Measuring the effects of reinsurance by the adjustment coefficient in the Sparre Anderson model

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Abstract

We study the insurer's adjustment coefficient as a function of retention levels for combinations of quota-share with excess of loss reinsurance in the Sparre Anderson (1957) model. We show that the insurer's adjustment coefficient is a unimodal function of the retention levels when the quota-share reinsurance premium is calculated on original terms and when the excess of loss premium is calculated according to the expected value principle.

Keywords

Sparre Anderson model; adjustment coefficient; reinsurance; excess of loss; quota-share; combinations of excess of loss and quota-share reinsurance.

1 Introduction

Several studies about the effect of reinsurance on the ultimate probability of ruin (for example Waters (1979), Gerber (1979), Waters (1983), Centeno (1986), and Hesselager (1990)) have concentrated their attention on the effect of reinsurance on the adjustment coefficient (or Lundberg exponent).

Waters (1983) proved that the adjustment coefficient is a unimodal function of the retention level in case of proportional reinsurance, without any restrictive assumptions on the distribution of the annual claims. He also investigated non-proportional reinsurance. He proved that the adjustment coefficient is a unimodal function of the retention limit for excess of loss reinsurance, assuming that the reinsurance premium calculation principle is the expected value principle, and that the annual claims have a compound Poisson distribution. These two assumptions are also assumed in the other cited papers, namely in Centeno (1986), where combinations of quota-share with excess of loss were considered.

In this paper we study the adjustment coefficient as a function of the retention levels for combinations of quota-share with excess of loss reinsurance, generalizing some of the results of Centeno (1986), when the number of claims are described by an ordinary renewal process. We prove that the adjustment coefficient is a unimodal function of the retention levels when the quota share premium is calculated on original terms with a commission and the excess of loss premium calculation principle used is the expected value principle.

2 Assumptions and Preliminaries

We assume that the number of claims $\{N(t)\}_{t\geq 0}$ follows an ordinary renewal process, i.e. the number of claims, $N(t)$, that occur in the time interval $(0,t]$ can be written as

$$N(t) = \sup\{n : S_n \leq t\} \tag{1}$$

with $S_0 = 0$, $S_n = T_1 + T_2 + \ldots + T_n$ for $n \geq 1$, where $\{T_i\}_{i=1}^{\infty}$ are independent and identically distributed non-negative random variables. S_n denotes the epoch of the nth claim and T_i is the time between the $i-1$th and the ith claim. Let the expected value of T_i be $1/\gamma$.

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed random variables, independent of $\{T_i\}_{i=1}^{\infty}$, where X_i denotes the amount of the ith claim. We assume that: F, the distribution function of X_i, is such that $F(0) = 0$, so that negative claims are not possible; $0 < F(x) < 1$ for $0 < x < +\infty$ (these assumptions could be relaxed); $dF(x)/dx$ exists and is continuous; the moment generating function of $F(x)$, $M_X(r)$, exists for $r \in (-\infty, \tau)$ for some $0 < \tau \leq +\infty$ and

$$\lim_{r \longrightarrow \tau} M_X(r) = \lim_{r \longrightarrow \tau} E\left[\mathrm{e}^{rX}\right] = +\infty. \tag{2}$$

Let μ be the expected value of X_i.

The risk process $\{Y(t)\}_{t\geq 0}$, is defined by

$$Y(t) = (1-e)P\,t - \sum_{i=1}^{N(t)} X_i, \qquad \left(\sum_{i=1}^{0} X_i \stackrel{\text{def}}{=} 0\right), \tag{3}$$

where P is the insurer's premium income per unit of time and eP is the amount used to cover the insurer's expenses.

$$Y_i = -\left[Y(S_i) - Y(S_{i-1})\right] = X_i - (1-e)P\,T_i. \tag{4}$$

Obviously $\{Y_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed random variables. The expected loss between two claims is

$$E[Y_i] = E[X_i] - (1-e)P\,E[T_i] = \mu - \frac{(1-e)P}{\gamma}, \tag{5}$$

and it is natural to define the relative safety loading by

$$\rho = \frac{\frac{(1-e)P}{\gamma} - \mu}{\mu} = \frac{(1-e)P}{\gamma\mu} - 1. \tag{6}$$

We assume that $\rho > 0$. Let $Z_0 = 0$ and $Z_n = \sum_{i=1}^{n} Y_i$, be the loss immediately after the nth claim. As ruin can only occur at claim epochs the ultimate ruin probability $\psi(u)$, for initial surplus $u \geq 0$, is

$$\psi(u) = \Pr\left\{u + Y(t) < 0, \text{ for some } t > 0\right\} = \Pr\left\{\max_{n\geq 1} Z_n > u\right\}. \tag{7}$$

Let

$$g(r) = M_{Y_i}(r) = E\left[\mathrm{e}^{rY_i}\right] = E\left[\mathrm{e}^{r(X_i-(1-e)PT_i)}\right] = E\left[\mathrm{e}^{rX}\right]E\left[\mathrm{e}^{-r(1-e)PT}\right] \tag{8}$$

where X and T have the same distribution than X_i and T_i respectively. The adjustment coefficient R is, in the renewal case, the unique positive solution of

$$g(r) = 1, \tag{9}$$

when such a root exists, or zero otherwise, and the Lundberg's inequality

$$\psi(u) \leq \mathrm{e}^{-Ru}, \tag{10}$$

is still valid (note that this inequality has to be modified for the stationary renewal case). Lundberg's inequality in the ordinary renewal case was first proved by Sparre Anderson (1957) and can be find using a martingale approach in Grandell (1992).

Let us consider that the insurer has a choice of reinsuring this risk either by a pure quota-share treaty, or by a pure excess of loss treaty or by any combination of quota share with excess of loss formed, as follows. First, the insurer chooses a quota-share retention level, which we denote a, paying a premium calculated on a proportional basis with a commission payment (see Carter (1979), p.87). More precisely, for retention level a, the insurer pays the reinsurer premium $(1-a)P$ less a commission $c(1-a)P$. Secondly, the insurer chooses an excess of loss retention limit M, so that, when a claim of size X occurs, the insurer retains $X_{a,M} = \min(aX, M)$ and transfers to the reinsurer through this arrangement $X - X_{a,M} = \max(0, aX - M)$. In return for this arrangement the insurer pays a reinsurance premium, which we assume to be calculated according to the expected value principle, with loading coefficient $\alpha > 0$. We assume that

$$e > c, \tag{11}$$

and

$$(1-e)P - (1+\alpha)\gamma\mu < 0, \tag{12}$$

which implies that the insurer can not reinsurer the hole risk with a certain profit.

After this reinsurer arrangement the insurer's net (of expenses and reinsurance) risk at time t is

$$Y_{a,M}(t) = ((1-e)P - P_{a,M})\,t - \sum_{i=1}^{N(t)} \min(aX_i, M), \tag{13}$$

where $P_{a,M}$, the reinsurance premium, is

$$P_{a,M} = (1-c)(1-a)P + (1+\alpha)\gamma \int_{M/a}^{\infty} (ax - M)dF(x). \tag{14}$$

For given (a, M), the adjustment coefficient, $R_{a,M}$, is now the unique positive root of

$$g_{a,M}(r) = 1, \tag{15}$$

when such a root exists, or zero otherwise, with

$$g_{a,M}(r) = E\left[\mathrm{e}^{rX_{a,M}}\right] E\left[\mathrm{e}^{-((1-e)P - P_{a,M})rT}\right]. \tag{16}$$

Let $E[W(a, M)]$ denote the insurer's expected net profit per period of time, after reinsurance and expenses, i.e.

$$E[W(a, M)] = (1-e)P - P_{a,M} - \gamma E[X_{a,M}].$$

Let L be the set of points for which the insurer's net expected profit is positive, i.e.

$$L = \{(a, M) : 0 \leq a \leq 1,\ M \geq 0 \text{ and } E[W(a, M)] > 0\}\,, \tag{17}$$

and let

$$\chi_{a,M}(r) = \ln E\left[\mathrm{e}^{rX_{a,M}}\right],$$

$$\kappa(r) = \ln E\left[\mathrm{e}^{-rT}\right],$$

and

$$H_{a,M}(r) = \ln(g_{a,M}(r)) = \chi_{a,M}(r) + \kappa(((1-e)P - P_{a,M})r). \tag{18}$$

Lemma 1 *(i) The adjustment coefficient is positive if and only if $(a, M) \in L$.*

(ii) For any $(a, M) \in L$, $H'_{a,M}(r)$ is positive at $r = R_{a,M}$.

Proof.

(i) Let us consider that (a, M) is fixed. Differentiating $\chi_{a,M}(r)$ and $\kappa(r)$ we get

$$\chi'_{a,M}(r) = \frac{E[X_{a,M}\mathrm{e}^{rX_{a,M}}]}{E[\mathrm{e}^{rX_{a,M}}]}, \tag{19}$$

$$\chi''_{a,M}(r) = \frac{E[X^2_{a,M}\mathrm{e}^{rX_{a,M}}]}{E[\mathrm{e}^{rX_{a,M}}]} - \left(\frac{E[X_{a,M}\mathrm{e}^{rX_{a,M}}]}{E[\mathrm{e}^{rX_{a,M}}]}\right)^2, \tag{20}$$

$$\kappa'(r) = -\frac{E\left[T\mathrm{e}^{-rT}\right]}{E\left[\mathrm{e}^{-rT}\right]} \tag{21}$$

and

$$\kappa''(r) = \frac{E\left[T^2\mathrm{e}^{-rT}\right]}{E\left[\mathrm{e}^{-rT}\right]} - \left(\frac{E\left[T\mathrm{e}^{-rT}\right]}{E\left[\mathrm{e}^{-rT}\right]}\right)^2. \tag{22}$$

The functions $\chi_{a,M}(r)$ and $\kappa(r)$ are both convex functions of r. This follows because $\chi''_{a,M}(r)$ and $\kappa''(r)$ are the variances of two Esscher transforms (the Esscher transforms of the distributions of the random variables $X_{a,M}$ and T, respectively). Hence for fixed (a, M), $H_{a,M}(r)$ is a convex function of r and

$$\begin{aligned} H'_{a,M}(r) &= \chi'_{a,M}(r) + ((1-e)P - P_{a,M})\kappa'(((1-e)P - P_{a,M})r) = \\ &= \frac{E[X_{a,M}\mathrm{e}^{rX_{a,M}}]}{E[\mathrm{e}^{rX_{a,M}}]} - \frac{((1-e)P - P_{a,M})E\left[T\mathrm{e}^{-r\left((1-e)P-P_{a,M}\right)T}\right]}{E\left[\mathrm{e}^{-r\left((1-e)P-P_{a,M}\right)T}\right]}. \end{aligned} \tag{23}$$

Let

$$\xi = \begin{cases} +\infty & \text{if} \quad M < +\infty \\ \tau & \text{if} \quad M = +\infty \end{cases}$$

where $M = +\infty$ means no excess of loss reinsurance. Noticing that

$$H_{a,M}(0) = 0$$

and

$$\lim_{r \longrightarrow \xi} H_{a,M}(r) = +\infty \tag{24}$$

and given the convexity of $H_{a,M}(r)$ we can say that the adjustment coefficient is positive if and only if

$$H'_{a,M}(0) < 0. \tag{25}$$

But by calculating (23) at $r = 0$, we get that (25) is equivalent to

$$E[X_{a,M}] - ((1-e)P - P_{a,M})/\gamma < 0 \tag{26}$$

and the first part of the Lemma is proved.

(ii) The second part of the lemma follows from the proof of (i).

■

Note that the above prove does not depend on the reinsurance premium calculation principles used for both the arrangements.

Let

$$a_0 = (e-c)P/[(1-c)P - \gamma E[X]] \tag{27}$$

and

$$A = \{a : 0 < a \leq 1 \text{ and there exists an } M \text{ such that } E[W(a,M)] = 0\}. \tag{28}$$

The proof of the following Lemma can be seen in Centeno (1985).

Lemma 2 *Under our assumptions on the reinsurance premium* $P_{a,M}$*,*

(i) $A = (a_0, 1]$

(ii) For each $a \in A$ *there is a unique* M *such that* $E[W(a,M)] = 0$*, i.e. there is a function* Φ *mapping* A *into* $(0, \infty)$ *such that* $M = \Phi(a)$ *is equivalent to* $E[W(a,M)] = 0$.

(iii) $\Phi(a)$ *is convex.*

(iv) $Lim_{a \to a_0} \Phi(a) = +\infty$

Hence the first part of Lemma 1 is equivalent to saying that the adjustment coefficient is positive if and only if $a > a_0$ and $M > \Phi(a)$.

3 The adjustment coefficient as function of the retention levels

Result 1 *(i) For a fixed value of $a \in (a_0, 1]$, $R_{a,M}$ is a unimodal function of M, attaining its maximum value at the only point satisfying*

$$M = \frac{1}{R_{a,M}} \left(\ln(1+\alpha) + \ln\left(\gamma \frac{E\left[T \mathrm{e}^{-R_{a,M}\left((1-e)P - P_{a,M}\right)T}\right]}{E^2\left[\mathrm{e}^{-R_{a,M}\left((1-e)P - P_{a,M}\right)T}\right]} \right) \right) \tag{29}$$

where $R_{a,M}$ is the only positive solution of (15). Let $\widehat{R}_a$ be the maximum of $R_{a,M}$.

(ii) $\widehat{R}_a$ is a unimodal function of a, for $a \in (a_0, 1]$, attaining its maximum value at $a = 1$, if and only if

$$\lim_{a \to 1^-} \frac{d}{da} \widehat{R}_a \geq 0.$$

Proof.

(i) The adjustment coefficient, $R_{a,M}$, is, for fixed $a \in (a_0, 1]$ and $M > \Phi(a)$, the only positive root to (15) or in an equivalent way the only positive root to

$$H_{a,M}(r) = 0. \tag{30}$$

Let us consider now $R_{a,M}$ as a function of $(a, M) \in L$. From the implicit function theorem it follows that

$$\frac{\partial}{\partial M} R_{a,M} = - \left. \frac{\frac{\partial}{\partial M} H_{a,M}(r)}{\frac{\partial}{\partial r} H_{a,M}(r)} \right|_{r = R_{a,M}}. \tag{31}$$

By Lemma 1 we know that the denominator of the right hand side of (31) is positive, so $\partial R_{a,M}/\partial M = 0$ if and only if $\partial H_M(r)/\partial M|_{r=R_{a,M}} = 0$.

Considering that

$$\frac{\partial}{\partial M} E\left[\mathrm{e}^{r X_{a,M}}\right] = r \mathrm{e}^{rM}(1 - F(M/a)) \tag{32}$$

and that

$$\frac{\partial}{\partial M} E\left[\mathrm{e}^{-((1-e)P - P_{a,M})rT}\right] = -r(1+\alpha)\gamma(1 - F(M/a)) E\left[T \mathrm{e}^{-((1-e)P - P_{a,M})rT}\right], \tag{33}$$

it follows that

$$\frac{\partial H_{a,M}(r)}{\partial M} = \frac{r(1-F(M))}{E\left[\mathrm{e}^{rX_{a,M}}\right]E\left[\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]}\times \\ \times\left\{\mathrm{e}^{rM}E\left[\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]-(1+\alpha)\gamma E\left[T\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]E\left[\mathrm{e}^{rX_{a,M}}\right]\right\}. \tag{34}$$

Hence $\partial H_{a,M}(r)/\partial M|_{r=R_{a,M}}=0$ is equivalent, for finite M, to

$$\mathrm{e}^{R_{a,M}M}E\left[\mathrm{e}^{-((1-e)P-P_{a,M})R_{a,M}T}\right]=(1+\alpha)\gamma E\left[T\mathrm{e}^{-((1-e)P-P_{a,M})R_{a,M}T}\right]E\left[\mathrm{e}^{R_{a,M}X_{a,M}}\right], \tag{35}$$

which is, given the definition of $R_{a,M}$ equivalent to

$$\mathrm{e}^{R_{a,M}M}E^2\left[\mathrm{e}^{-((1-e)P-P_{a,M})R_{a,M}T}\right]=(1+\alpha)\gamma E\left[T\mathrm{e}^{-((1-e)P-P_{a,M})R_{a,M}T}\right], \tag{36}$$

from where (29) follows.

Calculating the second derivative with respect to M of $R_{a,M}$, at the points where the first derivative is null, we get

$$\left.\frac{\partial^2}{\partial M^2}R_{a,M}\right|_{\frac{\partial}{\partial M}R_{a,M}=0} = -\left.\frac{\frac{\partial^2}{\partial M^2}H_{a,M}(r)}{\frac{\partial}{\partial r}H_{a,M}(r)}\right|_{r=R_{a,M},\frac{\partial}{\partial M}R_{a,M}=0}. \tag{37}$$

But the denominator in (37) is positive by Lemma 1, and (note that the denominator of (34) is 1 for $r=R_{a,M}$)

$$\left.\frac{\partial^2}{\partial M^2}H_{a,M}(r)\right|_{r=R_{a,M},\frac{\partial}{\partial M}R_{a,M}=0} = R^2_{a,M}(1-F(M/a))A_M(R_{a,M})\Big|_{\frac{\partial}{\partial M}R_{a,M}=0} \tag{38}$$

where

$$A_M(R_{a,M}) = \mathrm{e}^{R_{a,M}M}\left\{E\left[\mathrm{e}^{-((1-e)P-P_{a,M})R_{a,M}T}\right]\right. \\ \left.-2(1+\alpha)\gamma(1-F(M/a))E\left[T\mathrm{e}^{-((1-e)P-P_{a,M})R_{a,M}T}\right]\right\} \\ +(1+\alpha)^2\gamma^2(1-F(M/a))E\left[T^2\mathrm{e}^{-((1-e)P-P_{a,M})R_{a,M}T}\right]E\left[\mathrm{e}^{R_{a,M}X_{a,M}}\right].$$

But considering that $\frac{\partial}{\partial M}R_{a,M}=0$ whenever (35) holds we get

$$A_M(R_{a,M})|_{\frac{\partial}{\partial M}R_{a,M}=0} = (1+\alpha)\gamma E\left[T\mathrm{e}^{-((1-e)P-P_{a,M})R_{a,M}T}\right]\int_0^{M/a}\mathrm{e}^{R_{a,M}ax}dF(x) \\ +(1+\alpha)^2\gamma^2(1-F(M/a))\ \kappa''(((1-e)P-P_{a,M})R_{a,M}), \tag{39}$$

with κ'' given by (22) (which is strictly positive as we have seen), and hence (39) is strictly positive. Hence the second derivative with respect to M of $R_{a,M}$ is negative whenever the first derivative is zero, which implies that for fixed $a \in (a_0, 1]$, $R_{a,M}$ has at most one turning point, and that when such a point exists it is a maximum. The maximum will exist (and it will be finite) if we can guarantee that there is a finite solution to equation (36).

For fixed $a \in (a_0, 1]$, let

$$D_{a,M}(R_{a,M}) = \mathrm{e}^{R_{a,M}M} E^2 \left[\mathrm{e}^{-((1-e)P-P_{a,M})R_{a,M}T}\right] - (1+\alpha)\gamma E\left[T\mathrm{e}^{-((1-e)P-P_{a,M})R_{a,M}T}\right]. \tag{40}$$

Noticing that $\lim_{M\to\Phi(a)} R_{a,M} = 0$ (the expected profit is zero at $(a, \Phi(a))$), we get

$$\lim_{M\to\Phi(a), R_{a,M}\to 0} D_{a,M}(R_{a,M}) = -\alpha,$$

which is negative, and

$$\lim_{M\to+\infty, R\to R_{a,+\infty}} D_{a,M}(R_{a,M}) = +\infty,$$

because $R_{a,+\infty}$ exists given our assumptions (it is the adjustment coefficient before the excess of loss reinsurance takes place). Hence, for fixed $a \in (a_0, 1]$, it must exist at least one solution to (36). But we have already proved that if such a root exists it must be unique.

Note that in the classical model, i.e. when T is exponential distributed,

$$\gamma \frac{E\left[T\mathrm{e}^{-((1-e)P-P_{a,M})R_{a,M}T}\right]}{E^2\left[\mathrm{e}^{-((1-e)P-P_{a,M})R_{a,M}T}\right]} = 1,$$

and (29) is equivalent to $M = \frac{1}{R_{a,M}} \ln(1+\alpha)$, as it is known.

We have proved that, for fixed $a \in (a_0, 1]$, $R_{a,M}$, is a unimodal function of M, and that the maximum is attained at the unique finite solution of

$$D_{a,M}(R_{a,M}) = 0, \tag{41}$$

with $D_{a,M}(R_{a,M})$ given by (40). (41) defines M as a function of a. Let it be $\Upsilon(a)$. Let $\widehat{R}_a = R_{a,\Upsilon(a)}$.

(ii) Calculating the derivative of both sides of (30) at $M = \Upsilon(a)$ and $r = \widehat{R}_a$ we get

$$\frac{d}{da}\widehat{R}_a = -\left.\frac{\frac{\partial H_{a,M}(r)}{\partial a}}{\frac{\partial H_{a,M}(r)}{\partial r}}\right|_{M=\Upsilon(a);\; r=\widehat{R}_a} \tag{42}$$

and

$$\left.\frac{d^2}{da^2}\widehat{R}_a\right|_{\frac{d}{da}\widehat{R}_a=0} = -\left.\frac{\frac{\partial^2 H_{a,M}(r)}{\partial a^2}\frac{\partial^2 H_{a,M}(r)}{\partial M^2} - \left(\frac{\partial^2 H_{a,M}(r)}{\partial a \partial M}\right)^2}{\frac{\partial^2 H_{a,M}(r)}{\partial M^2}\frac{\partial H_{a,M}(r)}{\partial r}}\right|_{M=\Upsilon(a);\ r=\widehat{R}_a;\frac{d}{da}\widehat{R}_a=0} . \tag{43}$$

We have already shown in part (i) that $\frac{\partial^2 H_{a,M}(r)}{\partial M^2}$ calculated at $M = \Upsilon(a)$ and $r = \widehat{R}_a$ is positive and by Lemma 1 $\frac{\partial H_{a,M}(r)}{\partial r}$ calculated at the same point is also positive. Hence we can conclude that (43) is negative if and only if

$$\left.\frac{\partial^2 H_{a,M}(r)}{\partial a^2}\frac{\partial^2 H_{a,M}(r)}{\partial M^2} - \left(\frac{\partial^2 H_{a,M}(r)}{\partial a \partial M}\right)^2\right|_{r=R_{a,M};\frac{\partial}{\partial a}H_{a,M}(r)=0;\frac{\partial}{\partial M}H_{a,M}(r)=0} \tag{44}$$

is positive.

Noticing that

$$\frac{\partial}{\partial a}E[\mathrm{e}^{rX_{a,M}}] = r\int_0^{M/a} x\mathrm{e}^{rax}dF(x) \tag{45}$$

and that

$$\frac{\partial}{\partial a}E\left[\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right] = r\left[-(1-c)P + (1+\alpha)\gamma\int_{M/a}^{\infty} xdF(x)\right]E\left[T\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right], \tag{46}$$

we can say that (42) is zero if and only if

$$\begin{aligned}\left.\frac{\partial}{\partial a}H_{a,M}(r)\right|_{r=R_{a,M};\frac{\partial}{\partial M}H_{a,M}(r)=0} &= E\left[\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]\frac{\partial}{\partial a}E[\mathrm{e}^{rX_{a,M}}] + \\ &\quad + \left.E[\mathrm{e}^{rX_{a,M}}]\frac{\partial}{\partial a}E\left[\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]\right|_{\substack{r=R_{a,M};\\ \frac{\partial}{\partial M}H_{a,M}(r)=0}}\end{aligned} \tag{47}$$

is zero, which is to say that

$$\frac{\partial}{\partial a}E\left[\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right] = -E^2\left[\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]\frac{\partial}{\partial a}E[\mathrm{e}^{rX_{a,M}}]. \tag{48}$$

Calculating

$$\frac{\partial^2}{\partial a^2}E[\mathrm{e}^{rX_{a,M}}] = r^2\int_0^{M/a} x^2\mathrm{e}^{rax}dF(x) - r\frac{M^2}{a^3}\mathrm{e}^{rM}f(M/a), \tag{49}$$

where $f(x) = dF(x)/dx$, and

$$\begin{aligned}\frac{\partial^2}{\partial a^2}E\left[\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right] &= r^2 E\left[T^2\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right] \\ &\times\left[-(1-c)P+(1+\alpha)\gamma\int_{M/a}^{\infty} x dF(x)\right]^2 \\ &+r(1+\alpha)\gamma\frac{M^2}{a^3}f(M/a)E\left[T\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]\end{aligned} \tag{50}$$

we get that (considering (48))

$$\begin{aligned}&\left.\frac{\partial^2}{\partial a^2}H_{a,M}(r)\right|_{r=R_{a,M};\frac{\partial}{\partial a}H_{a,M}(r)=0;\frac{\partial}{\partial M}H_{a,M}(r)=0} = \\ &r^2E\left[\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]\int_0^{M/a}x^2\mathrm{e}^{rax}dF(x)+ \\ &+r^2\left[-(1-c)P+(1+\alpha)\gamma\int_{M/a}^{\infty}xdF(x)\right]^2 E\left[T^2\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]E[\mathrm{e}^{rX_{a,M}}]- \\ &\left.-2r^2E^2\left[\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]\left(\int_0^{M/a}x\mathrm{e}^{rax}dF(x)\right)^2\right|_{r=R_{a,M};\frac{\partial}{\partial a}H_{a,M}(r)=0;\frac{\partial}{\partial M}H_{a,M}(r)=0}\end{aligned} \tag{51}$$

which having in mind (48) and after some manipulation is equivalent to

$$\begin{aligned}&\left.\frac{\partial^2}{\partial a^2}H_{a,M}(r)\right|_{r=R_{a,M};\frac{\partial}{\partial a}H_{a,M}(r)=0;\frac{\partial}{\partial M}H_{a,M}(r)=0} = \\ &r^2E^2\left[\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]\mathrm{e}^{rM}(1-F(M/a))\int_0^{M/a}x^2\mathrm{e}^{rax}dF(x)+ \\ &+r^2\frac{E^4\left[\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]}{E^2\left[T\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]}\left(\int_0^{M/a}x\mathrm{e}^{rax}dF(x)\right)^2\kappa''(((1-e)P-P_{a,M})r) \\ &\left.+\,r^2E^2\left[\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]\left(\int_0^{M/a}\mathrm{e}^{rax}dF(x)\right)^2\varkappa_{a,M}(r)\right|_{r=R_{a,M};\frac{\partial}{\partial a}H_{a,M}(r)=0;\frac{\partial}{\partial M}H_{a,M}(r)=0},\end{aligned} \tag{52}$$

with κ'' given by (22) and

$$\varkappa_{a,M}(r) = \frac{\int_0^{M/a}x^2\mathrm{e}^{rax}dF(x)}{\int_0^{M/a}\mathrm{e}^{rax}dF(x)} - \left(\frac{\int_0^{M/a}x\mathrm{e}^{rax}dF(x)}{\int_0^{M/a}\mathrm{e}^{rax}dF(x)}\right)^2, \tag{53}$$

which are positive (they are both the variances of Esscher transforms), and hence (52) is positive.

Differentiating now $\partial H_{a,M}(r)/\partial M$ with respect to a, at the points where the first derivatives are zero, we get

$$\frac{\partial^2}{\partial a \partial M} H_{a,M}(r)\Big|_{r=R_{a,M};\frac{\partial}{\partial a}H_{a,M}(r)=0;\frac{\partial}{\partial M}H_{a,M}(r)=0} =$$

$$= \frac{r^2(1+\alpha)\gamma(1-F(M/a))\int_0^{M/a} x\mathrm{e}^{rax}dF(x)}{E\left[T\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]} \left\{E\left[\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right] E\left[T^2\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]\right.$$

$$\left. -2E^2\left[T\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]\right\}\Big|_{r=R_{a,M};\frac{\partial}{\partial a}H_{a,M}(r)=0;\frac{\partial}{\partial M}H_{a,M}(r)=0} =$$

$$= r^2(1+\alpha)\gamma(1-F(M/a))\frac{E^2\left[\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]}{E\left[T\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]}\int_0^{M/a} x\mathrm{e}^{rax}dF(x)\ \kappa''(((1-e)P-P_{a,M})r)$$

$$-r^2(1+\alpha)\gamma(1-F(M/a))E\left[T\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]\int_0^{M/a} x\mathrm{e}^{rax}dF(x)\Big|_{\substack{r=R_{a,M};\\ \frac{\partial}{\partial a}H_{a,M}(r)=0;\\ \frac{\partial}{\partial M}H_{a,M}(r)=0.}} \tag{54}$$

Hence

$$\frac{\partial^2 H_{a,M}(r)}{\partial a^2}\frac{\partial^2 H_{a,M}(r)}{\partial M^2} - \left(\frac{\partial^2 H_{a,M}(r)}{\partial a \partial M}\right)^2\Big|_{r=R_{a,M};\frac{\partial}{\partial a}H_{a,M}(r)=0;\frac{\partial}{\partial M}H_{a,M}(r)=0} =$$

$$\Big\{r^2(1-F(M/a))\mathrm{e}^{rM}E^2\left[\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]\int_0^{M/a} x^2\mathrm{e}^{rax}dF(x)$$

$$+r^2\frac{E^4\left[\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]}{E^2\left[T\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]}\left(\int_0^{M/a} x\mathrm{e}^{rax}dF(x)\right)^2\kappa''(((1-e)P-P_{a,M})r)$$

$$+ r^2E^2\left[\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]\left(\int_0^{M/a}\mathrm{e}^{rax}dF(x)\right)^2\varkappa_{a,M}(r)\Big\}$$

$$\times\Big\{r^2(1+\alpha)\gamma(1-F(M/a))E\left[T\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]\int_0^{M/a}\mathrm{e}^{rax}dF(x)$$

$$+r^2(1+\alpha)^2\gamma^2(1-F(M/a))^2\kappa''(((1-e)P-P_{a,M})r)\}$$

$$-r^4(1+\alpha)^2\gamma^2(1-F(M/a))^2\left(\frac{E^2\left[\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]}{E\left[T\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]}\int_0^{M/a} x\mathrm{e}^{rax}dF(x)\ \kappa''(((1-e)P-P_{a,M})r)\right.$$

$$\left.-E\left[T\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]\int_0^{M/a} x\mathrm{e}^{rax}dF(x)\right)^2\Big|_{\substack{r=R_{a,M};\\ \frac{\partial}{\partial a}H_{a,M}(r)=0;\\ \frac{\partial}{\partial M}H_{a,M}(r)=0.}} \tag{55}$$

Note that there are only two negative terms in the development of (55). The terms in $(\kappa'')^2$ cancel, and the sum of the first with the last term, having (36) in consideration, is equal to

$$\begin{aligned} & r^4(1+\alpha)^2\gamma^2(1-F(M/a))^2E^2\left[T\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]\times \\ & \times\left[\int_0^{M/a}x^2\mathrm{e}^{rax}dF(x)\int_0^{M/a}\mathrm{e}^{rax}dF(x)-\left(\int_0^{M/a}x\mathrm{e}^{rax}dF(x)\right)^2\right]\Bigg|_{r=R_{a,M};\frac{\partial}{\partial a}H_{a,M}(r)=0;\frac{\partial}{\partial M}H_{a,M}(r)=0} \\ & = r^4(1+\alpha)^2\gamma^2(1-F(M/a))^2E^2\left[T\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right]\left(\int_0^{M/a}\mathrm{e}^{rax}dF(x)\right)^2\varkappa_{a,M}(r)\Bigg|_{\substack{r=R_{a,M};\\ \frac{\partial}{\partial a}H_{a,M}(r)=0;\\ \frac{\partial}{\partial M}H_{a,M}(r)=0,}} \end{aligned} \tag{56}$$

with $\varkappa_{a,M}(r)$ given by (53), which is positive. Hence (55) is positive.

On the other hand, when $a\to a_0$, $\widehat{R}_a$ goes to zero and we can say that the maximum of $\widehat{R}_a$ is 1, if and only if

$$\lim_{a\to 1^-}\frac{d}{da}\widehat{R}_a\geq 0,$$

and the Result is proved.

■

Corollary 1 *(i) If*

$$(1-c)P\geq(1+\alpha)\gamma\left(\int_{\Upsilon(1)}^{\infty}xdF(x)+\int_0^{\Upsilon(1)}x\mathrm{e}^{R_{1,\Upsilon(1)}(x-\Upsilon(1))}dF(x)\right), \tag{57}$$

where $(r,M)=(R_{1,\Upsilon(1)},\Upsilon(1))$ *is the only solution to*

$$\begin{cases} E\left[\mathrm{e}^{rX_{1,M}}\right]E\left[\mathrm{e}^{-((1-e)P-P_{1,M})rT}\right]=1 \\ M=\frac{1}{r}\left(\ln(1+\alpha)+\ln\left(\gamma\frac{E\left[T\mathrm{e}^{-r\left((1-e)P-P_{1,M}\right)T}\right]}{E^2\left[\mathrm{e}^{-r\left((1-e)P-P_{1,M}\right)T}\right]}\right)\right) \end{cases}, \tag{58}$$

$R_{a,M}$ *attains its maximum value at* $(a^*,M^*)=(1,\Upsilon(1))$.

(ii) If (57) does not hold, then the pair (a^*, M^*) which maximizes $R_{a,M}$ is such that (r, a, M) is the solution to

$$\left\{\begin{array}{l} E\left[\mathrm{e}^{rX_{a,M}}\right] E\left[\mathrm{e}^{-((1-e)P-P_{a,M})rT}\right] = 1 \\ M = \frac{1}{r}\left(\ln(1+\alpha) + \ln\left(\gamma\frac{E\left[T\mathrm{e}^{-r\left((1-e)P-P_{a,M}\right)T}\right]}{E^2\left[\mathrm{e}^{-r\left((1-e)P-P_{a,M}\right)T}\right]}\right)\right) \\ (1+\alpha)\gamma\left(\int_{M/a}^{\infty} x dF(x) + \int_0^{M/a} x\mathrm{e}^{R_{a,M}(ax-M)} dF(x)\right) = (1-c)P \end{array}\right. \tag{59}$$

Proof. We only have to consider (42), (47) and that $M = \Upsilon(a)$ satisfies (29). ■

Corollary 2 *If $(1-c)P \geq (1+\alpha)\gamma E[X]$, then $R_{a,M}$ attains its maximum value at $(a^*, M^*) = (1, \Upsilon(1))$.*

Proof. We only have to notice that in that case

$$\begin{array}{l} (1-c)P - (1+\alpha)\gamma\left(\int_{\Upsilon(1)}^{\infty} x dF(x) + \int_0^{\Upsilon(1)} x\mathrm{e}^{R_{1,\Upsilon(1)}(x-\Upsilon(1))} dF(x)\right) \geq \\ (1+\alpha)\gamma\left(\int_0^{\Upsilon(1)} x\left(1-\mathrm{e}^{R_{1,\Upsilon(1)}(x-\Upsilon(1))}\right) dF(x)\right) \geq 0. \end{array}$$

■

Note that this corollary just means that if quota-share is at least as expensive as excess of loss (for the reinsurance of the hole risk), then excess of loss is optimal.

4 Examples

Example 1 *Let the individual claim amount distribution be Pareto(2,1), i.e. $F(x) = 1 - 1/(1+x)^2$, $x > 0$, so that $E[X] = 1$. Let $P = 1.6$, $e = 0.3$, $c = 0.2$ and $\alpha = 0.8$. We shall consider that the inter arrival times have mean 1, i.e. $\gamma = 1$. For these data the condition in Corollary 2 is not satisfied, i.e. quota-share is not, in the obvious sense, cheaper than excess of loss. We consider that the inter arrival time T is Gamma(n, β) distributed, i.e. with density function given by,*

$$p(t) = \frac{\beta^n}{\Gamma(n)}\mathrm{e}^{-\beta t} t^{n-1},\ t > 0.$$

For a Gamma(n, β) we have,

$$\gamma\frac{E\left[T\mathrm{e}^{-sT}\right]}{E^2\left[\mathrm{e}^{-sT}\right]} = \left(\frac{\beta+s}{\beta}\right)^{n-1}$$

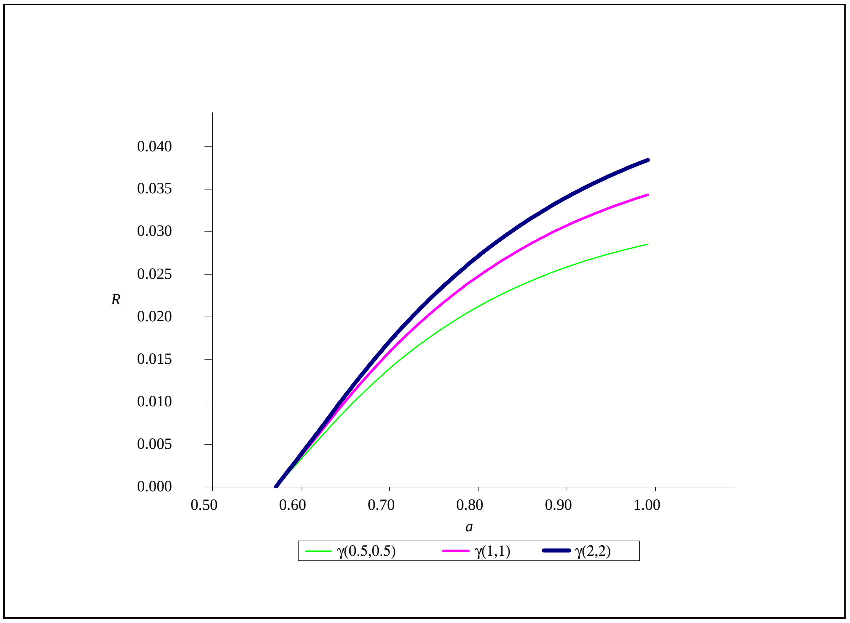

Figure 1: R as a function of $(a, \Upsilon(a))$; $c = 0.2$; $\alpha = 0.8$

and hence (29) is equivalent to

$$M = \frac{1}{R_{a,M}} \left(\ln(1+\alpha) + (n-1) \ln \left(\frac{\beta + ((1-e)P - P_{a,M})R_{a,M}}{\beta} \right) \right). \tag{60}$$

We shall consider three different situations: (n, β) equal respectively to (0.5,0.5), (1,1) (the classical model) and (2,2). Fig. 1 shows the adjustment coefficient as function of the retention level a, and the excess of loss limit M calculated according to (60). As we can see the optimal quota-share level is $a = 1$, for the three situations. The optimal excess of loss retention limit is 19.4524, 16.9804 and 15.6673 for the Gamma(0.5,0.5), Gamma(1,1) and Gamma(2,2) respectively, and the adjustment coefficient is equal to 0.0287357, 0.0346157 and 0.0387563 respectively. Fig. 2 shows the adjustment coefficient as a function of M when $a = 1$.

Example 2 *All the information as in the previous example, with the following two exceptions: $c = 0.25$ and $\beta = 1.2$.*

Fig. 3 is the analogue to Fig. 1. In this case, the optimum is a mixture of quota-share with excess of loss, namely (a, M) equal to (0.90215, 31.18843), (0.92791, 27.66260) and (0.94610, 25.82807) for the Gamma(0.5,0.5), Gamma(1,1) and Gamma(2,2) respectively.

As we would expect, for both examples, from the three inter arrival time distributions, the Gamma(0.5,0.5) induces smaller values for the adjustment coefficient, followed by the exponential.

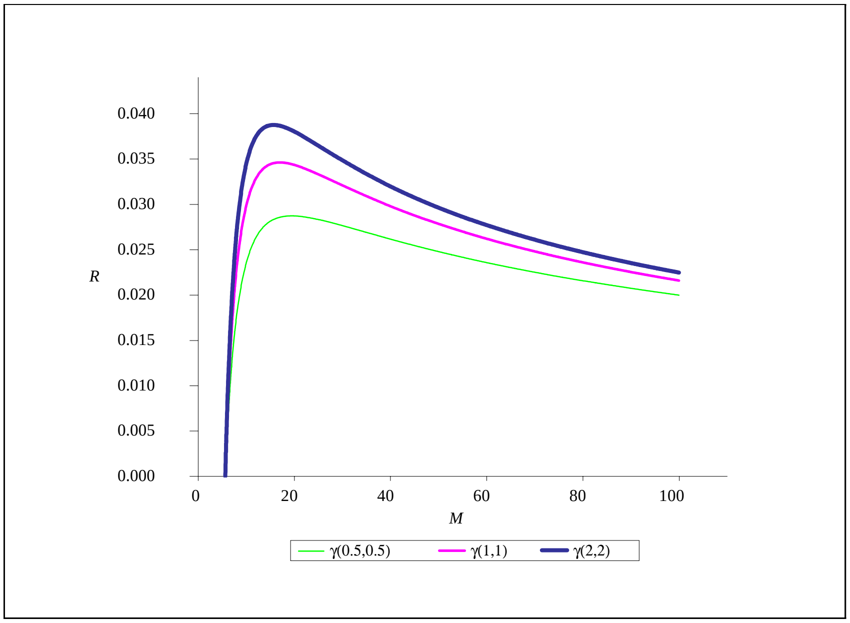

Figure 2: R as a function of M for $a = 1$

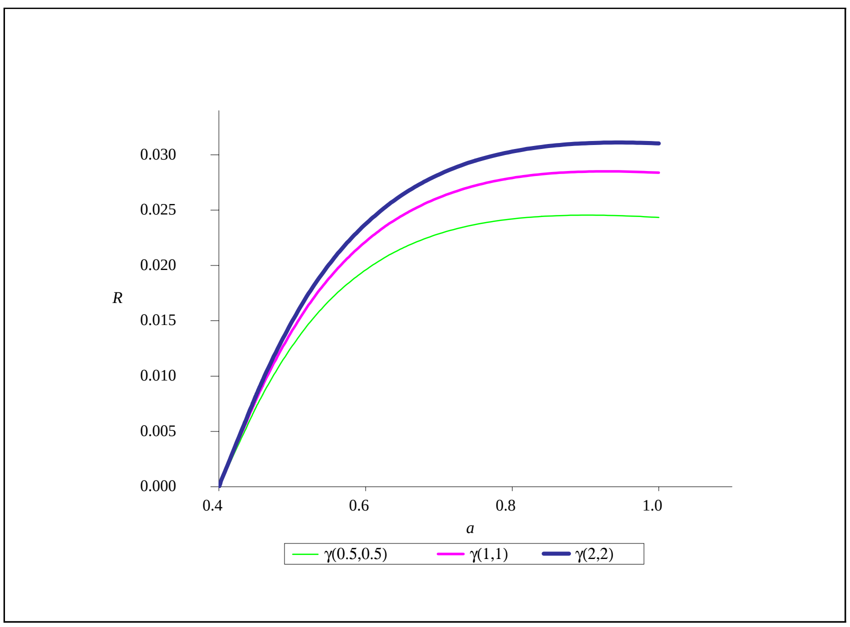

Figure 3: R as a function of $(a, \Upsilon(a))$; $c = 0.25$; $\alpha = 1.2$

References

Carter, R. (1979). *Reinsurance*, Kluwer Pub. in association with the Mercantile and General Reinsurance Co., London.

Centeno, L. (1985). On combining quota-share and excess of loss, *ASTIN Bulletin* **15**: 49–63.

Centeno, M. L. (1986). Measuring the effects of reinsurance by the adjustment coefficient, *Insurance: Mathematics and Economics* **5**: 169–182.

Gerber, H. U. (1979). *An Introduction to Mathematical Risk Theory*, S.S. Huebner Foundation Monographs, University of Pensylvania.

Grandell, J. (1992). *Aspects of Risk Theory*, Springer-Verlag, New York.

Hesselager, O. (1990). Some results on optimal reinsurance in terms of the adjustment coefficient, *Scandinavian Actuarial Journal* pp. 80–95.

Sparre Anderson, E. (1957). On the collective theory of risk in the case of contagious between the claims, *Transactions XV International Congress of Actuaries*.

Waters, H. (1979). Excess of loss reinsurance limits, *Scandinavian Actuarial Journal* pp. 37–43.

Waters, H. (1983). Some mathematical aspects of reinsurance, *Insurance: Mathematics and Economics* **2**: 17–26.

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