SOME INDICES FOR HEAVY-TAILED DISTRIBUTIONS

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Abstract

It is known that, for certain probability distributions, some classical risk-indices, depending on moments on the first and the second order, may not exist. In this paper some alternative indices are presented in order to give a classification of both the most common distributions and the only heavy-tailed (or subexponential) ones.

1 Introduction

We start by defining some basic tools in the extreme value theory and its applications. A positive function h on $(0, \infty)$ is *regularly varying* at ∞ of real index α (we write $h \in R_\alpha$) if

$$\lim_{x \to \infty} \frac{h(tx)}{h(x)} = t^\alpha, \quad t > 0.$$

Furthermore, a positive function h on $(0, \infty)$ is said to be *rapidly varying* at ∞ (we write $h \in R_{-\infty}$) if

$$\lim_{x \to \infty} \frac{h(tx)}{h(x)} = \begin{cases} 0, & if\ t > 1 \\ \infty, & if\ 0 < t < 1. \end{cases}$$

Typical examples of regularly varying functions of index α are

$$x^\alpha, x^\alpha \ln(1+x), \ (x \ln(1+x))^\alpha.$$

An example of rapidly varying function is, obviously, e^{-x}.

Let now X be a continuous random variable with finite expectation; then

$$e(x) = E(X - x | X > x), \quad x > 0$$

is called the *mean excess function* of X. Using the definition of $e(x)$ and partial integration, the following formulae are easily checked:

$$e(x) = \frac{\int_x^{+\infty}(t-x)dF(t)}{\bar{F}(x)} = \frac{\int_x^{+\infty}\bar{F}(t)dt}{\bar{F}(x)}$$

where $F(x)$ is the distribution function $p(X \leq x)$ and $\bar{F}(x) = 1 - F(x)$ is the tail of the distribution itself.

2 Asymptotic properties and subexponential distributions

For the sequel and, in particular, for the applications in the next paragraph, we take into account the following probability distributions:

i) Pareto with $\bar{F}(x) = (1+x)^{-\alpha}$ (i.e. $\bar{F} \in R_{-\alpha}$) and $e(x) = (1+x)/(\alpha-1)$, $\alpha > 1$;

ii) Standard Exponential with $\bar{F}(x) = e^{-x}$ (i.e. $\bar{F} \in R_{-\infty}$) and $e(x) = 1$;

iii) Weibull with $\bar{F}(x) = e^{-x^{\tau}}$, $0 < \tau < 1$;

iv) Standard Normal with density function $f(x) = (\sqrt{2\pi})^{-1}e^{-x^2/2}$;

v) Standard Lognormal with density function $f(x) = (\sqrt{2\pi}x)^{-1}e^{-(\ln x)^2/2}$.

Unfortunately there is no explicit form either for the mean excess functions in the last three cases or for the tails in the last two ones. However these asymptotic results hold:

a) Weibull distribution: $e(x) \sim x^{1-\tau}/\tau$ where $\sim$ means equivalent for x tending to infinity.
This result is an application of the following theorem (see [1]):

$$e(x) \sim \frac{x^{1-\alpha}}{\alpha L(x)} if -\ln \bar{F}(x) \sim x^{\alpha}L(x), L(x) \in R_0.$$

Moreover, the tail of the Weibull distribution is a rapidly varying function since $\bar{F} \in R_{-\infty}$ iff $\lim_{x\to\infty} e(x)/x = 0$ (see [2]).

b) Standard Normal distribution: $\bar{F}(x) \sim f(x)/x$ (Mill's ratio).
It immediately follows by l'Hospital's rule applied to $\bar{F}(x)/(x^{-1}f(x))$.
Furthermore, since $F(x)$ is a Von Mises function (i.e. $\bar{F}(x) = c\exp(-\int_z^x a(t)^{-1}dt)$ with $c > 0$, $a(x) = \bar{F}(x)/f(x)$ and $\lim_{x\to\infty} a'(x) = 0$) and, in particular, $\bar{F} \in R_{-\infty}$ (because of the Von Mises condition $\lim_{x\to\infty} a(x)/x = 0$), a possible choice for the mean excess function (see [2]) is $e(x) \sim a(x)$.
Hence, using the Mill's ratio, $e(x) \sim \bar{F}(x)/f(x) \sim x^{-1}$.

c) Standard Lognormal distribution:
$\bar{F}_{LN}(x) = \bar{F}_N(\ln x) \sim (\sqrt{2\pi}\ln x)^{-1}\exp(-(\ln x)^2/2)$, where $\bar{F}_N$ and $\bar{F}_{LN}$ are respectively the tails of the Normal and Lognormal distributions.
Since $F_{LN}(x)$ is a Von Mises function and since the Von Mises condition holds, we get, as in the Normal case, $\bar{F}_{LN} \in R_{-\infty}$.
A possible choice for the mean excess function is therefore, as in the previous case,

$$e(x) \sim \frac{\bar{F}_{LN}(x)}{f_{LN}(x)} \sim \frac{x}{\ln x}.$$

Let us now define an important class of probability distributions. We say that a distribution function $F(x)$ of a random variable X is *subexponential* (we write $F \in S$) if

$$\lim_{x\to\infty} \frac{p(X_1 + ... + X_n > x)}{p(X > x)} = n, \quad \forall n \geq 2$$

where X, X_1, ..., X_n are independent and identically distributed random variables.
It is easy to prove that the previous definition is equivalent to

$$\lim_{x\to\infty} \frac{p(X_1 + ... + X_n > x)}{p(M_n > x)} = 1, \quad \forall n \geq 2$$

where $M_n = \max(X_1, ..., X_n)$.
In other words, if X_i is the i-th claim of an insurance portfolio, the tails of the distribution of the sum and of the maximum of the first n claims are asymptotically of the same order. This obviously shows the influence of the largest claim on the total claim amount.
Moreover, the following properties hold:

i) $\bar{F} \in R_{-\alpha}$ implies $F \in S$, $0 \leq \alpha < \infty$;

ii) $F \in S$ implies $ord_\infty \bar{F}(x) < ord_\infty e^{-\varepsilon x} \quad \forall\varepsilon > 0$.

So, by the second property, it is required, as a necessary condition for the subexponentiality, that the tail tends to infinity slower than the tail of any exponential distribution. However we can prove that the Weibull and the Lognormal are both subexponential distributions. Infact sufficient condition for $F \in S$ is that $\exp(xf(x)/\bar{F}(x))f(x)$ is an integrable function on $[0, \infty)$ (see [2]).

In conclusion we consider a further tool, very useful to discriminate the heavy-tailed (or, better, subexponential) distributions. Let us recall the parameter τ of the Weibull distribution. It is easy to check that, if $\bar{F}(x) = e^{-x^\tau}$,

$$x\frac{d}{dx}[\ln(-\ln \bar{F}(x))] = \tau$$

(in the sequel, for a generic tail $\bar{F}(x)$, we define $WF(x) = x\frac{d}{dx}[\ln(-\ln \bar{F}(x))]$ where WF means *Weibull factor*).
Note that the Weibull factor, written as the ratio

$$WF(x) = \frac{\frac{d}{dx}[\ln(-\ln \bar{F}(x))]}{\frac{d}{dx}[\ln(-\ln \bar{F}_{\exp}(x))]}$$

clearly suggests a useful geometric interpretation for the subexponential distributions for which it can be proved that asymptotically $WF(x) < 1$.
In the Pareto case the Weibull factor is

$$WF(x) = \frac{x}{(1+x)\ln(1+x)} \sim \frac{1}{\ln x}$$

whereas in the Normal and Lognormal cases we have, asymptotically,
$WF_N(x) \sim 2$ and $WF_{LN}(x) \sim 2/\ln x$.

The basic results of this paragraph are summarized in the following table

distributions	tails	$e(x)$	$WF(x)$	Reg. var.	Type
Pareto ($\alpha > 1$)	$\frac{1}{(1+x)^\alpha}$	$\frac{1+x}{\alpha-1}$	$\sim \frac{1}{\ln x}$	$R_{-\alpha}$	Yes
Lognormal	$\sim \frac{e^{-(\ln x)^2/2}}{\sqrt{2\pi}\ln x}$	$\sim \frac{x}{\ln x}$	$\sim \frac{2}{\ln x}$	$R_{-\infty}$	Yes
Weibull ($0 < \tau < 1$)	e^{-x^τ}	$\sim \frac{x^{1-\tau}}{\tau}$	τ	$R_{-\infty}$	Yes
Exponential	e^{-x}	1	1	$R_{-\infty}$	No
Normal	$\sim \frac{e^{-x^2/2}}{\sqrt{2\pi}x}$	$\sim \frac{1}{x}$	~ 2	$R_{-\infty}$	No

3 Some indices for heavy-tailed distributions

It is konwn (see [2]) that, for distributions with regularly varying tail, the following property, for positive α, holds: if $\bar{F} \in R_{-\alpha}$ then

i) $E(X^\beta) < +\infty, \quad \beta < \alpha;$

ii) $E(X^\beta) = +\infty, \quad \beta \geq \alpha.$

It follows that if $\bar{F} \in R_{-\infty}$ (as for the Exponential, Weibull, Normal and Lognormal distributions) the moments of any order are finite, whereas, if $\bar{F} \in R_{-\alpha}$ (as for the Pareto, Loggamma and T-Student distributions) and taken $\beta = 2$, we obtain

$$E(X^2) = +\infty, \quad \text{for } \alpha \leq 2.$$

Therefore it is clear that typical risk-indices, as the variance or the standard deviation, may not exist. So, we are going to consider some alternative risk-indices that are more sensitive to the tail of the distribution:

i) the q-quantile

It is simply defined as the value x_q such that $p(X > x_q) = 1 - q$ and it is nothing but the Value at Risk of the claim X relative to a one-year time horizon and corresponding to a ruin probability equal to $1 - q$ (for example 1%).

ii) the tail-expectation

It is defined as $E(X|X > x_q)$ and it represents a more general index that points out the behaviour of the tail. Obviously

$$E(X|X > x_q) = e(x_q) + x_q.$$

iii) the normalized tail-expectation (NTE)

It is defined as $E(X|X > x_q)/x_q$. This index could be preferred because of the independence of monetary values. It is clear that the higher is this ratio, the more the claim X is heavy-tailed. Moreover the equality

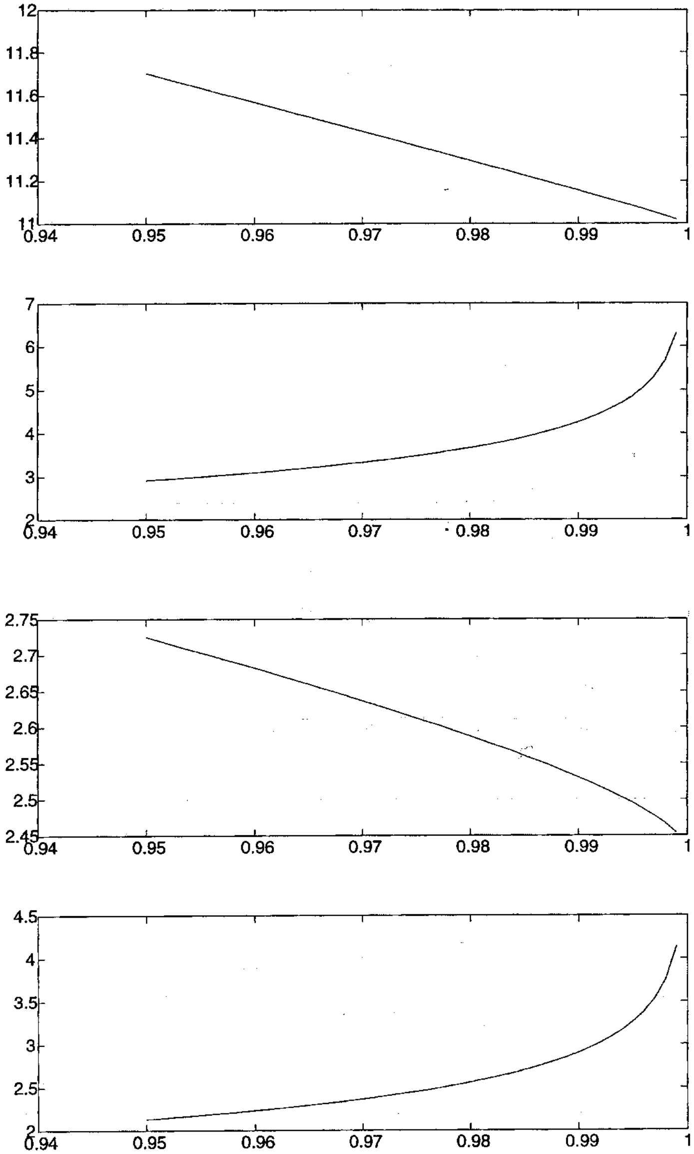
$$\frac{E(X|X > x_q)}{x_q} = 1 + \frac{e(x_q)}{x_q}$$

underlines the importance of the ratio $e(x)/x$.

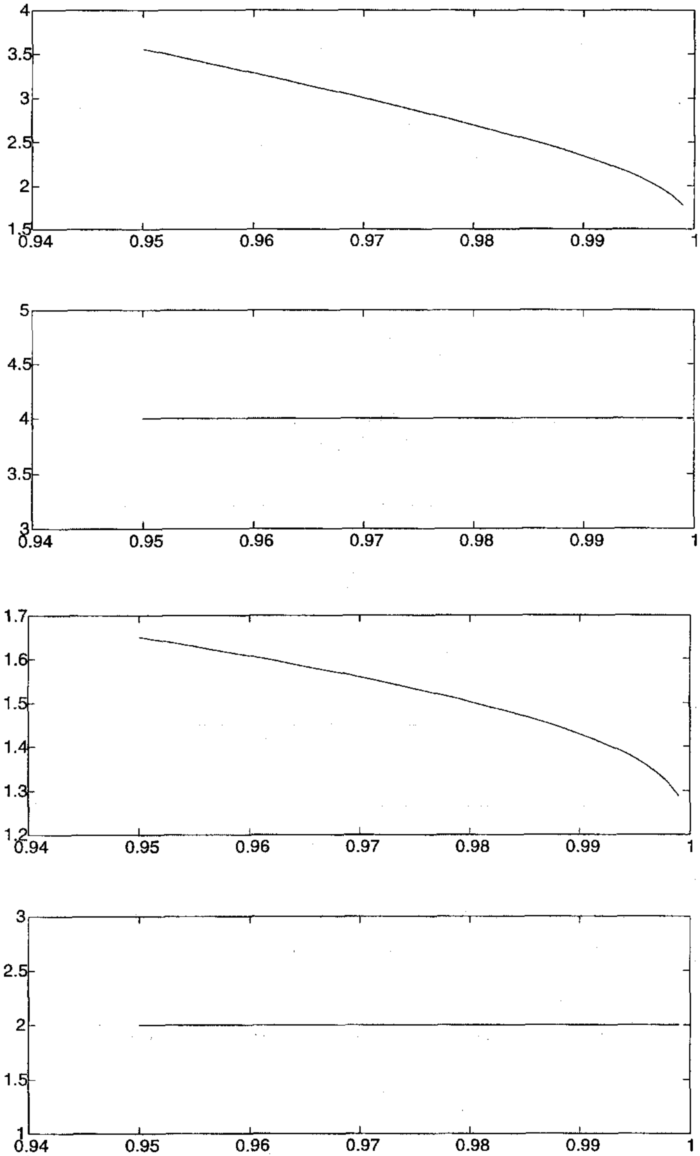
iv) the subexponential index (WF^{-1})

It is defined as $WF(x_q)^{-1}$. Since, as seen before, for a subexponential distribution we have $WF(x)^{-1} > 1$ (at least asymptotically), we can clearly use this index in order to check the subexponentiality of the heavy-tailed distributions.

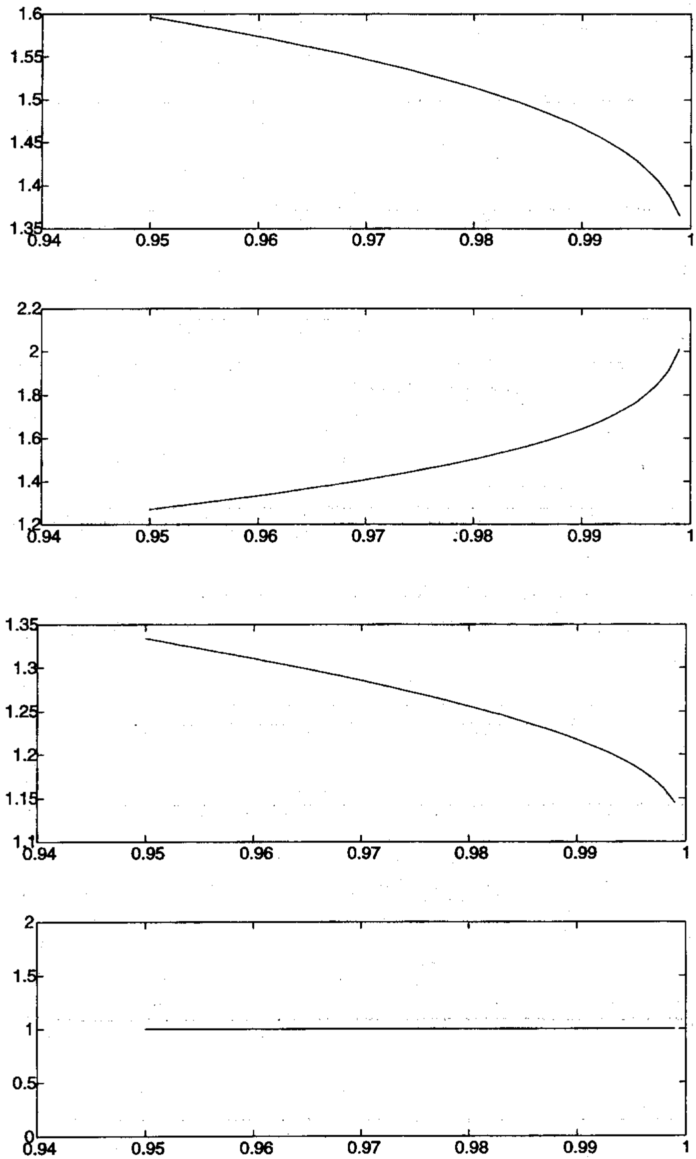
In the following figures the plots of NTE and WF^{-1}, depending on q, are reported.



Plots of NTE and WF^{-1} indices for Pareto distributions with $\alpha = 1.1$ (top figures) and $\alpha = 1.7$ (bottom figures).



Plots of NTE and WF^{-1} indices for Weibull distributions with $\tau = 0.25$ (top figures) and $\tau = 0.5$ (bottom figures).



Plots of NTE and WF^{-1} indices for Standard Lognormal (top figures) and for Standard Exponential (bottom figures) distributions.

Pareto (1.1)	NTE	WF^{-1}
$q = 0.99$	11.14	4.25
$q = 0.995$	11.08	4.85
$q = 0.999$	11.02	6.29
$q = 0.9999999$	11	14.65
$q \to 1$	11	∞

Pareto (1.7)	NTE	WF^{-1}
$q = 0.99$	2.53	2.90
$q = 0.995$	2.48	3.26
$q = 0.999$	2.45	4.12
$q = 0.9999999$	2.43	9.48
$q \to 1$	2.43	∞

Weibull (0.25)	NTE	WF^{-1}
$q = 0.99$	1.86	4
$q = 0.995$	1.75	4
$q = 0.999$	1.58	4
$q = 0.9999999$	1.24	4
$q \to 1$	1	4

Weibull (0.5)	NTE	WF^{-1}
$q = 0.99$	1.43	2
$q = 0.995$	1.37	2
$q = 0.999$	1.29	2
$q = 0.9999999$	1.12	2
$q \to 1$	1	2

Lognormal	NTE	WF^{-1}
$q = 0.99$	1.42	1.71
$q = 0.995$	1.38	1.94
$q = 0.999$	1.32	2
$q = 0.9999999$	1.19	3.48
$q \to 1$	1	∞

Normal	NTE	WF^{-1}
$q = 0.99$	1.18	0.69
$q = 0.995$	1.14	0.67
$q = 0.999$	1.10	0.64
$q = 0.9999999$	1	0.57
$q \to 1$	1	0.50

Some relevant values of NTE and WF^{-1} indices for Pareto, Weibull, Lognormal and Normal distributions. Note how WF^{-1} is more discriminant in the asymptotic cases.

References

[1] Beirlant J., Teugels J.L., *Modelling large claims in non-life insurance*, Insurance: Math. Econom., **11**, 17-29 (1992).

[2] Embrechts P., Kluppelberg C., Mikosch T., *Modelling Extremal Events for Insurance and Finance,* Springer, Berlin (1997).