

THE MEAN TIME FOR A NET PROFIT AND THE PROBABILITY OF RUIN PRIOR TO THAT PROFIT IN THE CLASSICAL RISK MODEL

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ABSTRACT

For the classical risk model, we consider the expected amount of time until the insurer makes a net profit exceeding a prescribed amount a . Since ruin may occur prior to that event, we also discuss the probability of net profit for the insurer of at least a before ruin occurs. Finally, we derive an expression for the conditional expectation of the time to reach a certain profit given that this happens prior to ruin.

KEYWORDS

Stopping time, independent stationary increments, random walk, ruin probability.

1. INTRODUCTION

We consider the classical risk model with risk reserve process

$$R(t) = u + ct - \sum_{i=1}^{N(t)} X_i.$$

Here u is the initial reserve, c is the premium income per unit time, $N(t)$ is the number of claims within $(0, t]$. These claims arrive to the insurer according to a Poisson process with intensity λ . The successive sizes of the claims, $\{X_i\}_{i \geq 1}$, are be i.i.d. random variables and we further assume that the claim amounts are independent of the claim times. For the claim size distribution, we assume throughout the paper that it is a non-lattice distribution on $(0, \infty)$ with a finite second moment. We consider throughout in the sequel the case where $c - \lambda E(X_1) > 0$, otherwise ruin is certain to occur. Denote

$$\tau_u = \begin{cases} \inf\{t > 0 : R(t) < 0\} \\ \infty \text{ if no such } t \text{ exists.} \end{cases}$$

Traditionally, the probability of ruin, $P(\tau_u < \infty)$, has been the primary quantity of interest in this model. Since Lundberg's formula which gives an upper bound for this probability, a large number of approximations and asymptotic expressions have been proposed both for finite and infinite time ruin probabilities using a variety of mathematical tools, such as Fourier analysis, Laplace transforms, renewal theory and martingale methods; see for instance, Cramér (1955), von Bahr (1974), Grandell (1977), (1991), Asmussen (1984), while a comprehensive list of references is given in Embrechts *et al* (1999).

A second key quantity for this model, which is related to the probability of ruin and which has attracted considerable interest recently is the *severity of ruin* (see e.g Gerber *et al* 1987, Picard 1994). With our notation, this can be expressed as $R(\tau_u)$ and it represents the magnitude of the first entry for the process $R(t)$ to the negative half line $(-\infty, 0)$. Here we consider the problem at the other end; namely, the first entry of the *net profit process* $B(t) = ct - \sum_{i=1}^{N(t)} X_i$ to an interval $[a, \infty)$ for some $a > 0$. Since this entails essentially considering a stopped sum of random variables, we use methods from sequential analysis to study the expected amount of time until the process $B(t)$ enters the interval $[a, \infty)$. This expected time for the process, the (first) *hitting time*, is closely related to the first moment of the step distribution in a random walk associated with $B(t)$. The distribution for the maximum of this random walk has been studied in the past using Spitzer's identity, see e.g. Embrechts *et al* (1999), Ch. 1. Here, we introduce the hitting time of the process $\{B(t), t > 0\}$ on the set $[a, \infty)$ for some fixed positive a as

$$\zeta_a = \inf\{t \geq 0 : B(t) \geq a\}$$

and in section 2 we consider the expectation $E(\zeta_a)$ through an associated random walk. Note however that it is clearly possible in this model that ruin occurs before the net profit of the insurer exceeds a prescribed amount a , in which case consideration of the time until the net profit enters $[a, \infty)$ becomes of little practical significance. Considering the process $B(t)$, this means that the process hits the lower bound $-u$ before the upper bound a (note that, in terms of $B(t)$, $\tau_u = \inf\{t > 0 : B(t) < -u\}$). In section 3, we discuss the probability of ruin before ζ_a and we give an expression for $E(\zeta_a | \zeta_a < \tau_u)$, the conditional expectation of the gain time without an occurrence of ruin before.

2. THE MEAN TIME FOR THE FIRST HITTING

Note that $\{B(t), t \geq 0\}$ is a process whose paths consist of a group of lines having the same upward slope and are right-continuous. From this path property, it is clear that

$$\zeta_a = \inf\{t \geq 0 : B(t) = a\}.$$

Let $T_i, i \geq 1$ be the interarrival times between claims, i.e. T_i is the time between the $(i-1)$ th claim X_{i-1} and the i th claim X_i (T_1 is the time before the first claim). Then $T_i, i \geq 1$ are i.i.d. random variables and independent of $X_i, i \geq 1$. We will discuss the question through consideration of the process $\{B(t), t \geq 0\}$ and the associated random walk $S_n = \sum_{i=1}^n Y_i$, where $Y_i = cT_i - X_i, i \geq 1$. This random walk (or $\sum_{i=1}^n (-Y_i), n \geq 1$) is frequently encountered in the context of the classical risk model, because the probability of ruin is related to the distribution of its minimum (maximum if we refer to $\sum_{i=1}^n (-Y_i), n \geq 1$) which can be analyzed e.g. by Spitzer's identity and Wiener-Hopf factorization. Define now

$$\rho_a = \inf\{n : S_n > a\}$$

and

$$\zeta_a^+ = T_1 + \cdots + T_{\rho_a}.$$

The variable ζ_a^+ represents the first time that, after a claim is cleared, the process $B(t)$ is still over a . It is obvious that $\zeta_a < \zeta_a^+$. Here we mention that for any fixed $a \in (0, \infty)$, ζ_a^+ is a proper random variable (i.e. $P(\zeta_a^+ < \infty) = 1$) and has a finite expectation; see for instance, Theorem 3.1 on page 78 of Gut (1988). We will attack the expectation $E(\zeta_a)$ by expressing it as the difference $E(\zeta_a^+) - E(\zeta_a^+ - \zeta_a)$.

To achieve this, we introduce two classes of nondecreasing σ -fields (this is often called filtration in the stochastic process literature; see, e.g. Karatzas and Shreve, 1991) $\{\mathcal{F}_t, t \geq 0\}$ and $\{\mathcal{G}_n, n \geq 1\}$ where $\mathcal{F}_t = \sigma\{B(s), 0 \leq s \leq t\}$ and $\mathcal{G}_n = \sigma\{(T_1, X_1), \dots, (T_n, X_n)\}$. $\{B(t), \mathcal{F}_t, t \geq 0\}$ and $\{(X_n, T_n), \mathcal{G}_n, n \geq 1\}$ are therefore adapted processes, i.e. for any Borel sets $A \subset R$ and $B \subset R^2$, $\{B(t) \in A\} \in \mathcal{F}_t$ and $\{(X_n, T_n) \in B\} \in \mathcal{G}_n$. A well-known important property of a Poisson process is that it has stationary and independent increments. Since the claims $\{X_i\}_{i \geq 1}$ are i.i.d. and independent of claim times, $\{B(t), t \geq 0\}$ is then still a stationary independent increment process, i.e. $B(t) - B(s)$ is independent of \mathcal{F}_s for any $t > s \geq 0$ and the distribution of $B(t) - B(s)$ is the same as that of $B(t-s)$.

Consider the expectation $E(\zeta_a^+)$ first. It is apparent that ρ_a is a stopping time with respect to the σ -class $\{\mathcal{G}_n, n \geq 1\}$ and T_{n+1} is independent of \mathcal{G}_n for all n . Thus, by Wald's lemma, we have

$$E(\zeta_a^+) = E(T_1 + \cdots + T_{\rho_a}) = \lambda^{-1}E(\rho_a).$$

For $E(\rho_a)$, a result in Woodroffe (1982) asserts that

$$E(\rho_a) = \frac{1}{E(Y_1)} \left(a + \frac{E(S_\kappa^2)}{2E(S_\kappa)} \right) + o(1),$$

where

$$\kappa = \inf\{n \geq 1 : S_n > 0\}$$

is the first (strict) ascending ladder epoch (for a renewal process on $(0, \infty)$, κ is simply equal to one) and $o(1)$ represents a quantity which tends to zero as $a \rightarrow \infty$. This is the second order expression for the well known result (see for example Theorem 4.1 on page 83 of Gut 1988) $\lim_{a \rightarrow \infty} E(\rho_a)/a = 1/E(Y_1)$. Therefore

$$E(\zeta_a^+) = \frac{1}{c - \lambda E(X_1)} \left(a + \frac{E(S_\kappa^2)}{2E(S_\kappa)} \right) + o(1). \quad (1)$$

The distribution of S_κ , which is the first ladder height, is generally hard to determine exactly. But the following two formulae (see Corollary 2.7 and Theorem 2.6 of Woodroffe, 1982) can help us to produce an approximation for $E(S_\kappa^2)/2E(S_\kappa)$.

$$(i) \quad \frac{E(S_\kappa^2)}{2E(S_\kappa)} = \frac{[E(Y_1)]^2 + E(Y_1 - E(Y_1))^2}{2E(Y_1)} - \sum_{r=1}^{\infty} \frac{1}{r} E(S_r^-)$$

where $-$ denotes the negative part, and

$$(ii) \quad \frac{E(S_\kappa^2)}{2E(S_\kappa)} = \frac{[E(Y_1)]^2 + E(Y_1 - E(Y_1))^2}{4E(Y_1)} + \frac{1}{\pi} \int_0^{\infty} s^{-2} [\mathcal{R}\xi(s) + \log(E(Y_1)s)] ds$$

where

$$\xi(s) = \log \left[\frac{1}{1 - \phi(s)} \right] \quad (\text{principal branch of the logarithm}),$$

$\phi(s)$ here is the characteristic function of Y and $\mathcal{R}\xi(s)$ denotes the real part of $\xi(s)$. For formula (ii), an extra requirement for the distribution of X_1 that it is a strongly nonlattice distribution is needed. A distribution is said to be strongly nonlattice if its characteristic function $\phi(s)$ satisfies

$$\limsup_{|s| \rightarrow \infty} |\phi(s)| < 1.$$

Next, we consider $E(\zeta_a^+ - \zeta_a)$. Since $\{B(t), t > 0\}$ has stationary independent increments, $\{B(t+s) - B(s), \mathcal{F}_{t+s}, t > 0\}$ is also a process with stationary and independent increments for any fixed $s \in (0, \infty)$. This property also holds if we replace the constant s above with a $\{\mathcal{F}_t, t \geq 0\}$ -stopping time η . Recall that a random variable η is a $\{\mathcal{F}_t, t \geq 0\}$ -stopping time if for any $b \in [0, \infty)$, the event $\{\eta \leq b\}$ belongs to \mathcal{F}_b . For a process with a discrete time parameter, say $\{B(n), \mathcal{F}_n, n \geq 0\}$, the following result appears in Chow and Teicher (1978).

Lemma 2.1 Let η be a $\{\mathcal{F}_t, t \geq 0\}$ -stopping time such that $P(\eta < \infty) = 1$. Then $\{B(t+\eta) - B(\eta), \mathcal{F}_{\eta+t}, t > 0\}$ is a process with stationary and independent increments, and $B(t+\eta) - B(\eta)$ has the same distribution as $B(t)$. Here $\mathcal{F}_{\eta+t}$ represents the σ -algebra of events prior to the stopping time $\eta + t$.

Proof: It is sufficient to prove that, for any $t > s \geq 0$ and any Borel sets A, B , the events $\{B(t+\eta) - B(s+\eta) \in A\}$ and $\{B(s+\eta) - B(\eta) \in B\}$ are independent, and $P(B(t+\eta) - B(s+\eta) \in A) = P(B(t-s) \in A)$. Note that

$$\begin{aligned} & P(B(t+\eta) - B(s+\eta) \in A, B(s+\eta) - B(\eta) \in B) \\ &= E\{P[B(t+\eta) - B(s+\eta) \in A, B(s+\eta) - B(\eta) \in B \mid \eta]\} \\ &= \int P[B(t+\eta) - B(s+\eta) \in A, B(s+\eta) - B(\eta) \in B \mid \eta = b] dF_\eta(b) \\ &= \int P[B(t+b) - B(s+b) \in A, B(s+b) - B(b) \in B \mid \eta = b] dF_\eta(b) \end{aligned}$$

where $F_\eta(b)$ is the distribution of η . But $\{\eta = b\} \in \mathcal{F}_b$ since η is a $\{\mathcal{F}_t, t \geq 0\}$ -stopping time. So both $\{B(t+b) - B(s+b) \in A\}$ and $\{B(s+b) - B(b) \in B\}$ are independent of $\{\eta = b\}$. Now, using the stationarity and independence for the increments of the process $B(t)$, we have

$$\begin{aligned} & P[B(t+b) - B(s+b) \in A, B(s+b) - B(b) \in B \mid \eta = b] \\ &= P(B(t+b) - B(s+b) \in A, B(s+b) - B(b) \in B) \\ &= P(B(t+b) - B(s+b) \in A)P(B(s+b) - B(b) \in B) \\ &= P(B(t-s) \in A)P(B(s) \in B). \end{aligned}$$

Thus

$$P(B(t+\eta) - B(s+\eta) \in A, B(s+\eta) - B(\eta) \in B)$$

$$\begin{aligned}
&= \int P(B(t-s) \in A)P(B(s) \in B)dF_\eta(c) \\
&= P(B(t-s) \in A)P(B(s) \in B).
\end{aligned}$$

A similar procedure through conditioning will give

$$P(B(t+\eta) - B(s+\eta) \in A) = P(B(t-s) \in A)$$

and

$$P(B(s+\eta) - B(\eta) \in B) = P(B(s) \in B).$$

The result then follows. \square

We next note that ζ_a is a $\{\mathcal{F}_t, t \geq 0\}$ -stopping time (the proof of this can be found in the Appendix). Thus, we obtain from the lemma that the process $\{B(\zeta_a + t) - B(\zeta_a), t \geq 0\}$ is probabilistically the same with $\{B(t), t \geq 0\}$. We can thus consider a random walk which corresponds to $\{B(\zeta_a + t) - B(\zeta_a), t \geq 0\}$ in the same way that S_n is associated with $\{B(t), t \geq 0\}$. In other words, the random walk induced by the process $\{B(\zeta_a + t) - B(\zeta_a), t \geq 0\}$ is a (probabilistically equivalent) version of S_n , when we consider the stopping time ζ_a (rather than time zero) as the starting time of the process. From the definitions of ζ_a^+ and ζ_a , it is easy to see that $\zeta_a^+ - \zeta_a$ is the time for the random walk to move up one ladder. Thus we have

$$E(\zeta_a^+ - \zeta_a) = E(T_1 + \dots + T_\kappa) = \frac{E(\kappa)}{\lambda}. \quad (2)$$

From (1) and (2) we therefore obtain the following

Proposition 2.2 The mean time for the first hitting on $[a, \infty)$ satisfies

$$E(\zeta_a) = \frac{1}{c - \lambda E(X_1)} \left(a + \frac{E(S_\kappa^2)}{2E(S_\kappa)} \right) - \frac{E(\kappa)}{\lambda} + o(1). \quad (3)$$

By applying a Wiener-Hopf factorization (see, for instance, page 22 of Prabhu, 1980), the value for $E(\kappa)$ above can be found to be $[(1 - P(\bar{\kappa} < \infty))]^{-1}$, where $\bar{\kappa}$ is the first descending ladder epoch, i.e.

$$\bar{\kappa} = \inf\{n \geq 1 : S_n \leq 0\}.$$

Suppose now that the claim size X_1 has a continuous distribution. Then $\bar{\kappa}$ is probabilistically the same as $\inf\{n \geq 1 : S_n < 0\}$. Thus the probability $P(\bar{\kappa} < \infty)$ is in fact the

probability of ruin when the initial reserve is 0. This probability is well-known to be equal to $\lambda E(X_1)/c$, see e.g. Feller (1971). This gives that

$$E(\kappa) = \frac{c}{c - \lambda E(X_1)}.$$

With this result, $E(S_\kappa)$ can be then found by Wald's lemma to be

$$E(S_\kappa) = E(Y_1)E(\kappa) = \frac{c}{\lambda}.$$

3. THE PROBABILITY OF RUIN BEFORE THE NET PROFIT

Section 2 considers the hitting time ζ_a without any restriction. But the process $\{B(t), t \geq 0\}$ may hit the lower bound $-u$ (so that ruin occurs) before hitting the upper bound a . Therefore the probability $P(\tau_u < \zeta_a)$, i.e. the probability of ruin before net profit at least a , and the conditional expectation $E(\zeta_a | \tau_u > \zeta_a)$ deserve consideration. Write $\psi(u) = P(\tau_u < \infty)$ for the probability of eventual ruin. Since an exact expression for $\psi(u)$ is not available in general, this is typically calculated using a variety of existing numerical methods. Here we present an expression for $P(\tau_u < \zeta_a)$ in terms of the function $\psi(u)$.

It is apparent that $P(\tau_u < \zeta_a) = P(\tau_u < \infty) - P(\zeta_a < \tau_u < \infty)$ (note that the event $\{\tau_u = \zeta_a\}$ cannot happen). By Lemma 2.1, the probability that ruin occurs after time ζ_a can simply be viewed as the probability of eventual ruin when the initial reserve is $a + u$. Thus we obtain

Proposition 3.1 The probability of ruin before the net gain exceeds a is $P(\tau_u < \zeta_a) = \psi(u) - \psi(u + a)$.

Now we consider the conditional expectation $E(\zeta_a | \zeta_a < \tau_u)$. Write $E(\zeta_a)$ as

$$E(\zeta_a) = E(\zeta_a I_{\{\zeta_a > \tau_u\}}) + E(\zeta_a I_{\{\zeta_a < \tau_u\}}),$$

where $I_{\{\cdot\}}$ is the indicator function for the event $\{\cdot\}$. Further,

$$E(\zeta_a I_{\{\zeta_a > \tau_u\}}) = E(\tau_u I_{\{\zeta_a > \tau_u\}}) + E[(\zeta_a - \tau_u) I_{\{\zeta_a > \tau_u\}}] > E[(\zeta_a - \tau_u) I_{\{\zeta_a > \tau_u\}}].$$

So

$$E(\zeta_a I_{\{\zeta_a < \tau_u\}}) < E(\zeta_a) - E[(\zeta_a - \tau_u) I_{\{\zeta_a > \tau_u\}}]. \quad (4)$$

Suppose that at time b the lower bound is crossed and this happens before the random walk hits the upper bound a , i.e. the event $\{\tau_u = b, \zeta_a > b\}$ occurs. Then, to reach the stopping time ζ_a from time $\tau_u = b$, the process $\{B(t+b) - B(b), t > 0\}$ must have a stride more than $a+u$ (for there is a “overshoot” down below level $-u$ at time $\tau_u = b$). Thus we have

$$\zeta_a - b > \zeta'_{a+u} := \inf\{t > 0 : B(t+b) - B(b) = a+u\}.$$

However, the process $\{B(t+b) - B(b), t > 0\}$ is independent of \mathcal{F}_b , and hence, it is independent of the event $\{\tau_u = b, \zeta_a > b\}$ since $\{\tau_u = b, \zeta_a > b\} \in \mathcal{F}_b$ (like ζ_a , τ_u is also a $\{\mathcal{F}_t, t \geq 0\}$ -stopping time; see Appendix for a proof of this). Therefore, by this independence and by the probabilistic equivalence of the processes $\{B(t+b) - B(b), t > 0\}$ and $\{B(t), t > 0\}$, we get

$$\begin{aligned} E[(\zeta_a - \tau_u)I_{\{\tau_u=b, \zeta_a>b\}}] &= E[(\zeta_a - b)I_{\{\tau_u=b, \zeta_a>b\}}] \\ &> E(\zeta'_{a+u}I_{\{\tau_u=b, \zeta_a>b\}}) \\ &= E(\zeta'_{a+u})P(\tau_u = b, \zeta_a > b) \\ &= E(\zeta_{a+u})P(\tau_u = b, \zeta_a > b), \end{aligned}$$

whence

$$E[(\zeta_a - \tau_u)I_{\{\zeta_a > \tau_u\}}] > E(\zeta_{a+u})P(\zeta_a > \tau_u). \quad (5)$$

From (4), (5), we have

Proposition 3.2 The conditional expectation of the first hitting time before ruin $E(\zeta_a | \zeta_a < \tau_u)$ satisfies

$$\begin{aligned} E(\zeta_a | \zeta_a < \tau_u) &= \frac{E(\zeta_a I_{\{\zeta_a < \tau_u\}})}{P(\zeta_a < \tau_u)} \\ &< \frac{E(\zeta_a) - E(\zeta_{a+u})P(\zeta_a > \tau_u)}{P(\zeta_a < \tau_u)}. \end{aligned} \quad (6)$$

Combining Propositions 2.2 and 3.1, it is straightforward to obtain the following expression for this upper bound as

$$\frac{1}{c - \lambda E(X_1)} \left(a + \frac{E(S_\kappa^2)}{2E(S_\kappa)} \right) - \frac{E(\kappa)}{\lambda} - \frac{u}{c - \lambda E(X_1)} \frac{\psi(u) - \psi(u+a)}{1 - \psi(u) + \psi(u+a)} + o(1).$$

APPENDIX

In general, the hitting time of a stochastic process in continuous time to a certain set in its state space is not necessarily a stopping time. But for the process $\{B(t), t \geq 0\}$, we can prove that both ζ_a and τ_u are stopping times with respect to the filtration $\{\mathcal{F}_t, t \geq 0\}$.

$\{B(t), t \geq 0\}$ is a process with right-continuous paths. Each path of this process is composed of a group of half-open lines with the same upward slope c . Thus

$$\{\zeta_a \leq b\} = \left\{ \sup_{0 \leq t < b} B(t) > a \right\} \cup \left\{ \sup_{0 \leq t < b} B(t) \leq a, B(b) = a \right\}.$$

Let D be a countable set which is dense in $[0, b)$. Then, by the right continuity of the process $\{B(t), t > 0\}$, we have

$$\begin{aligned} & \left\{ \sup_{0 \leq t < b} B(t) > a \right\} \cup \left\{ \sup_{0 \leq t < b} B(t) \leq a, B(b) = a \right\} \\ &= \left\{ \sup_{t \in D} B(t) > a \right\} \cup \left\{ \sup_{t \in D} B(t) \leq a, B(b) = a \right\}. \end{aligned}$$

The event on the right side of equality is in \mathcal{F}_b since D is countable. So ζ_a is a $\{\mathcal{F}_t, t \geq 0\}$ -stopping time.

In order to show that τ_u is also a $\{\mathcal{F}_t, t \geq 0\}$ -stopping time, we simply note that

$$\{\tau_u \leq b\} = \left\{ \inf_{0 \leq t \leq b} B(t) < -u \right\}.$$

The event on the right hand side is in \mathcal{F}_b with the similar arguments as above.

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