Risk-minimization with mortality derivatives: mixed dynamic and static hedging

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PFA Pension

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Focus on

- **Mixed dynamic and static hedging:**

- **Valuation and hedging with traditional bonds:**

- **Dynamic hedging with mortality derivatives:**
Financial market and discounted price processes

Savings account $B^* = 1$

Usual risky assets $X = (X^1, \ldots, X^d)$. Traded dynamically

Illiquid asset $Y$. Traded at fixed discrete times
Example: mortality derivative (survivor swaps)

Martingale measure $Q$ for assets $(X, Y)$

Goals:

- Derive optimal mixed dynamic and static risk-minimizing strategies
- Compare results with usual dynamic hedging
- Assess efficiency
Application to survivor swaps

Consider portfolio of insured lives

Survivor swap payment rate =
  current number of survivors – “expected” number of survivors

(“expected” number fixed at time 0. May include market price of risk)

Market price of survivor swap at future times involves
  – conditional expected number of survivors (new information)
  – current zero coupon bond prices

How should we use the survivor swap for hedging?
Illiquid asset. Not realistic with dynamic trading
  Maybe trade one or two times
From continuous to discrete trading

Liability with payment process $A(t), \; t \in [0, T]$
Determine full dynamic strategy $(\xi(t), \vartheta(t), \eta(t)), \; t \in [0, T]$

What if $Y$ only traded at times $0 = t_0 < \cdots < t_n = T$?

Naive guess: use $\vartheta(t) = \vartheta(t_i), \; t \in (t_i, t_{i+1}],$ and $\xi(t)$ unchanged

Not optimal if trend in $\vartheta(t)$ or if $X$ and $Y$ are correlated...
Risk-minimization - full dynamic hedging

(Föllmer/Sondermann, Schweizer, Møller)
Financial market: $X(t), Y(t), B^*(t) = 1$

Trading strategy $\varphi = (\xi, \vartheta, \eta)$

Value process $V(t, \varphi) = \xi(t)X(t) + \vartheta(t)Y(t) + \eta(t)$

Payment process $A = (A(t))_{0 \leq t \leq T}$

Cost process
$C(t, \varphi) = V(t, \varphi) - \int_0^t \xi(u) dX(u) - \int_0^t \vartheta(u) dY(u) + A^*(t)$

Risk process $R(t, \varphi) = E^Q \left[ (C(T, \varphi) - C(t, \varphi))^2 \right| \mathcal{F}(t)]$

Criterion of risk-minimization
Minimize $R(t, \varphi)$ over $\varphi$ for all $t$
Market value decomposition

\[ V^*(t) := E^Q[A^*(T) \mid \mathcal{F}(t)] \]

\[ = V^*(0) + \int_0^t \xi^A(u) \, dX(u) + \int_0^t \vartheta^A(u) \, dY(u) + L^A(t) \]

where

- \( L^A \) is a \( Q \)-martingale
- \((X, Y)\) and \( L^A \) are orthogonal

**Theorem.** Risk-minimizing strategy \( \varphi = (\xi, \vartheta, \eta) \), \( V(T, \varphi) = 0 \):

\[ \xi(t) = \xi^A(t) \quad \text{dynamic strategy for } X \]

\[ \vartheta(t) = \vartheta^A(t) \quad \leftarrow \text{not possible with illiquid asset!} \]

\[ \eta(t) = V^*(t) - A^*(t) - \xi^A(t)X(t) - \vartheta^A(t)Y(t) \]

Minimal risk process

\[ R(t, \varphi) = E^Q \left[ (L^A(T) - L^A(t))^2 \mid \mathcal{F}(t) \right] \]
Obtaining a decomposition with an illiquid asset

"Project" (decompose) illiquid asset on liquid assets:

\[ dY(t) = \xi^Y(t)dX(t) + dL^Y(t) \]

**First term:** Financial risk, e.g. interest rate risk

**Second term:** Non-hedgeable insurance risk

Insert in market value decomposition

\[
\begin{align*}
V^*,Q(t_i) &= V^*,Q(t_0) + \int_0^{t_i} \left( \xi^A(u) + \vartheta^A(u)\xi^Y(u) \right) dX(u) \\
&\quad + \sum_{j=1}^{i} \left( \int_{t_{j-1}}^{t_j} \vartheta^A(u) dL^Y(u) + \Delta L^A(t_j) \right)
\end{align*}
\]

with \( \Delta L^A(t_j) = L^A(t_j) - L^A(t_{j-1}) \).

**Leads to decomposition with orthogonal terms**
The mixed discrete and continuous strategy

Minimizing the risk process

\[ R(t, \varphi) = \mathbb{E}^Q \left[ (C(T, \varphi) - C(t, \varphi))^2 \mid \mathcal{F}(t) \right] \]

leads to minimization of

\[ \mathbb{E}^Q \left[ \left( \int_{t_i}^{t_{i+1}} \vartheta^A(u) dL^Y(u) - \vartheta(t_i) \Delta L^Y(t_{i+1}) \right)^2 \mid \mathcal{F}(t_i) \right] \]

Usual quadratic problem with solution

\[ \hat{\vartheta}(t_i) = \frac{\mathbb{E}^Q \left[ \int_{t_i}^{t_{i+1}} \vartheta^A(u) dL^Y(u) \Delta L^Y(t_{i+1}) \mid \mathcal{F}(t_i) \right]}{\mathbb{E}^Q \left[ (\Delta L^Y(t_{i+1}))^2 \mid \mathcal{F}(t_i) \right]} \]

**Interpretation:** Optimal position \( \hat{\vartheta}(t_i) \) is a risk-adjusted average of the future dynamic strategy \( \vartheta^A(t) \) on interval \((t_i, t_{i+1}]\).
Mixed dynamic and static hedging

**Theorem**
The unique mixed discrete- and continuous-time risk-minimizing strategy associated with $A^*$ is

$$
\hat{\vartheta}^*(t) = \hat{\vartheta}^A(t_{j-1}) = \frac{E^Q \left[ \int_{t_{j-1}}^{t_j} \vartheta^A(u)dL^Y(u)\Delta L^Y(t_j) \bigg| \mathcal{F}(t_{j-1}) \right]}{E^Q \left[ (\Delta L^Y(t_j))^2 \bigg| \mathcal{F}(t_{j-1}) \right]}
$$

$$
\hat{\xi}^*(t) = \hat{\xi}^A(t) = \xi^A(t) + \xi^Y(t)(\vartheta^A(t) - \hat{\vartheta}^A(t_{j-1}))
$$

$$
\eta(t) = V^{*,Q}(t) - A^*(t) - \hat{\xi}^A(t)X(t) - \hat{\vartheta}^A(t_{j-1})Y(t)
$$

$t \in (t_{j-1}, t_j]$.

**Note**: Correction term for $\xi(t)$ arises to adjust for modified risk from illiquid asset
Modeling the mortality intensity:

**Known at time 0:**
\[ \mu^\circ(x + t) \] is mortality intensity “today” at all ages \( x + t \)

**Unknown at time 0:**
\[ \zeta(x, t) \] is relative change in the mortality from 0 to \( t \), age \( x \)

**Stochastic mortality intensity:**
\[ \mu(x, t) = \mu^\circ(x + t) \zeta(x, t) \]

True survival probability from \( t \) to \( T \) given information \( \mathcal{I}(t) \):
\[
S(x, t, T) = E^P \left[ e^{-\int_t^T \mu(x, \tau) d\tau} \middle| \mathcal{I}(t) \right]
\]
The model:

2-dim. time-inhomogeneous CIR model known from finance:

\[ d\zeta(x, t) = (\gamma(x, t) - \delta(x, t)\zeta(x, t))dt + \sigma(x, t)\sqrt{\zeta(x, t)}dW^\mu(t) \]

Survival probability in affine model

\[ S(x, t, T) = e^{A^\mu(x, t, T) - B^\mu(x, t, T)\mu(x, t)} \]

Forward mortality intensity

\[ f^\mu(x, t, T) = -\frac{\partial}{\partial T} \log S(x, t, T) \]

Financial market

Vasiček model for short rate:

\[ dr(t) = (\gamma^{r,\alpha} - \delta^{r,\alpha} r(t)) dt + \sqrt{\gamma^{r,\sigma}} dW^r(t) \]

Zero coupon bond prices

\[ P(t, T) = e^{A^r(t, T) - B^r(t, T)r(t)} \]
Mortality improvement distribution

Parameters:

<table>
<thead>
<tr>
<th></th>
<th>Pf. 1</th>
<th>Pf. 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_j$</td>
<td>0.0001800</td>
<td>0.0001805</td>
</tr>
<tr>
<td>$\delta_j$</td>
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<tr>
<td>$\sigma_{j,1}$</td>
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<td>0.000</td>
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<tr>
<td>$\sigma_{j,2}$</td>
<td>0.018</td>
<td>0.019</td>
</tr>
</tbody>
</table>

Initial values:

$\zeta_j(x, 0) = 1$

Mean reversion levels:

$\frac{\gamma_j}{\delta_j}$

- Portfolio 1: 0.0225
- Portfolio 2: 0.0223
Two portfolios of insured lives

Counting processes and martingales

\[ N_j(x, t) = \sum_{i=1}^{n_j} 1(T_{j,i} \leq t) \]

\[ M_j(x, t) = N_j(x, t) - \int_0^t (n_j - N_j(x, u-)) \mu_j(x, u) du \]

Insurance payment process (Benefits – premiums on pf 1)

\[ dA(t) = (n_1 - N_1(x, \bar{T})) \Delta A_0(\bar{T}) d1_{t \geq \bar{T}} \]

\[ + a_0(t)(n_1 - N_1(x, t)) dt + a_1(t)dN_1(x, t) \]

\((a_i, A_0 \text{ deterministic functions})\)
Market value process

\[
V^{*,Q}(t) = E^Q \left[ \int_{[0,T]} e^{-\int_0^\tau r(u)du} dA(\tau) \mid \mathcal{F}(t) \right]
\]

\[
= \int_{[0,t]} e^{-\int_0^\tau r(u)du} dA(\tau)
\]

\[
+ e^{-\int_0^t r(u)du} (n - N_1(x, t)) V^Q(t, r(t), \mu_1(x, t))
\]

where

\[
V^Q(t, r(t), \mu_1(x, t))
\]

\[
= \int_{t}^{T} P(t, \tau) S_1^Q(x, t, \tau) \left( a_0(\tau) + a_1(\tau) f^{\mu_1,Q}(x, t, \tau) \right) d\tau
\]

\[
+ P(t, \bar{T}) S_1^Q(x, t, \bar{T}) \Delta A_0(\bar{T}) 1(t < \bar{T})
\]

Survivor swap price process

\[
\tilde{Z}_j^*(x, t) = (n_j - N_j(x, t)) \int_t^T P^*(t, u) S_j^Q(x, t, u) du
\]

\[
- n_j t \tilde{\rho}_x \int_t^T P^*(t, u)_{u-t} \tilde{\rho}_{x+t} du
\]
Price and market value dynamics

Zero coupon bond price dynamics:

\[ dP^*(t, T) = -\sigma^r B^r(t, T)P^*(t, T)dW^{r, Q}(t) \]

Survivor swap price dynamics:

\[ dZ^*_{j, Q}(x, t) = \nu^Z_{j, Q}(t)dM^Q_j(x, t) + \eta^Z_{j, Q}(t)dW^{r, Q}(t) + \rho^Z_{j, Q}(t)dW^{\mu, Q}(t) \]

**Market risk:** \( \eta^Z_{j, Q}(t)dW^{r, Q}(t) \)

**Remaining risk:** Unsystematic and systematic mortality risk

Market value dynamics:

\[ dV^*_{j, Q}(t) = \nu^V_{j, Q}(t)dM^Q_1(x, t) + \eta^V_{j, Q}(t)dW^{r, Q}(t) + \rho^V_{j, Q}(t)dW^{\mu, Q}(t) \]

Market value decomposition:

\[ dV^*_{j, Q}(t) = \xi^A_j(t)dX(t) + \vartheta^A_j(t)dZ^*_j(x, t) + dL^A_j(t) \]
Dependence on the number of policy-holders \( n_1 \)

Survivor swaps on portfolio 1:

\[
\vartheta^A_1(t) = \frac{\nu^{V,Q}(t) + \rho_1^{V,Q}(t)(\kappa_{1,1}^Q(t))^{-1} + \rho_2^{V,Q}(t)(\kappa_{1,2}^Q(t))^{-1}}{\nu_1^{Z,Q}(t) + \rho_{1,1}^{Z,Q}(t)(\kappa_{1,1}^Q(t))^{-1} + \rho_{1,2}^{Z,Q}(t)(\kappa_{1,2}^Q(t))^{-1}}
\]

Here: \( \rho_j^{V,Q}(t) \) and \( \rho_{1,j}^{Z,Q}(t) \) are proportional to \( n_1 \)
Comparison of discrete and continuous strategy

Discrete strategy:

\[
\hat{\vartheta}_j^A(0) = \frac{\mathbb{E}_Q \left[ \int_0^T \vartheta_j^A(u) dL_j^*, Q(u) \Delta L_j^*, Q(T) \bigg| \mathcal{F}(0) \right]}{\mathbb{E}_Q \left[ \left( \Delta L_j^*, Q(T) \right)^2 \bigg| \mathcal{F}(0) \right]}
\]
Intrinsic risk - continuous hedging

<table>
<thead>
<tr>
<th>$n_1$</th>
<th>$n_2$</th>
<th>$\sqrt{R(0,\varphi^*<em>V)}</em>{n_1}$</th>
<th>$\sqrt{R(0,\varphi^*<em>B)}</em>{n_1}$</th>
<th>$\sqrt{R(0,\varphi^*<em>1)}</em>{n_1}$</th>
<th>$\sqrt{R(0,\varphi^*<em>2)}</em>{n_1}$</th>
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<td>0.111</td>
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<tr>
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<td>0.627</td>
<td>0.062</td>
<td>0.032</td>
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<td>0.035</td>
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<tr>
<td>10,000</td>
<td>100,000</td>
<td>0.626</td>
<td>0.055</td>
<td>0.013</td>
<td>0.020</td>
</tr>
</tbody>
</table>

Notation

$\varphi^*_V$: No hedging (only savings account)

$\varphi^*_B$: Bond market

$\varphi^*_1$: Bond market + survivor swap on portfolio 1

$\varphi^*_2$: Bond market + survivor swap on portfolio 2
Intrinsic risk - mixed hedging

<table>
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<tr>
<th></th>
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<th>( \sqrt{R(0,\varphi_{D_j}^*)} )</th>
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<th>( \sqrt{R(0,\varphi_{D_j}^*)} )</th>
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<tr>
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<td>0.105</td>
<td>0.104</td>
<td>0.073</td>
<td>0.104</td>
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<tr>
<td>100</td>
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<td>0.105</td>
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<td>0.073</td>
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<tr>
<td>1,000</td>
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<td>0.045</td>
<td>0.040</td>
<td>0.038</td>
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<td>0.037</td>
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<tr>
<td>10,000</td>
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<td>0.022</td>
<td>0.026</td>
<td>0.019</td>
<td>0.024</td>
</tr>
</tbody>
</table>

**Notation**

- \( D_j^1 \): Constant swap for portfolio \( j \) on \((0, 60]\)
- \( D_j^2 \): Trading swap for portfolio \( j \) on \((0, 30]\) and \((30, 60]\)