An Extension of the Bühlmann Credibility Model

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Summary

In the classical credibility model the conditional independence of the claims amounts is given up and replaced by a more general assumption. A formula for the credibility estimator is given and a practical parameter estimation procedure proposed.

Keywords

Credibility, Bühlmann model, Regression model.

1. Introduction

Credibility theory is a very old branch of risk theory and/or Nonlife Insurance Mathematics. First results were given by Mowbray and Witney (1914/1918). The today used approach goes back to the papers by Bühlmann (1967) and Bühlmann & Straub (1970). The models of both papers were generalized into many directions e.g. to the so-called regression credibility models, what gives as important special case the linear trend credibility model (see Hachemeister (1975)). Though the whole theory is already quite far developed, there can still be given new results. In the present note the classical Bühlmann model of 1967 is extended a little bit. The assumption of conditionally independent claims amounts is given up and replaced by a certain autoregressive model for the claims amounts. Interpreting the model as special case of the regression credibility model, easily a formula for the credibility estimator can be given. Also a reasonable parameter estimation procedure can be written down by considering conclusions of econometrics.

2. Preliminaries

Suppose all is based on a probability space (Ω, Δ, P) and consider a collective of risks during periods with indices $i = 1, 2, 3, \dots, n+1$. The claims behaviour of a risk over all periods let be described by a parameter ϑ.

Suppose that the value of this parameter is unknown and interprete it as a realization of a random variable

$$\theta : (\Omega, \Delta, P) \to (\Theta, \tau)$$

with parameter space Θ and σ-algebra τ on Θ.

The observed claims amounts of a risk under consideration let be described by the non-negative random variables:

$$X_i\,, \quad \text{with} \quad i = 1, 2, \dots, n+1\,,$$

all defined on (Ω, Δ, P). It is assumed that all X_i are square integrable, meaning that they lay in the L_2. The main problem of credibility theory consists in forecasting optimally the net premium:

$$\mu_{n+1}(\theta) = E(X_{n+1}|\theta)$$

from the (past) claims amounts $X_i, i = 1, \dots, n$. For that one restricts on linear-affine forecasts i.e. on forecasts with structure

$$f_n = a_o + \sum_{i=1}^{n} a_i \cdot X_i \quad . \tag{1}$$

An optimal forecast $\hat{\mu}_{n+1}$ would be one of structure (1) with:

$$||\mu_{n+1}(\theta) - \hat{\mu}_{n+1}|| \leq ||\mu_{n+1}(\theta) - f_n||$$

for all f_n of type (1), where $||\cdot||$ is just the L_2-norm:

$$||f|| = E(f^2)^{1/2} \quad .$$

The $\hat{\mu}_{n+1}$ is the so-called credibility estimator of the net premium $\mu_{n+1}(\theta)$. Mathematically seen it is nothing else but the (L_2)-projection of $\mu_{n+1}(\theta)$ on the linear (sub)space of all forecasts f_n of structure (1).

In general the credibility estimator can be determined by solving certain normal equations. For many models handy formulas for the credibility estimator were derived. For more details the reader is referred e.g. to the textbook Kremer (1985), chapter 2. In the following a further model will be investigated.

3. The model

In the setting of the previous section assume more special:

(A.1) $E(X_i|\theta) =: \mu(\theta)$ is independent of the period no. i,

(A.2) $e_i = X_i - \mu(\theta)$ follows an autoregressive model of type:

$$\boxed{e_i = \rho \cdot e_{i-1} + \varepsilon_i}$$

where the parameter ρ satisfies:

$$|\rho| < 1$$

and the error term ε_i:

$$\begin{aligned} E(\varepsilon_i|\theta) &= 0 \\ Var(\varepsilon_i|\theta) &= \sigma^2(\theta) \\ Cov(\varepsilon_i, \varepsilon_j|\theta) &= 0 \end{aligned}$$

for all i, j and with a function

$$\sigma^2 : \Theta \to [0, \infty)$$

such that $\sigma^2(\theta)$ is integrable.

The ε_j let be independent of e_i for $j > i$ given θ.

This model is a generalization of the classical Bühlmann model of 1967. That classical model is in principal the special case $\rho = 0$ with more stronger:

$$\varepsilon_1, \dots, \varepsilon_n \qquad \text{i.i.d. given } \theta\,.$$

Because of technical reasons take instead of $i \in \mathbb{N}_0$ for the sequel $i \in \mathbb{Z}$.

It is no problem to give a formula for the credibility estimator.

4. Credibility estimator

Theorem:
One has under $(A,1)$, $(A.2)$ the credibility formula:

$$\hat{\mu}_{n+1} = Z_n \cdot \overline{X}_n^{\rho} + (1 - Z_n) \cdot \mu \tag{2}$$

with

$$\overline{X}_n^{\rho} = \left(\frac{X_1 + X_n + (1-\rho) \cdot \sum_{i=2}^{n-1} X_i}{n \cdot (1-\rho) + 2 \cdot \rho} \right)$$

$$\mu = E(\mu(\theta)) = E(X_i)$$

and the credibility factor:

$$Z_n = \left(\frac{a \cdot [n \cdot (1-\rho) + 2\rho]}{\phi/(1-\rho) + a \cdot [n \cdot (1-\rho) + 2\rho]} \right)$$

where

$$\begin{aligned} a &= Var(\mu(\theta)) \\ \phi &= E(\sigma^2(\theta))\,. \end{aligned}$$

Proof:
Just apply e.g. the Theorem 2.24 in Kremer (1985) for the special choices:

$$q = 1,\ Y_{(i)} := (1)$$

$$b(\theta) = \mu(\theta)$$

$$\Lambda := a, \quad \beta := \mu\,.$$

One has to calculate $\Phi = E(Cov(X|\theta))$ (with $X = (X_1, \ldots, X_n)^T$) .
One has:

$$\begin{aligned} E(e_i^2|\theta) &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \rho^j \rho^k \cdot E(\varepsilon_{i-j}\varepsilon_{i-k}|\theta) = \\ &= \sum_{j=0}^{\infty} \rho^{2j} \cdot \sigma^2(\theta) = \left(\tfrac{\sigma^2(\theta)}{1-\rho^2}\right) \end{aligned}$$

$$\begin{aligned} E(e_{i+k}\, e_i|\theta) &= \rho \cdot E(e_{i+k-1}\, e_i|\theta) + E(e_{i+k}\, e_i|\theta) \\ &= \rho \cdot E(e_{i+k-1}\, e_i|\theta) = \ldots \quad \text{induction} \quad \ldots = \\ &= \rho^k \cdot E(e_i^2|\theta) \end{aligned}$$

for $k \geq 1$. This gives:

$$\Phi = (\frac{\phi}{1-\rho^2}) \cdot \begin{pmatrix} 1 & \rho & \rho^2 & \ldots & \rho^{n-1} \\ \rho & 1 & \ddots & & \vdots \\ \rho^2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \rho \\ \rho^{n-1} & \ldots & \ldots & \rho & 1 \end{pmatrix}$$

and

$$\Phi^{-1} = \phi^{-1} \cdot \begin{pmatrix} 1 & -\rho & 0 & \ldots & \ldots & 0 \\ -\rho & 1+\rho^2 & -\rho & 0 & \ldots & 0 \\ 0 & -\rho & 1+\rho^2 & & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots & & \vdots \\ 0 & & \ddots & 1+\rho^2 & & -\rho \\ 0 & \ldots & 0 & -\rho & & 1 \end{pmatrix} \qquad (3)$$

One gets with $Y = (1, \ldots, 1)^T$:

$$\begin{aligned} (Y^T\Phi^{-1}Y)^{-1}Y^T\Phi^{-1}X &= \overline{X}_n^{\rho} \\ \Lambda Y^T\Phi^{-1}Y(I + \Lambda \cdot Y^T\Phi^{-1}Y)^{-1} &= Z_n\,. \end{aligned}$$

□

Obviously for $\rho = 0$ the classical credibility formula of Bühlmann (1967) results as special case.

5. Parameter estimation

In the above credibility formula (2) the **structural parameters** μ, a, ϕ, ρ are unknown. One likes to estimate them in advance out of the whole collective. Insertion of the estimators into (2) would give a so-called **empirical credibility estimator** $\hat{\hat{\mu}}_{n+1}$.

For giving an adequate procedure for estimating the structural parameters supppose that one has the claims amounts

$$X_{ki}\,, \quad i = 1, \ldots, n, \quad k = 1, \ldots, K$$

of periods $i = 1, \ldots, n$ and risks no. $k = 1, \ldots, K$. The corresponding risk parameters shall be denoted by:

$$\theta_k\,, \quad k = 1, \ldots, K$$

For the sequel assume:

(B.1) the risk parameter θ_k, $k = 1, \ldots, k$ are i.i.d.,

(B.2) the random vectors $(X_{k1}, \ldots, X_{k,n})$ $k = 1, \ldots, K$ are independent,

(B.3) (A.1) and (A.2) hold for each risk with the same structural parameters μ, a, ϕ, ρ.

It is obvious to estimate μ like in the classical Bühlmann model according:

$$\hat{\mu}_0 = \frac{1}{K \cdot n} \cdot \sum_{k=1}^{K} \sum_{i=1}^{n} X_{ki}\,.$$

Since:

$$E(Var(X_{ki}|\theta_k)) = E(E(e_i^2|\theta)) = \left(\frac{\phi}{1-\rho^2}\right)$$

the classical parameter estimation procedure (see Kremer (1985), Theorem 2.22) implies as reasonable estimator of a:

$$\hat{a}_0 = \frac{1}{K-1} \cdot \sum_{k=1}^{K} \left(\frac{1}{n} \cdot \sum_{i=1}^{n} X_{ki} - \mu\right)^2 - \frac{1}{n} \cdot \left(\frac{\hat{\phi}_0}{1-\hat{\rho}_0^2}\right)$$

when having estimators $\hat{\phi}_0, \hat{\rho}_0$ of ϕ, ρ.

It is not possible to give explicit estimators $\hat{\phi}_0, \hat{\rho}_0$. The author proposes an interactive procedure as follows:

Start with $\hat{\rho}_0 = \hat{\rho} = 0$ and take as estimator of ϕ the classical one:

$$\hat{\phi}_0 = \frac{1}{K(n-1)} \cdot \sum_{k=1}^{K} \sum_{i=1}^{n} \left(X_{ki} - \frac{1}{n} \cdot \sum_{l=1}^{n} X_{kl} \right)^2 .$$

Compute the empirical credibility estimators $\hat{\hat{\mu}}^k_{n+1}$ for each risk (with no. k) according to (2) (with $\hat{\mu}_0, \hat{a}_0, \hat{\phi}_0, \hat{\rho}_0 = 0$ inserted for μ, a, ϕ, ρ) and then the estimated residuals:

$$\hat{e}_{ki} = X_{ki} - \hat{\hat{\mu}}^k_{n+1} .$$

A reasonable first estimate for ρ would be:

$$\hat{\rho}_1 = \frac{1}{K} \cdot \sum_{k=1}^{K} \left(\frac{\sum_{i=2}^{n} \hat{e}_{ki} \cdot \hat{e}_{k,i-1}}{\sum_{i=2}^{n} \hat{e}^2_{k,i-1}} \right) \tag{4}$$

and a new, more reasonable estimate of ϕ:

$$\hat{\phi}_1 = \frac{1}{K(n-2)} \cdot \sum_{k=1}^{K} \sum_{i=2}^{n} (\hat{e}_{ki} - \hat{\rho}_1 \cdot \hat{e}^2_{k,i-1})^2 . \tag{5}$$

For the parameter μ one is willing to take as new estimate:

$$\hat{\mu}_1 = \frac{1}{K} \cdot \sum_{k=1}^{K} \hat{\hat{\mu}}^k_{n+1} \tag{6}$$

and for the parameter a:

$$\hat{a}_1 = \frac{1}{K-1} \cdot \sum_{k=1}^{K} \left(\hat{\hat{\mu}}^k_{n+1} - \hat{\mu}_1 \right)^2 - \frac{1}{n} \cdot \frac{\hat{\phi}_1}{1 - \hat{\rho}_1^2} . \tag{7}$$

With the new estimates compute the new empirical credibility estimators according to (2), new estimated residuals and according to (4), (5), (6), (7) (with indices $1 \to 2$) new estimates $\hat{\rho}_2, \hat{\phi}_2, \hat{a}_2, \hat{\mu}_2$ for ρ, ϕ, a, μ.

Repeat the whole procedure until there is sufficiently good convergence in the estimates $\hat{\rho}_m, \hat{\phi}_m, \hat{a}_m, \hat{\mu}_m$ and use the last values as final estimates of ρ, ϕ, a, μ.

6. Final remarks

The idea for the estimation procedure of part 4 the author got from econometrics (s. Toutenbourg & Röder (1978), p. 31). The new model often will be more realistic than the classical Bühlmann model. The assumption (A.2) can be used (modified) also for the general regression model. Then one only has to insert the Φ^{-1} of (3) into the credibility formula (see Kremer (1985), pp. 58–59). The adequate parameter estimation procedure would be similar to that of the present part 4.

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Resumée

— Une généralisation du modèle crédibilité de Bühlmann —
Dans le modèle classique de la crédibilité la condition de l'indépendence conditionelle est abandonnée et remplacée par une supposition plus générale. Une formule pour l'estimateur de la crédibilité est donnée et un processus de l'estimation des paramètres proposé.

Zusammenfassung

— Eine Erweiterung des Bühlmann Glaubwürdigkeitsmodelles —
Im klassischen Glaubwürdigkeitsmodell wird die bedingte Unabhängigkeit der Schadenbeiträge aufgegeben und durch eine allgemeinere Annahme ersetzt. Eine Formel für den Glaubwürdigkeitsschätzer wird gegeben und eine praktische Parameterschätzprozedur vorgeschlagen.

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