios analysed using suggested models
  - simple reserve development ≡ alternative I
  - run off model ≡ alternative II
- Obtained results compared with SCL and MCL approach
- Confirmed better fit between Paid and Incurred data using ”alternative” models
- Results on 3 different portfolios presented as follows
  1. Example presented in original paper of MCL
  2. Not finalised but ”smooth” triangle
  3. Not finalised and volatile data with increase in accident year direction
Other Approaches to model Paid and Incurred data

Causality for Paid data

- We know that $P_{i,j+1} = P_{i,j} + P^d_{i,j}$
- So far we have presented 2 basic models for future Paid developments
  1. Standard Chain Ladder: $P_{i,j+1} = f_j \cdot P_{i,j} + \varepsilon_{i,j}$
  2. Alternative model 1: $P_{i,j+1} = P_{i,j} + \alpha_j R_{i,j} + \varepsilon_{i,j}$

- Why not try to combine these two approaches?

\[
\begin{pmatrix}
P_{i,j+1} \\
R_{i,j+1}
\end{pmatrix} = 
\begin{pmatrix}
f_j & \alpha_j \\
\delta_j & \beta_j
\end{pmatrix} \cdot 
\begin{pmatrix}
P_{i,j} \\
R_{i,j}
\end{pmatrix} + 
\begin{pmatrix}
\varepsilon^P_{i,j} \\
\varepsilon^R_{i,j}
\end{pmatrix}
\]

- Two simple models could be understood as special cases
  - $\alpha_j = 0 \Rightarrow$ obtain SCL model
  - $f_j = 1 \Rightarrow$ obtain ”alternative” model 1

- We can expect $\delta_j = 0$ if paid compensation is not informative for future reserving
Estimates of Parameters

- Usual Estimates of matrix parameters used for vector regression models

\[
\hat{\Pi}_j = \left( \sum_{i=1}^{n-j} Y_i X_i' \right) \left( \sum_{i=1}^{n-j} X_i X_i' \right)^{-1}
\]

- We use notation

\[
Y_i \equiv \begin{pmatrix} P_{i,j+1} \\ R_{i,j+1} \end{pmatrix}, \quad \Pi_j \equiv \begin{pmatrix} f_j & \alpha_j \\ \delta_j & \beta_j \end{pmatrix}, \quad X_i \equiv \begin{pmatrix} P_{i,j} \\ R_{i,j} \end{pmatrix}, \quad \Sigma \equiv \text{Var} \begin{pmatrix} \varepsilon_{i,j}^P \\ \varepsilon_{i,j}^R \end{pmatrix}
\]

- Estimate of Variance matrix

\[
\hat{\Sigma} = \frac{1}{n-j-1} \sum \hat{\varepsilon}_i \hat{\varepsilon}_i'
\]

where \( \hat{\varepsilon}_i = Y_i - \hat{\Pi}_j' X_i \)
References 1


Other Approaches to model Paid and Incurred data

References 2


Thank you very much for your attention