

ON XL-SL REINSURANCE

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Abstract.

A combination of excess-loss and stop-loss reinsurance is considered. It belongs to an extended class of so-called perfectly hedged experience rating contracts studied in insurance and finance by the author. To price this contract, a CAPM fair principle is proposed. A concrete implementation requires knowledge about the covariance between the excess-loss and stop-loss reinsurance components. Increasing the number of risks arbitrarily, an exact covariance evaluation appears impractical or quite difficult. For practical purposes some simple and refined covariance bounds are derived.

Keywords : excess-loss reinsurance, stop-loss reinsurance, perfect hedge, CAPM fair principle, covariance bounds

1. Introduction.

Many of the conventional insurance pricing principles like the standard deviation and variance principle, the Esscher principle and the more recent PH-transform principle contain an unknown parameter, which describes the level of the required security loading and/or the expected profit of the business. The evaluation of this parameter is usually based on a stability or ruin criterion (e.g. Schnieper(1990)). In the recent literature one encounters also some pricing principles with prescribed loading factors, among others the Karlsruhe principle (e.g. Heilmann(1988), Hürlimann(1994a/97a)), the special Dutch principle (e.g. Heerwaarden and Kaas(1992), Hürlimann(1991/94b/95b/c)), the Hardy-Littlewood principle (Hürlimann(1998a)), a CAPM fair principle (Hürlimann(1994a/98b)), and the entropy distortion principle (Hürlimann(1999a)). The purpose of the present study is to illustrate the use of the CAPM recipe for pricing the XL-SL reinsurance contract. Parts of our results should also be useful for the risk control of portfolios of derivative risks encountered in Asset and Liability Management.

Recall what is meant by XL-SL reinsurance. Given are n independent risks represented by random variables X_1, \dots, X_n defined on some probability space. Let $X = \sum_{i=1}^n X_i$ be the aggregate risk and consider the random variable

$$U = U(L) = U(L_1, \dots, L_n) = \sum_{i=1}^n (X_i - L_i)_+, \quad (1.1)$$

which represents the payoff of an *XL (excess-loss) reinsurance* with individual limits $L_i, i = 1, \dots, n$, and called here anticipated excess-loss. Then the random variable

$$X(L) = X(L_1, \dots, L_n) = \sum_{i=1}^n X_i \wedge L_i, \quad (1.2)$$

where $x \wedge y = \min(x, y)$, represents the *anticipated aggregate risk* after XL-reinsurance. Consider further a *SL (stop-loss) reinsurance* on the anticipated aggregate risk with deductible d , whose payoff is defined by

$$V = V(L, d) = (X(L) - d)_+. \quad (1.3)$$

Then the sum of the XL and SL reinsurance payoffs $Z = Z(L, d) = U(L) + V(L, d)$ is viewed as payoff of a combined excess-loss/stop-loss reinsurance, called *XL-SL reinsurance*. Note that the limiting special case $L_i = L \rightarrow \infty, i = 1, \dots, n$, identifies Z with the payoff $(X - d)_+$ of a *SL* reinsurance on the aggregate risk. The above notations will be used throughout. It is assumed that all random variables have finite means and variances, and the covariances between random variables exist.

An important question of practical significance concerns the separate pricing of the XL and SL reinsurance payoffs and its relationship with the overall pricing of the XL-SL reinsurance payoff. Mathematically, if H denotes a pricing principle, which is a real functional acting on a suitable space of risks, then one is interested in the relationship between $H[U], H[V]$ and $H[Z] = H[U + V]$ as well as in concrete pricing rules for use in insurance and finance practice, where such constructions occur. This question is discussed in Section 2, where the CAPM fair principle is proposed for pricing. A practical implementation requires knowledge of the covariance $Cov[U, V]$ between the components of the XL-SL reinsurance payoff. By increasing the number of risks arbitrarily, an exact computation turns out to be rather impractical, or at least quite complex. For this reason possibly tight bounds for this covariance are welcome. Section 3 settles the general framework for our covariance calculations and displays some analytical formulas. Simple covariance bounds are derived in Section 4. Refined covariance bounds are proposed in Section 5. Their application to XL-SL reinsurance is summarized in Section 6. We conclude with a numerical example in Section 7.

2. XL-SL reinsurance pricing.

In general, at least the *subadditive property* $H[Z] = H[U + V] \leq H[U] + H[V]$ should hold. Indeed, if $H[Z] > H[U] + H[V]$, then a market participant could choose to insure Z and reinsurer the components U and V separately. Its asset position equals $H[Z] - (H[U] + H[V]) > 0$ while its liability $-Z + (U + V) = 0$ vanishes, an undesired riskless profit. Let us illustrate at two practical pricing principles.

If $H[X] = E[X] + \theta \sigma[X]$ is the *standard deviation principle* with $\sigma[X] = \sqrt{Var[X]}$, then subadditivity holds because $\sigma^2[U + V] = \sigma^2[U] + \sigma^2[V] + 2 \cdot Cov[U, V] \leq (\sigma[U] + \sigma[V])^2$. A practical implementation would require an evaluation of the covariance, a question thoroughly analyzed later on. However, note that the standard deviation principle is rejected because it does not preserve the usual stochastic order (e.g. van Heerwaarden(1991)).

If $H[X] = E[X] + \theta\sigma^2[X]$ is the *variance principle*, then subadditivity fails in our situation because $\sigma^2[U+V] \geq \sigma^2[U] + \sigma^2[V]$ (Theorem 4.1, inequality (4.3)) with a strict inequality in general (refined lower bounds of Section 5).

On the other side, if strict subadditivity holds, that is $H[Z] < H[U] + H[V]$, then a market player, which insurers U and V separately and reinsurers Z makes the riskless profit $(H[U] + H[V]) - H[Z] > 0$. Therefore, to avoid arbitrage opportunities, the *additive property* $H[U+V] = H[U] + H[V]$ should be satisfied.

Among the various additive principles (e.g. Bühlmann(1980), Venter(1991), Aase(1993)), the simplest one is a *CAPM covariance principle* according to which the prices of the components are set equal to (Borch(1982)) :

$$\begin{aligned} H[U] &= E[U] + \frac{\text{Cov}[Z,U]}{\text{Var}[Z]}(H[Z] - E[Z]), \\ H[V] &= E[V] + \frac{\text{Cov}[Z,V]}{\text{Var}[Z]}(H[Z] - E[Z]). \end{aligned} \quad (2.1)$$

To implement this solution, there remains to determine the price $H[Z]$ of XL-SL reinsurance and the covariance $\text{Cov}[U,V]$. Fortunately, there exists a recent approach, which proposes a method to price such reinsurance contracts.

One needs the additional payoff $D = D(L, d) = (d - X(L))_+$, which represents a *stop-loss dividend* on the anticipated aggregate risk. This nomenclature stems from the identity

$$d + (X(L) - d)_+ = X(L) + (d - X(L))_+, \quad (2.2)$$

which tells us that the amount d plus the payoff V of the SL reinsurance allows one to finance the anticipated aggregate risk plus the stop-loss dividend.

Lemma 2.1. One has the identity

$$d + U(L) + V(L, d) = X + D(L, d). \quad (2.3)$$

Proof. Since $X_i \wedge L_i = X_i - (X_i - L_i)_+$, $i = 1, \dots, n$, one has $X(L) = X - U(L)$. Inserted in (2.2) the identity (2.3) follows. \diamond

The relationship $d + Z = X + D$ identifies XL-SL reinsurance as further main example of an extended class of “perfectly hedged experience rating contracts”, whose story begins with Hürlimann(1994a) (see also Hürlimann(1995d/98b)). According to the proposed *CAPM fair principle* the XL-SL reinsurance price should be set equal to

$$H[Z] = E[Z] + \frac{\text{Cov}[X,Z]}{\text{Var}[X]}(P - E[X]), \quad (2.4)$$

where $P = H[X]$ is the price of the aggregate risk. Furthermore, if individual variance prices $H[X_i] = E[X_i] + \theta\text{Var}[X_i]$, $i = 1, \dots, n$, are appropriate, hence

$P = H[X] = \sum_{i=1}^n H[X_i] = E[X] + \theta \cdot \text{Var}[X]$ (additivity for independent risks), then a perfect hedging argument (Hürlimann(1994a), Section 4) yields the unique CAPM fair price

$$P = P(L, d) = H[X] = E[X] + \frac{\text{Var}[X]}{\text{Cov}[X, Y]} \cdot E[D], \quad \text{with } Y = d - D, \quad (2.5)$$

which does not depend on an unknown loading factor, but on the deductible d , the limits L_i , and the mean and covariance properties of the involved random variables. Clearly, the choice of d and the L_i 's has a big influence upon the price (2.5) (e.g. Hürlimann(1994a), Table 2). If the price P is known (e.g. tariff guarantee, or other pricing principles for the X_i 's) the choice of d and the L_i 's will be restricted to those values, which satisfy the equation (2.5). If there is no such restriction, one should look at optimal values. For example, by given individual limits L_i , an *optimal deductible* d for SL reinsurance could be obtained following the proposal in Hürlimann(1999b). The generalized problem of an optimal simultaneous choice of the quantities (d, L_1, \dots, L_n) remains open for investigation.

For a concrete implementation of the above methodology, the covariances in the formulas (2.1), (2.4) and (2.5) must be determined or at least bounded appropriately. This more technical task is subject of the next Sections.

3. Some covariance formulas.

An ideal mathematical goal is the exact analytical and/or numerical evaluation of the relevant covariances $\text{Cov}[U, V]$, $\text{Cov}[X, Z]$ and $\text{Cov}[X, Y]$ in terms of the distributions $F_i(x) = \Pr(X_i \leq x)$ of the individual risks, $i = 1, \dots, n$. Besides the distribution $F_X(x) = \Pr(X \leq x)$ of an arbitrary risk X (with finite mean μ_X and variance σ_X^2), the following related quantities are always used :

$\bar{F}_X(x) = 1 - F_X(x)$: the decumulative distribution function or survival function

$\pi_X(t) = E[(X - t)_+] = \int_t^\infty \bar{F}_X(x) dx$: the stop-loss transform

$\bar{\pi}_X(t) = E[(t - X)_+] = t - \mu_X + \pi_X(t)$: the “conjugate” stop-loss transform

$\sigma_X^2(t) = \text{Var}[(X - t)_+] = E[(X - t)_+]^2 - \pi_X(t)^2$, with

$E[(X - t)_+]^2 = 2 \cdot \int_t^\infty \pi_X(x) dx$: the stop-loss transform of degree two

$\bar{\sigma}_X^2(t) = \text{Var}[(t - X)_+] = \sigma_X^2 - 2 \cdot \pi_X(t) \cdot \bar{\pi}_X(t) - \sigma_X^2(t)$

The determination of these quantities is classical and assumed to be known. Under partial information on X , there exist more or less sharp bounds for these quantities. For example, best upper and lower bounds for the partial means $\pi_X(t)$ are well-known (e.g. Hürlimann(1997b)), and the partial variances $\sigma_X^2(t)$ can be bounded using the generalized inequalities of Kremer and Schmitter (e.g. Hürlimann(1994a/97c/97d)).

With the definitions and notations of Section 1 and 2, recall the relations

$$d + Z = X + D, \quad Z = U + V, \quad X = Y + Z, \quad Y = d - D, \quad X = X(L) + U. \quad (3.1)$$

Proposition 3.1. The random variables U, V, X, Y, Z associated to XL-SL reinsurance satisfy the following covariance formulas :

$$\text{Cov}[X, Y] = \text{Var}[X] - \text{Cov}[X, Z] \quad (3.2)$$

$$\text{Cov}[X, Z] = \text{Cov}[X, U] + \text{Cov}[X, V] \quad (3.3)$$

$$\text{Cov}[X, U] = \sum_{i=1}^n [\sigma_i^2(L_i) + \pi_i(L_i) \cdot \bar{\pi}_i(L_i)] \quad (3.4)$$

$$\text{Cov}[X, V] = \sum_{i=1}^n \text{Cov}[X_i, V] \quad (3.5)$$

$$\text{Cov}[U, V] = \text{Cov}[X, V] - \sigma_{X(L)}^2(d) - \pi_{X(L)}(d) \cdot \bar{\pi}_{X(L)}(d) \quad (3.6)$$

Proof. The relations (3.3) and (3.5) are trivial, (3.2) follows from (3.1) using that $Y = X - Z$. For (3.4), note that by independence of the X_i 's one has

$$\text{Cov}[X, U] = \sum_{i=1}^n \text{Cov}[X_i, (X_i - L_i)_+],$$

and use formula (2) in Hürlimann(1993). Finally (3.6) follows using $U = X - X(L)$ and the same mentioned formula (2). \diamond

The structure of the remaining covariances $C_i(L, d) := \text{Cov}[X_i, V]$, $i = 1, \dots, n$, is more complex. Setting

$$X(L) = X_i \wedge L_i + Y_i, \quad (3.7)$$

where $Y_i := \sum_{j \neq i} X_j \wedge L_j$ is independent of X_i , one has

$C_i(L, d) = \text{Cov}[X_i, (X_i \wedge L_i + Y_i - d)_+]$. Therefore, it remains to discuss the evaluation of quantities $C(L, d) = \text{Cov}[X, (X \wedge L + Y - d)_+]$, where X, Y are independent random variables and L, d are non-negative real numbers. Using the identity $(X \wedge L + Y - d)_+ = X \wedge L + Y - d + (d - X \wedge L - Y)_+$ one gets

$$C(L, d) = \sigma_X^2 - \sigma_X^2(L) - \pi_X(L) \cdot \bar{\pi}_X(L) + \bar{C}(L, d), \quad \text{with} \quad (3.8)$$

$$\bar{C}(L, d) = \text{Cov}[X, (d - X \wedge L - Y)_+]. \quad (3.9)$$

It is possible to derive integral representations for this latter quantity.

Proposition 3.2. Let X and Y be non-negative independent random variables and let d, L be non-negative real numbers. Two cases must be distinguished :

Case 1 : $d \leq L$

$$\bar{C}(L, d) = \int_0^d [(t - \mu_X) F_X(t) - \bar{\pi}_X(t)] \cdot F_Y(d - t) dt \quad (3.10)$$

Case 2 : $d > L$

$$\begin{aligned} \bar{C}(L, d) &= \int_0^L [(t - \mu_X)F_X(t) - \bar{\pi}_X(t)] \cdot F_Y(d - t) dt \\ &+ [\pi_X(L) + (L - \mu_X)\bar{F}_X(L)] \cdot \bar{\pi}_Y(d - L). \end{aligned} \quad (3.11)$$

Proof. With $I(\cdot)$ the indicator function, one has the identity

$$\begin{aligned} (d - x \wedge L - y)_+ &= (d - x - y)_+ \cdot I(x \leq L) + (d - L - y)_+ \cdot I(x > L) \\ &= \int_0^L I(x \leq t, y \leq d - t) dt + (d - L - y) \cdot I(x > L) \cdot I(y \leq d - L). \end{aligned} \quad (3.12)$$

Inserting this into $\bar{C}(L, d) = E[(X - \mu_X)(d - X \wedge L - Y)_+]$, the result follows through calculation. \diamond

By knowledge of the distribution of Y , which in fact stands for $Y_i = \sum_{j \neq i} X_j \wedge L_j$ in our situation, the exact covariance can in principle be calculated. However, increasing the number of risks arbitrarily, an increasing number of exact computations is required, which is impractical. For this reason, various lower and upper bounds will be derived.

4. Simple covariance bounds.

Let us begin with a simple but often overlooked general covariance formula.

Lemma 4.1. (Wallenius(1971)) Let X and Y be two real random variables. Then one has the covariance formula $Cov[X, Y] = Cov[X, E[Y|X]]$.

Proof. It is well-known that $Cov[X, Y] = E[Cov[X, Y|X]] + Cov[E[X|X], E[Y|X]]$. Since the first term vanishes, the formula is immediate. \diamond

Corollary 4.1. Let X and Y be independent random variables and d, L real numbers. Then the covariance $Cov[(X - L)_+, (X \wedge L + Y - d)_+]$ is non-negative.

Proof. Applying Lemma 4.1 one has using the independence assumption that

$$\begin{aligned} Cov[(X - L)_+, (X \wedge L + Y - d)_+] &= Cov[(X - L)_+, E[(Y - (d - X \wedge L))_+ | X]] \\ &= Cov[(X - L)_+, \pi_Y(d - X \wedge L)]. \end{aligned}$$

Since $f(x) = (x - L)_+$ and $g(x) = \pi_Y(d - x \wedge L)$ are non-decreasing functions of x , the result follows from Chebyshev's inequality (e.g. Hardy, Littlewood and Polya(1934), no. 43). Alternatively, a random pair (X, Y) is positively quadrant dependent if, and only if, the property $Cov[f(X), g(Y)] \geq 0$ holds for all non-decreasing real functions f and g for which the covariance exists (e.g. Jogdeo(1982)). Since (X, X) is trivially positive quadrant dependent, the result follows. \diamond

In a first step, we show that the main covariance formulas of interest can be roughly bounded by quantities involving only first and second order partial moments of the individual risks $X_i, i = 1, \dots, n$, the aggregate risk X and the anticipated aggregate risk $X(L)$ after XL-reinsurance. The obtained bounds will be useful for a crude estimation of the situation.

Theorem 4.1. The random payoff $Z = U + V$ of the XL-SL reinsurance program associated to a collection of independent risks $X_i, i = 1, \dots, n$, satisfies the following covariance properties :

$$\begin{aligned} & \frac{1}{2}(\text{Var}[X] - \text{Var}[X(L)]) + \text{Cov}[X(L), V] \\ & \leq \text{Cov}[X, U] + \text{Cov}[X(L), V] \\ & \leq \text{Cov}[X, Z] \leq \text{Cov}[X, U] + \text{Cov}[X, (X - d)_+] \end{aligned} \quad (4.1)$$

with

$$\begin{aligned} \text{Cov}[X, U] &= \sum_{i=1}^n \{ \sigma_i^2(L_i) + \pi_i(L_i) \bar{\pi}_i(L_i) \} \\ \text{Cov}[X(L), V] &= \sigma_{X(L)}^2(d) + \pi_{X(L)}(d) \cdot \bar{\pi}_{X(L)}(d) \\ \text{Cov}[X, (X - d)_+] &= \sigma_X^2(d) + \pi_X(d) \cdot \bar{\pi}_X(d) \end{aligned} \quad (4.2)$$

$$0 \leq \text{Cov}[U, V] \leq \text{Cov}[X, (X - d)_+] - \text{Cov}[X(L), V] \quad (4.3)$$

Proof. The relations (4.2) are immediate (e.g. Hürlimann(1993), formula (2)). Since $d + Z = X + D$ and $X(L) = X - U$ one obtains $\text{Cov}[X, Z] = \sigma_X^2 + C(d)$ with $C(d) = \text{Cov}[X, (d + U - X)_+]$. By Lemma 4.1 one has

$$C(d) = E[(X - \mu_X)E[(d + U - X)_+ | X]].$$

Inserting the straightforward inequality

$$\begin{aligned} E[(d + U - X)_+ | X] &= d + E[U | X] - X + E[(X - d - U)_+ | X] \\ &\leq d + E[U | X] - X + (X - d)_+ = E[U | X] + (d - X)_+, \end{aligned}$$

one obtains the upper bound

$$C(d) \leq C(d)_u := E[(X - \mu_X)E[U | X]] + E[(X - \mu_X)(d - X)I(X \leq d)].$$

With Lemma 4.1 and the decomposition $X - \mu_X = (X - d) + (d - \mu_X)$ one gets further

$$C(d)_u = \text{Cov}[X, U] + (d - \mu_X) \bar{\pi}_X(d) - \bar{\sigma}_X^2(d) - \bar{\pi}_X(d)^2 = \text{Cov}[X, U] - \bar{\sigma}_X^2(d) - \pi_X(d) \bar{\pi}_X(d).$$

Using the relation $\bar{\sigma}_X^2(d) = \sigma_X^2 - 2\pi_X(d) \bar{\pi}_X(d) - \sigma_X^2(d)$, one obtains

$$\text{Cov}[X, Z] \leq \sigma_X^2 + C(d)_u = \text{Cov}[X, U] + \sigma_X^2(d) + \pi_X(d) \bar{\pi}_X(d),$$

which shows the last upper bound in (4.1). On the other side one has

$$\text{Cov}[X, Z] = \text{Cov}[X, U] + \text{Cov}[X(L), V] + \text{Cov}[U, V]. \quad (4.4)$$

First, let us show the lower bound in (4.3). For each $i = 1, \dots, n$ set (as in (3.7))

$$X(L) = X_i \wedge L_i + Y_i \quad (4.5)$$

where $Y_i := \sum_{j \neq i} X_j \wedge L_j$ is independent of X_i . Then one has

$$\text{Cov}[U, V] = \sum_{i=1}^n C_i(L_i, d), \text{ with } C_i(L_i, d) = \text{Cov}[(X_i - L_i)_+, (X_i \wedge L_i + Y_i - d)_+]. \quad (4.6)$$

Since $C_i(L_i, d) \geq 0$ by Corollary 4.1, this shows $\text{Cov}[U, V] \geq 0$ and the middle lower bound in (4.1) follows by (4.4). Comparing (4.4) with the last upper bound in (4.1) shows the upper bound in (4.3). Finally, to get the first lower bound in (4.1) note that

$$\text{Var}[U] = \text{Var}[X] + \text{Var}[X(L)] - 2\text{Cov}[X, X(L)] = \text{Var}[X] + \text{Var}[X(L)] - 2\text{Var}[X] + 2\text{Cov}[X, U] \geq 0,$$

which implies the desired inequality for $\text{Cov}[X, U]$. \diamond

Corollary 4.2. The correlation coefficient between the components of the XL-SL reinsurance payoff $Z = U + V$ satisfies the simple bounds

$$0 \leq \rho(U, V) \leq \frac{(\sigma_X^2(d) - \sigma_{X(L)}^2(d)) + (\pi_X(d)\bar{\pi}_X(d) - \pi_{X(L)}(d)\bar{\pi}_{X(L)}(d))}{\sigma_{X(L)}(d) \cdot \sqrt{\sigma_{X(L)}^2 - \sigma_X^2 + 2\text{Cov}[X, U]}} \quad (4.7)$$

with

$$\text{Cov}[X, U] = \sum_{i=1}^n \{\sigma_i^2(L_i) + \pi_i(L_i)\bar{\pi}_i(L_i)\}. \quad (4.8)$$

Remark 4.1. If necessary, the second order partial moments $\sigma_i^2(L_i)$, $\sigma_X^2(d)$, $\sigma_{X(L)}^2(d)$ can be bounded using the inequalities of Kremer and Schmitter (e.g. Hürlimann(1994a/97c/d)). Of interest is also an extrapolation formula by Kremer(1998) (see also Hürlimann(1999c)).

5. Refined covariance bounds.

The derivation of various tighter bounds for $\text{Cov}[X, Z]$ is possible. By (4.4) it suffices to obtain tighter bounds for $\text{Cov}[U, V]$. Using the decomposition (4.6), it suffices to obtain tight bounds for a covariance of the general form

$$C(L, d) = \text{Cov}[(X - L)_+, (X \wedge L + Y - d)_+], \quad (5.1)$$

where X and Y are independent random variables and d, L are non-negative real numbers.

Lemma 5.1. One has

$$C(L, d) = \pi_X(L)\bar{\pi}_X(L) + \bar{C}(L, d), \text{ with} \quad (5.2)$$

$$\bar{C}(L, d) = \text{Cov}[(X - L)_+, (d - X \wedge L - Y)_+]. \quad (5.3)$$

Proof. This follows immediately using the identity

$$\begin{aligned} (X \wedge L + Y - d)_+ &= X \wedge L + Y - d + (d - X \wedge L - Y)_+ \\ &= X - (X - L)_+ + Y - d + (d - X \wedge L - Y)_+. \quad \diamond \end{aligned}$$

The following integral representations will be useful.

Lemma 5.2. With Stieltjes integrals one has the following formulas. For arbitrary values of d, L one has

$$C(L, d) = \pi_X(L) \cdot \left[\bar{\pi}_X(L) + \bar{\pi}_Y(d - L)F_X(L) - \int_{-\infty}^L \bar{\pi}_Y(d - x)dF_X(x) \right]. \quad (5.4)$$

If $d > L$ one has furthermore

$$C(L, d) = \pi_X(L) \cdot \left[\int_{-\infty}^L [\bar{\pi}_Y(d - L) - \bar{\pi}_Y(d - x)]dF_X(x) + \int_{-\infty}^L (L - x)\bar{F}_Y(d - x)dF_X(x) \right]. \quad (5.5)$$

Proof. The formula (5.4) follows from (5.2) and the calculation

$$\begin{aligned} \bar{C}(L, d) &= E\left[[(X - L)_+ - \pi_X(L)] \cdot (d - X \wedge L - Y)_+ \cdot [I(X \leq L) + I(X > L)]\right] \\ &= E\left[[X - L - \pi_X(L)](d - L - Y)_+ I(X > L)\right] - \pi_X(L) \cdot E[(d - X - Y)_+ I(X \leq L)] \\ &= \pi_X(L) \cdot \left[(1 - \bar{F}_X(L))\bar{\pi}_Y(d - L) - E[(d - X - Y)_+ I(X \leq L)] \right]. \end{aligned}$$

To show (5.5), note first the rearrangements (use that $d > L$):

$$\begin{aligned} \bar{C}(L, d) &= E\left[[(X - L)_+ - \pi_X(L)] \cdot (d - X \wedge L - Y)_+ \cdot [I(X \leq L) + I(X > L)]\right] \\ &= \pi_X(L) \cdot E\left[\left[(d - L - Y)_+ - (d - X - Y)_+\right] \cdot I(X \leq L)\right] \\ &= \pi_X(L) \cdot \left(E[(X - L)I(X \leq L)I(Y \leq d - L)] + E[(X + Y - d)I(X \leq L)I(d - L \leq Y \leq d - X)] \right). \end{aligned}$$

Inserting this and the rearrangement

$$\bar{\pi}_X(L) = E\left[(L - X)I(X \leq L)[I(Y \leq d - L) + I(d - L < Y \leq d - X) + I(Y > d - X)]\right]$$

into (5.2), one obtains

$$C(L, d) = \pi_X(L) \cdot \left(\begin{aligned} &E[(Y - (d - L))I(X \leq L)I(d - L < Y \leq d - X)] \\ &+ E[(L - X)I(X \leq L)I(Y > d - X)] \end{aligned} \right),$$

which is equivalent with the integral representation (5.5). \diamond

Though it is possible to derive in a similar manner bounds for risks X_i with arbitrary interval supports, we restrict the discussion to the main situation of non-negative random variables with support $[0, \infty)$. Since in our application, where Y stands representatively for $Y_i := \sum_{j \neq i} X_j \wedge L_j$, the distribution $F_Y(y)$ is rather complex to evaluate, our emphasis is on bounds, which depend at most on the mean μ_Y and variance σ_Y^2 , and special values of $\bar{\pi}_Y(y)$ and $F_Y(y)$, which can be further bounded. Two cases are distinguished.

5.1. The case $d \leq L$.

First, let us formulate simple lower and upper bounds.

Proposition 5.1. Let X and Y be independent random variables with support $[0, \infty)$, and let d, L be non-negative real numbers such that $d \leq L$. Then $C(L, d)$ has the upper bound

$$C(L, d)_u = \pi_X(L) \cdot [\bar{\pi}_X(L) - \bar{\pi}_X(d - \mu_Y)] \quad (5.6)$$

and the lower bound

$$C(L, d)_l = \pi_X(L) \cdot [\bar{\pi}_X(L) - \bar{\pi}_X(d)]. \quad (5.7)$$

Proof. For non-negative random variables the formula (5.4) reads (note that $\bar{\pi}_Y(d - x) = 0$ for $x \geq d$):

$$C(L, d) = \pi_X(L) \cdot \left[\bar{\pi}_X(L) - \int_0^d \bar{\pi}_Y(d - x) dF_X(x) \right]. \quad (5.8)$$

The inequality of Jensen $\bar{\pi}_Y(d - x) \geq (d - \mu_Y - x)_+$ implies that

$$\int_0^d \bar{\pi}_Y(d - x) dF_X(x) \geq \int_0^{d - \mu_Y} (d - \mu_Y - x) dF_X(x) = \bar{\pi}_X(d - \mu_Y),$$

hence (5.6). The lower bound (5.7) follows similarly using the inequality $\bar{\pi}_Y(d - x) \leq (d - x)_+$. \diamond

More complex bounds can be obtained through application of the known analytical formulas for the extremal stop-loss transforms by given support and moments to order four (Jansen et al.(1986) for the maximal stop-loss transforms and Hürlimann(1997e) in general). The simplest case yields the following refined lower bound.

Proposition 5.2. Suppose X, Y, d, L are as in Proposition 5.1, and let $k_Y = \frac{\sigma_Y}{\mu_Y}$ be the coefficient of variation of Y . Then one has the improved lower bound

$$C(L, d)_\ell = \pi_x(L) \cdot \left[\bar{\pi}_x(L) - \left(\frac{k_Y^2}{1+k_Y^2} \right) \bar{\pi}_x(d) - \left(\frac{1}{1+k_Y^2} \right) \bar{\pi}_x\left(d - \frac{1}{2}(1+k_Y^2)\mu_Y\right) \right]. \quad (5.9)$$

Proof. From the mentioned works (or alternatively Goovaerts and De Vylder(1982), Goovaerts et al.(1984), Hürlimann(1997b)) one borrows the best upper bound

$$\max\{\bar{\pi}_Y(d-x)\} = \begin{cases} \left(\frac{k_Y^2}{1+k_Y^2} \right) (d-x), & d - \frac{1}{2}(1+k_Y^2)\mu_Y \leq x \leq d \\ \frac{1}{2}\mu_Y \left[\sqrt{k_Y^2 + \left(\frac{d-x-\mu_Y}{\mu_Y} \right)^2} + \left(\frac{d-x-\mu_Y}{\mu_Y} \right) \right], & x \leq d - \frac{1}{2}(1+k_Y^2)\mu_Y \end{cases} \quad (5.10)$$

If $d \leq \frac{1}{2}(1+k_Y^2)\mu_Y$ only the linear part in x is required and yields with (5.8) the lower bound

$$C(L, d)_\ell = \pi_x(L) \cdot \left[\bar{\pi}_x(L) - \left(\frac{k_Y^2}{1+k_Y^2} \right) \bar{\pi}_x(d) \right], \quad (5.11)$$

which coincides with (5.9) because $\bar{\pi}_x(x) = 0$ for $x \leq 0$. If $d > \frac{1}{2}(1+k_Y^2)\mu_Y$ replace (5.10) by the less tight upper bound, which is linear in x :

$$\bar{\pi}_Y(d-x)_u = \begin{cases} \left(\frac{k_Y^2}{1+k_Y^2} \right) (d-x), & d - \frac{1}{2}(1+k_Y^2)\mu_Y \leq x \leq d \\ d-x - \frac{1}{2}\mu_Y, & x \leq d - \frac{1}{2}(1+k_Y^2)\mu_Y \end{cases} \quad (5.12)$$

The result follows through calculation. \diamond

5.2. The case $d > L$.

The counterpart to Proposition 5.1 can only be formulated for the upper bound.

Proposition 5.3. Let X and Y be independent random variables with support $[0, \infty)$, and suppose that $d > L$. If $L \leq d - \mu_Y$ one has the upper bound

$$C(L, d)_u = \pi_x(L) F_X(L) \pi_Y(d-L), \quad (5.13)$$

and if $L > d - \mu_Y$ one has

$$C(L, d)_u = \pi_x(L) \cdot \left[F_X(L) \pi_Y(d-L) + (L-d+\mu_Y) \bar{F}_X(L) + \pi_x(L) - \pi_x(d-\mu_Y) \right]. \quad (5.14)$$

Proof. Rewrite (5.4) as

$$C(L, d) = \pi_X(L) \cdot [\bar{\pi}_X(L) + F_X(L)\bar{\pi}_Y(d-L) - A(L, d)], \quad (5.15)$$

with $A(L, d) = E[(d - X - Y)_+ I(X \leq L)]$. The inequality $E[(d - X - Y)_+ | X] \geq (d - \mu_Y - X)_+$ yields the lower bound $A(L, d)_\ell = E[(d - \mu_Y - X) I(X \leq L \wedge (d - \mu_Y))]$. Two sub-cases are distinguished. If $L \leq d - \mu_Y$ insert

$$A(L, d)_\ell = E[(d - \mu_Y - L + L - X) I(X \leq L)] = (d - \mu_Y - L)F_X(L) + \bar{\pi}_X(L)$$

into (5.15) to get after calculation (5.13). If $L > d - \mu_Y$ insert

$$A(L, d)_\ell = E[(d - \mu_Y - X) I(X \leq d - \mu_Y)] = \bar{\pi}_X(d - \mu_Y)$$

into (5.15) to get (5.14). The result is shown. \diamond

Remark 5.1. The calculation of a lower bound based on the inequality $\bar{\pi}_Y(d-x) \leq (d-x)_+$ leads to $C(L, d)_\ell = \pi_X(L)F_X(L) \cdot [\bar{\pi}_Y(d-L) - \mu_Y] < 0$. Since $C(L, d) \geq 0$ by Corollary 4.1, this is not a useful bound.

The calculation of a genuine lower bound and an alternative upper bound uses the integral representation (5.5), rewritten as

$$C(L, d) = \pi_X(L) \cdot [C^1(L, d) + C^2(L, d)], \quad \text{with} \quad (5.16)$$

$$C^1(L, d) = \int_0^L [\bar{\pi}_Y(d-L) - \bar{\pi}_Y(d-x)] dF_X(x) \quad (5.17)$$

$$C^2(L, d) = \int_0^L (L-x)\bar{F}_Y(d-x) dF_X(x) \quad (5.18)$$

Proposition 5.4. Under the assumptions of Proposition 5.3, a lower bound to $C^1(L, d)$ is described as follows :

Case 1 : $d > L + \mu_Y$

$$C^1(L, d)_\ell = 0 \quad (5.19)$$

Case 2 : $L < d \leq L + \mu_Y$

Sub-case 21 : $L^2 \leq \sigma_Y^2 + (\mu_Y - d)^2$

$$C^1(L, d)_\ell = \frac{1}{2}(L + \mu_Y - d)F_X(L) \quad (5.20)$$

Sub-case 22 : $L^2 \geq \sigma_Y^2 + (\mu_Y - d)^2$

$$C^1(L, d)_\ell = \frac{(L + \mu_Y - d)^2}{\sigma_Y^2 + (L + \mu_Y - d)^2} \left[\bar{\pi}_X(L) - \bar{\pi}_X\left(\frac{1}{2} \cdot \frac{L^2 - (L + \mu_Y - d)^2 - \sigma_Y^2}{L + \mu_Y - d}\right) \right] \quad (5.21)$$

A trivial lower bound to $C^2(L, d)$ is $\bar{\pi}_X(L)\bar{F}_Y(d)$ and a more sophisticated one is described as follows :

Case 1 : $d > L + \mu_Y$

$$C^2(L, d)_\ell = 0 \quad (5.22)$$

Case 2 : $L < d \leq L + \mu_Y$

$$\begin{aligned} C^2(L, d)_\ell &= \int_{d-\mu_Y}^L (L-x) \cdot \frac{[x-(d-\mu_Y)]^2}{\sigma_Y^2 + [x-(d-\mu_Y)]^2} dF_X(x) \\ &\geq \frac{1}{\sigma_Y^2 + [L-(d-\mu_Y)]^2} \cdot J_X(L, d), \end{aligned} \quad (5.23)$$

$$J_X(L, d) = (L-d)\pi_X^2(d) - \pi_X^3(d) + (L-d)^2\pi_X(L) + 2(L-d)\pi_X^2(L) + \pi_X^3(L), \quad (5.24)$$

where $\pi_X^k(z) = E[(X-z)_+^k]$ denotes the stop-loss transform of degree k .

Proof. The derivation of our lower bound to $C^1(L, d)$ uses a linear version of the minimum modified stop-loss transform $\pi_Y(d-L, d-x) := \pi_Y(d-L) - \pi_Y(d-x) = \int_{d-x}^{d-L} \bar{F}_Y(y) dy$, which one finds in Goovaerts et al.(1984), p. 357-58, or in a more structured form in Hürlimann(1997b/e). Recall that $\pi_*(d, L) := \min_{Y \in D_2} [\pi_Y(d, L)] = \min_{Y \in D_2} \left[\int_d^L \bar{F}_Y(y) dy \right]$ satisfies the following properties :

$$(1) \quad \pi_*(d, L) = 0 \quad \text{if } d > \mu$$

$$(2a) \quad \pi_*(d, L) = \frac{(d-\mu)^2}{\sigma^2 + (d-\mu)^2} (L-d) \quad \text{if } d \leq \mu \quad \text{and} \quad L \leq \frac{1}{2} \left(d + \mu + \frac{\sigma^2}{\mu-d} \right)$$

$$(2b)$$

$$\pi_*(d, L) = \frac{1}{2} \left[(L-d) + (\mu-d) - \sqrt{\sigma^2 + (L-\mu)^2} \right] \geq \pi_*(d, \frac{1}{2} \left(d + \mu + \frac{\sigma^2}{\mu-d} \right)) = \frac{1}{2} (\mu-d)$$

$$\text{if } d \leq \mu \quad \text{and} \quad L \geq \frac{1}{2} \left(d + \mu + \frac{\sigma^2}{\mu-d} \right)$$

Inserting the corresponding lower bound to $\pi_*(d-L, d-x)$, $x \in [0, L]$, into (5.27), one obtains the desired lower bound as follows :

$$(1) \quad C^1(L, d)_\ell = 0 \quad \text{if } d > L + \mu_Y$$

$$(21) \quad C^1(L, d)_\ell = \int_0^L \frac{1}{2}(L + \mu_Y - d) dF_X(x) = \frac{1}{2}(L + \mu_Y - d)F_X(L)$$

if $L < d \leq L + \mu_Y$ and $L^2 \leq \sigma_Y^2 + (\mu_Y - d)^2$

$$(22) \quad C^1(L, d)_\ell = \int_0^{\frac{1}{2}(d+L-\mu_Y-\frac{\sigma_Y^2}{\mu_Y+L-d})} \frac{1}{2}(L + \mu_Y - d) dF_X(x) \\ + \int_{\frac{1}{2}(d+L-\mu_Y-\frac{\sigma_Y^2}{\mu_Y+L-d})}^L \frac{(L + \mu_Y - d)^2}{\sigma_Y^2 + (L + \mu_Y - d)^2} (L - x) dF_X(x) \\ \text{if } L < d \leq L + \mu_Y \text{ and } L^2 \geq \sigma_Y^2 + (\mu_Y - d)^2.$$

The remaining calculation to get (5.21) is routine. The derivation of our lower bound to $C^2(L, d)$ uses the Cantelli inequalities for $\bar{F}_Y(y)$, $y \in [0, \infty)$, which are a special case of the general Chebyshev-Markov inequalities (e.g. Zelen(1954), Simpson and Welch(1960), Kaas and Goovaerts(1986), Hürlimann(1997e)). Recall that

$$\bar{F}_*(y) := \min_{Y \in D_2} [\bar{F}_Y(y)] = \begin{cases} \frac{(\mu - y)^2}{\sigma^2 + (\mu - y)^2}, & y \leq \mu \\ 0, & y \geq \mu \end{cases}$$

Inserted into (5.18) one obtains the desired bound as follows. If $d > L + \mu_Y$ then $C^2(L, d)_\ell = 0$ and if $L < d \leq L + \mu_Y$ one has

$$C^2(L, d)_\ell = \int_{d-\mu_Y}^L (L - x) \cdot \frac{[x - (d - \mu_Y)]^2}{\sigma_Y^2 + [x - (d - \mu_Y)]^2} dF_X(x) \\ \geq \frac{1}{\sigma_Y^2 + [L - (d - \mu_Y)]^2} \int_{d-\mu_Y}^L (L - x) \cdot [x - (d - \mu_Y)]^2 dF_X(x) \geq \frac{1}{\sigma_Y^2 + [L - (d - \mu_Y)]^2} \cdot J_X(L, d),$$

$$J_X(L, d) = \int_d^L (L - x)(x - d)^2 dF_X(x) \\ = \int_d^\infty [(L - d) + (d - x)](x - d)^2 dF_X(x) + \int_L^\infty (x - L)[(x - L) + (L - d)]^2 dF_X(x),$$

from which (5.24) follows. \diamond

A simple alternative to the upper bound of Proposition 5.3 is the following one. Bounds similar to Proposition 5.4 could also be obtained if required in practical work.

Proposition 5.5. Under the assumptions of Proposition 5.3, an upper bound to $C(L, d)$ is

$$C(L, d)_u = \pi_X(L) \cdot [F_X(L)[\pi_Y(d - L) - \pi_Y(d)] + \bar{\pi}_X(L)\bar{F}_Y(d - L)] \quad (5.25)$$

Proof. For $x \in [0, L]$ one has $\pi_Y(d - x) \geq \pi_Y(d)$ and $\bar{F}_Y(d - x) \geq \bar{F}_Y(d - L)$. The result follows from (5.16) to (5.18). \diamond

6. Practical refined covariance bounds.

The simple covariance bounds of Section 4 depend only on the first and second order partial moments of $X_i, i=1, \dots, n$, X and $X(L)$. By (4.4), (4.6) and Lemma 5.2, the exact value of $Cov[X, Z]$ depends on the distributions of $X_i, i=1, \dots, n$, on $\pi_{X(L)}(d), \sigma_{X(L)}^2(d)$, and on the distributions of $Y_i := \sum_{j \neq i} X_j \wedge L_j, i=1, \dots, n$. Increasing the number of risks arbitrarily, an exact computation of the distributions of all Y_i 's is impractical. Since $X_i \wedge L_i + Y_i = X(L), i=1, \dots, n$, a natural approximation to the exact value is intuitively obtained by setting $Y_i \approx X(L)$. However, this procedure may fail because nothing is known about the involved approximation errors.

When applied to (4.6), the refined covariance bounds of Section 5 will be made dependent only on the distributions of the X_i 's and partial information about $X(L)$. In this sense, the obtained bounds are of similar but improved usefulness when compared with the simple bounds of Section 4. Using the relations

$$\mu_{Y_i} = \mu_{X(L)} - \mu_i + \pi_i(L_i), \quad i=1, \dots, n, \quad (6.1)$$

$$\sigma_{Y_i}^2 = \sigma_{X(L)}^2 - \bar{\sigma}_i^2(L_i), \quad i=1, \dots, n, \quad (6.2)$$

$$k_{Y_i}^2 = \frac{\sigma_{Y_i}^2}{\mu_i^2}, \quad i=1, \dots, n, \quad (6.3)$$

$$\bar{F}_{Y_i}(x) \leq \bar{F}_{X(L)}(x), \quad \pi_{Y_i}(x) \leq \pi_{X(L)}(x), \quad i=1, \dots, n, \quad (6.4)$$

where the latter ones follow from the stochastic order relations $Y_i \leq_{st} X(L), i=1, \dots, n$, the following practical bounds for $Cov[U, V] = \sum_{i=1}^n C_i(L_i, d)$, $C_i(L_i, d)$ bounded according to Section 5, are obtained.

Case $d \leq L_i$:

From (5.6) and (5.9) one obtains the practical bounds :

$$C_i(L_i, d)_u = \pi_i(L_i) \cdot [\bar{\pi}_i(L_i) - \bar{\pi}_i(d - \mu_{Y_i})] \quad (6.5)$$

$$C_i(L_i, d)_\ell = \pi_i(L_i) \cdot \left[\bar{\pi}_i(L_i) - \bar{\pi}_i(d) + \frac{1}{1 + k_{Y_i}^2} \cdot [\bar{\pi}_i(d) - \bar{\pi}_i(d - \frac{1}{2}(1 + k_{Y_i}^2)\mu_{Y_i})] \right] \quad (6.6)$$

Case $d > L_i$:

From (6.4) and Proposition 5.3 one obtains the summarized upper bound :

$$C_i(L_i, d)_u \leq \pi_i(L_i) \cdot \left[F_i(L_i) \pi_{X(L)}(d - L_i) + (L_i - d + \mu_{Y_i})_+ \cdot [\bar{F}_i(L_i) + \pi_i(L_i) - \pi_i(d - \mu_{Y_i})] \right]. \quad (6.7)$$

Observing that $\pi_Y(d-L) - \pi_Y(d) = \int_{d-L}^d \bar{F}_Y(y) dy$ and using (6.4), one gets from Proposition 5.5 the alternate upper bound

$$C_i(L_i, d)_u \leq \pi_i(L_i) \cdot [F_i(L_i) \cdot [\pi_{X(L)}(d-L_i) - \pi_{X(L)}(d)] + \bar{\pi}_i(L_i) \bar{F}_{X(L)}(d-L_i)]. \quad (6.8)$$

Finally, Proposition 5.4 yields the following lower bound

$$C_i(L_i, d)_\ell = \pi_i(L_i) \cdot [C_i^1(L_i, d) + C_i^2(L_i, d)], \text{ where} \quad (6.9)$$

$$C_i^1(L_i, d)_\ell = \begin{cases} 0, & \text{if } d > L_i + \mu_{Y_i} \\ \frac{1}{2}(L_i + \mu_{Y_i} - d)F_i(L_i), & \\ \text{if } L_i < d \leq L_i + \mu_{Y_i} \text{ and } L_i^2 \leq \sigma_{Y_i}^2 + (\mu_{Y_i} - d)^2 & \\ \frac{(L_i + \mu_{Y_i} - d)^2}{\sigma_{Y_i}^2 + (L_i + \mu_{Y_i} - d)^2} \left[\bar{\pi}_i(L_i) - \bar{\pi}_i\left(\frac{1}{2} \cdot \frac{L_i^2 - (L_i + \mu_{Y_i} - d)^2 - \sigma_{Y_i}^2}{L_i + \mu_{Y_i} - d}\right) \right], & \\ \text{if } L_i < d \leq L_i + \mu_{Y_i} \text{ and } L_i^2 \geq \sigma_{Y_i}^2 + (\mu_{Y_i} - d)^2 & \end{cases} \quad (6.10)$$

and

$$C_i^2(L_i, d)_\ell = \begin{cases} 0, & \text{if } d > L_i + \mu_{Y_i} \\ \frac{1}{\sigma_{Y_i}^2 + (L_i + \mu_{Y_i} - d)^2} \cdot J_i(L_i, d), & \text{if } L_i < d \leq L_i + \mu_{Y_i} \end{cases} \quad (6.11)$$

with

$$J_i(L_i, d) = (L_i - d)\pi_i^2(d) - \pi_i^3(d) + (L_i - d)^2\pi_i(L_i) + 2(L_i - d)\pi_i^2(L_i) + \pi_i^3(L_i). \quad (6.12)$$

Remark 6.1. In case $d > L_i$ the upper bounds can be made only dependent on the mean $\mu_{X(L)}$ and variance $\sigma_{X(L)}^2$ of the anticipated aggregate loss. For this one may use the corresponding upper bounds by given mean, variance and support $[0, \infty)$, which are well-known (e.g. Hürlimann(1997e)). For convenience of the reader, the upper bounds are summarized as follows :

$$\pi_{X(L)}(d-L_i) \leq \begin{cases} \mu_{X(L)} - \frac{d-L_i}{1+k_{X(L)}^2}, & \text{if } d \leq L_i + \frac{1}{2}(1+k_{X(L)}^2)\mu_{X(L)} \\ \frac{1}{2}\mu_{X(L)} \left[\sqrt{k_{X(L)}^2 + \left(\frac{d-L_i-\mu_{X(L)}}{\mu_{X(L)}}\right)^2} - \left(\frac{d-L_i-\mu_{X(L)}}{\mu_{X(L)}}\right) \right], & \\ \text{if } d \geq L_i + \frac{1}{2}(1+k_{X(L)}^2)\mu_{X(L)} & \end{cases} \quad (6.13)$$

$$\pi_{X(L)}(d-L_i) - \pi_{X(L)}(d) \leq$$

$$\begin{aligned}
& L_i, \text{ if } d < \mu_{X(L)} \\
& \frac{L_i}{d} \mu_{X(L)}, \text{ if } \mu_{X(L)} \leq d \leq (1+k_{X(L)}^2) \mu_{X(L)} \\
& \mu_{X(L)} - \frac{d-L_i}{1+k_{X(L)}^2}, \text{ if } (1+k_{X(L)}^2) \mu_{X(L)} \leq d \leq L_i + \frac{1}{2}(1+k_{X(L)}^2) \mu_{X(L)} \\
& \frac{1}{2} \mu_{X(L)} \left[\sqrt{k_{X(L)}^2 + \left(\frac{d-L_i-\mu_{X(L)}}{\mu_{X(L)}} \right)^2} - \left(\frac{d-L_i-\mu_{X(L)}}{\mu_{X(L)}} \right) \right], \\
& \text{if } L_i + \frac{1}{2}(1+k_{X(L)}^2) \mu_{X(L)} \leq d \leq L_i + \frac{1}{2} \left(d + \mu_{X(L)} - \frac{\sigma_{X(L)}^2}{d - \mu_{X(L)}} \right) \\
& \frac{\sigma_{X(L)}^2}{\sigma_{X(L)}^2 + (d-L_i-\mu_{X(L)})^2} \cdot L_i, \text{ if } d \geq L_i + \frac{1}{2} \left(d + \mu_{X(L)} - \frac{\sigma_{X(L)}^2}{d - \mu_{X(L)}} \right)
\end{aligned} \tag{6.14}$$

$$\bar{F}_{X(L)}(d-L_i) \leq \begin{cases} 1, & d \leq L_i + \mu_{X(L)} \\ \frac{\mu_{X(L)}}{d-L_i}, & L_i + \mu_{X(L)} \leq d \leq L_i + (1+k_{X(L)}^2) \mu_{X(L)} \\ \frac{\sigma_{X(L)}^2}{\sigma_{X(L)}^2 + (d-L_i-\mu_{X(L)})^2}, & d \geq L_i + (1+k_{X(L)}^2) \mu_{X(L)} \end{cases} \tag{6.15}$$

7. A numerical illustration.

To illustrate our pricing method, consider a portfolio of n independent and identically distributed normal risks $X_i \sim N(\mu, \sigma^2)$ with mean $\mu = 100'000$ and standard deviation $\sigma = \mu$. Then $X \sim N(n \cdot \mu, n \cdot \sigma^2)$ is also normally distributed. Though more of academic interest, this simple situation has some predictive power, as will be seen.

All individual limits are set equal, that is $L_i = L, i = 1, \dots, n$, and $d > L$. According to (2.4) and (2.5) the price of the XL-SL reinsurance payoff equals

$$H[Z] = E[Z] + \frac{\text{Cov}[X, Z]}{\text{Cov}[X, Y]} \cdot \bar{\pi}_{X(L)}(d), \tag{7.1}$$

and the XL and SL components have by (2.1) the values

$$\begin{aligned}
H[U] &= E[U] + \frac{\text{Var}[U] + \text{Cov}[U, V]}{(\text{Var}[U] + \text{Cov}[U, V]) + (\text{Var}[V] + \text{Cov}[U, V])} (H[Z] - E[Z]), \\
H[V] &= E[V] + \frac{\text{Var}[V] + \text{Cov}[U, V]}{(\text{Var}[U] + \text{Cov}[U, V]) + (\text{Var}[V] + \text{Cov}[U, V])} (H[Z] - E[Z]).
\end{aligned} \tag{7.2}$$

From (3.2) one gets

$$\frac{\text{Cov}[X, Z]}{\text{Cov}[X, Y]} = \left(\frac{\sigma_X^2}{\text{Cov}[X, Z]} - 1 \right)^{-1}. \quad (7.3)$$

With (4.1) and (4.2), one obtains in our normal model the simple bound

$$\begin{aligned} \text{Cov}[X, Z] &\leq \text{Cov}[X, U] + \text{Cov}[X, (X - d)_+] \\ &= n \cdot [\sigma_i^2(L) + \pi_i(L)\bar{\pi}_i(L)] + \sigma_X^2(d) + \pi_X(d)\bar{\pi}_X(d) \\ &= n \cdot \sigma^2 \cdot \left[\bar{\Phi}\left(\frac{L - \mu}{\sigma}\right) + \bar{\Phi}\left(\sqrt{n} \cdot \left(\frac{d - \mu}{\sigma}\right)\right) \right], \end{aligned} \quad (7.4)$$

where $\bar{\Phi}(x) = 1 - \Phi(x)$, with $\Phi(x)$ the standard normal distribution. From (7.3) one gets

$$\frac{\text{Cov}[X, Z]}{\text{Cov}[X, Y]} \leq \frac{\bar{\Phi}\left(\frac{L - \mu}{\sigma}\right) + \bar{\Phi}\left(\sqrt{n} \cdot \left(\frac{d - \mu}{\sigma}\right)\right)}{\Phi\left(\frac{L - \mu}{\sigma}\right) - \bar{\Phi}\left(\sqrt{n} \cdot \left(\frac{d - \mu}{\sigma}\right)\right)}. \quad (7.5)$$

For a constant value of $\frac{d}{n}$ ($= (1.2)\mu, (1.5)\mu$ in our numerical example), this upper bound simplifies for sufficiently large n to

$$\frac{\text{Cov}[X, Z]}{\text{Cov}[X, Y]} \leq \frac{\bar{\Phi}\left(\frac{L - \mu}{\sigma}\right)}{\Phi\left(\frac{L - \mu}{\sigma}\right)}. \quad (7.6)$$

By the inequality of Bowers(1969) one has further the upper bound

$$\begin{aligned} \pi_{X(L)}(d) &\leq \frac{1}{2} \left[\sqrt{\sigma_{X(L)}^2 + (d - \mu_{X(L)})^2} - (d - \mu_{X(L)}) \right], \text{ with} \\ \mu_{X(L)} &= n \cdot [\mu - \pi_i(L)], \quad \sigma_{X(L)}^2 = n \cdot \bar{\sigma}_i^2(L), \text{ where for a normal distribution} \\ \pi_i(L) &= \sigma \cdot \left[\Phi\left(\frac{L - \mu}{\sigma}\right) - \left(\frac{L - \mu}{\sigma}\right) \bar{N}\left(\frac{L - \mu}{\sigma}\right) \right], \quad \Phi(x) = N'(x), \\ \bar{\sigma}_i^2(L) &= \sigma^2 \cdot N\left(\frac{L - \mu}{\sigma}\right) - \pi_i(L) \cdot \bar{\pi}_i(L), \quad \bar{\pi}_i(L) = L - \mu + \pi_i(L). \end{aligned} \quad (7.7)$$

This yields a safe bound for $E[V] = \pi_{X(L)}(d)$ as well as for $\bar{\pi}_{X(L)}(d) = d - \mu_{X(L)} + \pi_{X(L)}(d)$ in (7.1). Since $E[U] = n \cdot \pi_i(L)$, an upper bound for (7.1) has been found. It remains to approximate the variance/covariance ratios in (7.2). Usually, the SL reinsurance component is more risky than the XL component. For this reason, we prefer to bound $H[V]$ on the safe side. With (4.2) and (4.3) one gets

$$\begin{aligned} \text{Var}[U] + \text{Cov}[U, V] &\leq \sigma_x^2(d) + \pi_x(d)\bar{\pi}_x(d) = n \cdot \sigma^2 \cdot \bar{\Phi}\left(\sqrt{n} \cdot \left(\frac{d}{n} - \frac{\mu}{\sigma}\right)\right), \\ \text{Var}[U] + \text{Cov}[U, V] &\geq \text{Var}[U] = n \cdot \sigma_i^2(L) = n \cdot \left[\sigma^2 \cdot \bar{\Phi}\left(\frac{L - \mu}{\sigma}\right) - \pi_i(L) \cdot \bar{\pi}_i(L) \right]. \end{aligned} \quad (7.8)$$

Together with the Bowers upper bound for $E[V]$ this yields an upper bound for $H[V]$. The price of the XL component is chosen such that no arbitrage occurs, that is as the difference of safe bounds $H[U] = H[Z] - H[V]$.

Table 7.1 summarizes some typical prices for the limits $L = 2\mu, 3\mu$ and $d = (1.2)E[X], (1.5)E[X]$ in dependence of an increasing number of risks $n = 10, 100, 1000$. Some interesting observations can be made. If $d = (1.2)E[X]$ the size n of the portfolio is a critical parameter. It is only for $n \geq 100$ that the loading factor for $H[Z]$ becomes stable. This is not the case for $d = (1.5)E[X]$, where for $L = 3\mu$ one observes even an increased loading factor for $n = 1000$. In all cases with $n = 1000$ no loading is required to price (even on the safe side) the SL component. This result should be a surprise for more than one actuary. Our economic preference goes to the most stable choice $L = 200'000, d = (1.5)E[X]$.

Table 7.1 : XL-SL reinsurance prices for independent and identical normal risks

size n	Net reinsurance premiums			Loading factors		
	$E[U]$	$E[V]$	$E[Z]$	$\frac{H[U] - E[U]}{E[U]}$	$\frac{H[V] - E[V]}{E[V]}$	$\frac{H[Z] - E[Z]}{E[Z]}$
$L = 200'000, d = (1.2)E[X] = (1.2) \cdot n \cdot \mu$						
10	83'315	55'431	138'746	0.612	3.545	1.784
100	833'155	64'795	897'950	0.578	2.474	0.715
1000	8'331'547	66'122	8'397'669	0.643	0	0.638
$L = 300'000, d = (1.2)E[X] = (1.2) \cdot n \cdot \mu$						
10	8'491	82'495	90'886	0.291	1.385	1.283
100	84'907	109'396	194'303	0.247	0.765	0.538
1000	849'070	114'508	963'578	0.575	0	0.506
$L = 200'000, d = (1.5)E[X] = (1.5) \cdot n \cdot \mu$						
10	83'315	30'587	113'902	1.105	2.505	1.481
100	833'155	32'015	865'170	1.327	0	1.278
1000	8'331'547	32'173	8'363'720	1.321	0	1.316
$L = 300'000, d = (1.5)E[X] = (1.5) \cdot n \cdot \mu$						
10	8'491	43'488	51'979	0.512	0.999	0.919
100	84'907	46'778	131'685	1.407	0	0.907
1000	849'070	47'164	896'234	1.395	0	1.322

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References.

- Aase, K.K.* (1993). Equilibrium in a reinsurance syndicate; existence, uniqueness and characterization. *ASTIN Bulletin* 23, 185-211.
- Borch, K.* (1982). Additive insurance premiums : a note. *The Journal of Finance*, 1295-98. Reprinted in *Borch (1990)*, 192-97.
- Borch, K.* (1990). *Economics of Insurance*. Advanced Textbooks in Economics 29. North-Holland.
- Bowers, N.L.* (1969). An upper bound for the net stop-loss premium. *Transactions of the Society of Actuaries* XIX, 211-16.
- Bühlmann, H.* (1980). An economic premium principle. *ASTIN Bulletin* 11, 52-60.
- De Vylder, F. and M.J. Goovaerts* (1982). Upper and lower bounds on stop-loss premiums in case of known expectation and variance of the risk variable. *Bulletin of the Swiss Association of Actuaries*, 149-64.
- Goovaerts, M.J., DeVylder, F., Haezendonck, J.* (1984). *Insurance Premiums*. North-Holland.
- Hardy, G., Littlewood, J.E. and G. Polya* (1934). *Inequalities*. 8th reprint of 2nd edition (1989). Cambridge University Press.
- Heerwaarden, van A.E.* (1991). Ordering of risks : theory and actuarial applications. Ph.D. Thesis, Tinbergen Research Series no. 20, Amsterdam.
- Heerwaarden, van A.E. and R. Kaas* (1992). The Dutch premium principle. *Insurance : Mathematics and Economics* 11, 129-33.
- Heilmann, W.-R.* (1986). A premium calculation principle for large risks. In : *Operations Research Proceedings*, 342-51. Springer.
- Heilmann, W.-R.* (1988). *Fundamentals of Risk Theory*. Verlag Versicherungswirtschaft, Karlsruhe.
- Hürlimann, W.* (1991). Absicherung des Anlagerisikos, Diskontierung der Passiven und Portfoliotheorie. *Bulletin of the Swiss Association of Actuaries*, 217-50.
- Hürlimann, W.* (1993). An elementary upper bound for the variance of a stop-loss risk. *Bulletin of the Swiss Association of Actuaries*, 97-99.
- Hürlimann, W.* (1994a). Splitting risk and premium calculation. *Bulletin of the Swiss Association of Actuaries*, 167-97.
- Hürlimann, W.* (1994b). A note on experience rating, reinsurance and premium principles. *Insurance : Mathematics and Economics* 14, 197-204.
- Hürlimann, W.* (1995a). A distribution-free all-finance binomial valuation model. *Transactions 25th International Congress of Actuaries, Brussels*, vol. 3, 365-82.
- Hürlimann, W.* (1995b). On fair premium principles and Pareto-optimal risk-neutral portfolio valuation. *Transactions 25th International Congress of Actuaries*, vol. 1, 189-208.
- Hürlimann, W.* (1995c). Transforming, ordering and rating risks. *Bulletin of the Swiss Association of Actuaries*, 213-36.
- Hürlimann, W.* (1995d). CAPM, derivative pricing and hedging. *Proceedings 5th AFIR International Colloquium*, Brussels.
- Hürlimann, W.* (1997a). On quasi-mean value principles. *Blätter der Deutschen Gesellschaft für Versicherungsmathematik XXIII(1)*, 1-16.
- Hürlimann, W.* (1997b). Best bounds for expected financial payoffs (II) applications. *Journal of Computational and Applied Mathematics* 82, 213-227.

- Hürlimann, W.* (1997c). An elementary unified approach to loss variance bounds. *Bulletin of the Swiss Association of Actuaries*, 73-88.
- Hürlimann, W.* (1997d). Fonctions extrémales et gain financier. *Elemente der Mathematik* 52, 152-68.
- Hürlimann, W.* (1997e). Extremal Moment Methods and Stochastic Orders – Application in Actuarial Science. Monograph manuscript (available from the author).
- Hürlimann, W.* (1998a). On distribution-free safe layer-additive pricing. *Insurance : Mathematics and Economics* 22, 277-85.
- Hürlimann, W.* (1998b). Market premiums by the p-norm principle. *Proceedings 8th AFIR International Colloquium*, Cambridge.
- Hürlimann, W.* (1999a). Einführung in die theoretische Prämienkalkulation. Manuscript (available from the author).
- Hürlimann, W.* (1999b). Non-optimality of a linear combination of proportional and non-proportional reinsurance. *Insurance : Mathematics and Economics* 24, 219-227.
- Hürlimann, W.* (1999c). On the loading of a stop-loss contract : a correction on extrapolation and two stable price methods. *Blätter der Deutschen Gesellschaft für Versicherungsmathematik*.
- Jansen, K., Haezendonck, J. and M.J. Goovaerts* (1986). Analytical upper bounds on stop-loss premiums in case of known moments up to the fourth order. *Insurance : Mathematics and Economics* 5, 315-334.
- Jogdeo, K.* (1982). Dependence, concepts of. In : Johnson, N.L., Kotz, S. (Ed.). *Encyclopedia of Statistical Sciences*, vol. 2, 324-34.
- Kaas, R., Goovaerts, M.J.* (1986). Application of the problem of moments to various insurance problems in non-life. In Goovaerts et al.(eds.). *Insurance and Risk Theory*, 79-118. D. Reidel Publishing Company.
- Kremer, E.* (1998). Extrapolating the loading of a stoploss treaty. *Blätter der Deutschen Gesellschaft für Versicherungsmathematik XXIII(4)*, 623-24.
- Schnieper, R.* (1990). Insurance premiums, the insurance market and the need for reinsurance. *Bulletin of the Swiss Association of Actuaries*, 129-47.
- Simpson, J.H. and B.L. Welch* (1960). Table of the bounds of the probability integral when the first four moments are given. *Biometrika* 47, 399-410.
- Venter, G.G.* (1991). Premium calculation implications of reinsurance without arbitrage. *ASTIN Bulletin* 21, 223-30.
- Wallenius, K.T.* (1971). A conditional covariance formula with applications. *The American Statistician*, June, 32-33.
- Zelen, M.* (1954). Bounds on a distribution function that are functions of moments to order four. *Journal of Research of the National Bureau of Standards* 53(6), 377-81.