Pension Fund Management Based on
Constrained Consumption-Investment

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Based on work with Holger Kraft

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• Classical and modern asset management and LDI

• Terminal wealth problems with VaR constraints

• Intermediate payments

• Intermediate constraints
1 Classical and modern asset management and LDI

- Classical asset management: mean-variance-optimization, one-period-model

\[
\max_{\pi: \text{Var}[A] < k} E[A] \quad \text{versus} \quad \min_{\pi: E[A] > k} \text{Var}[A]
\]

Figure 1: Harry Markowitz (1927- )
• Modern asset management: utility optimization, dynamic strategies

$$\max \pi E [u (A)]$$

Figure 2: Robert C. Merton (1944- )
LDI (Liability-Driven Investment) is essentially involving $L$ in the objective and/or the constraint

\[
\begin{align*}
\max_{\pi : A > L} E[u(A)] \\
\max_{\pi} E[u(A - L)] \\
\max_{\pi : P(A > L) \geq 1 - \varepsilon} E[u(A)] \\
\max_{\pi : E^Q[(L - A)^+] \leq \varepsilon} E[u(A)]
\end{align*}
\]

What is the time horizon of the objectives and/or the constraint? And what about intermediate payments and/or constraints?
2 Terminal wealth problems with VaR constraints

\[
\max_{\pi: P(A(T) > L(T)) \geq 1 - \varepsilon} E[u(A(T))]
\]

- Invest first \(A'(0)\) in the optimal portfolio solving

\[
\max_{\pi} E[u(A'(T))]
\]

- Buy, for the residual amount, \(A(0) - A'(0)\) an option with payoff

\[
(L(T) - A'(T)) 1[l < A'(T) < L(T)]
\]

The two positions sum up to the claim (with value \(A(0)\))

\[
\max(A'(T), L(T) 1[l < A'(T)])
\]
Figure 3: Utility function, auxiliary utility function with Lagrange term, and concavifying line
Figure 4: Optimal wealth and utility of optimal wealth as a function of $A'$
3 Intermediate payments

\( A(T) \) finances also a payment of \( C \) at time \( T_1 < T_2 \) (generalizes to \( n \) periods):

\[
\max_{\pi: P(C(T_1) > K(T_1)) \geq 1 - \varepsilon_1, P(A(T_2) > L(T_2)) \geq 1 - \varepsilon_2} E [u(C(T_1)) + u(A(T_2))]
\]

- Allocate the initial capital \( A \) to the two 'projects': \( A^1(0) + A^2(0) \)

- Solve each of the two terminal value problems separately
• Calculate the allocation $A^1(0)$ such that

$$\frac{\partial}{\partial A^1(0)} \max_{P(A^1(T_1) > K(T_1)) \geq 1 - \epsilon_1} E \left[ u \left( A^1(T_1) \right) \right]$$

$$= \frac{\partial}{\partial A^2(0)} \max_{P(A^2(T_2) > L(T_2)) \geq 1 - \epsilon_2} E \left[ u \left( A^2(T_2) \right) \right]$$
4 Intermediate constraints

$A$ exceeds $L$ at time $T_1$ and time $T_2 > T_1$ (generalizes to $n$ periods):

$$\max_{\pi : P(A(T_1) > L(T_1)) > 1 - \varepsilon_1 \quad P(A(T_2) > L(T_2)) > 1 - \varepsilon_2} E[u(A(T_2))]$$

- Invest first $A''(0)$ in the optimal portfolio solving
  $$\max_{\pi} E[u(A''(T_2))]$$

- Buy, for the amount $A'(0) - A''(0)$ a put option leading to the total time $T_2$ payoff
  $$\max \left( A''(T_2) , L(T_2) \mathbf{1}[l_2 < A''(T_2)] \right)$$
with value process $A'(t)$

• Buy, for the residual amount $A(0) - A'(0)$ a put option leading to the total time $T_1$ payoff

$$\max(A'(T_1), L(T_1) \ 1[l_1 < A''(T_2)])$$

with value $A(0)$