QUASI-EXACT NUMERICAL EVALUATION OF SYNTHETIC CDO PRICES

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AGENDA

• Sub-prime mortgage crisis
  • Facts & aftermath / Wikipedia / New York Times
  • Lessons / nature & magnitude of credit risk transfer

• Pricing model
  • synthetic CDO tranche / fair spread / one-factor copula model / one-factor Gaussian copula model

• Evaluation
  • conditional tranche loss / pseudo compound Poisson approximation / Panjer recursive algorithm / Examples
SUB-PRIME MORTGAGE CRISIS

• Facts & aftermath

  – Wikipedia

    • Merrill Lynch's 2008 large losses partly due to drop in value of un-hedged portfolio of CDO’s after AIG ceased offering CDS’s on Merrill's CDO’s

  – New York Times (May 14, 2009)

    • “New Regulations Sought on Derivatives”: swaps and other derivatives should be traded on exchanges and backed by capital reserves (T.F. Geithner from Obama administration)
SUB-PRIME MORTGAGE CRISIS

• Lessons learnt from the crisis
  – Nature of credit risk transfer
    • Risk characteristics of CDO’s (Gibson(2004))
  – Magnitude of credit risk transfer
    • Loss distributions of synthetic CDO’s
    • Evaluation methods: analytical, semi-analytical, Monte-Carlo simulation, exact calculation
    • New quasi-exact numerical method: recursive algorithm for pseudo compound Poisson approximation (Hipp, Hürlimann, Michel)
Pricing model / synthetic CDO tranche

- synthetic collateralized debt obligation (CDO) transfers the credit risk on a portfolio of credit default swaps (CDS).
- synthetic CDO pool made up of \( m \) names with notional values \( N(k) \) and recovery rate \( R(k) \), \( k=1,…,m \). The loss-given-default of name \( k \) is \( \text{LGD}(k)=N(k)\cdot\{1-R(k)\} \).
- premium dates: \( 0=t(0)<t(1)<…<t(n)=T (=\text{maturity date}) \)
- discount factors: \( D(0), D(1),…, D(n) \)
- synthetic CDO tranche of size \( S \) and attachment point \( \ell \) has payoff function at time \( t(i) \): \( L(i)=\min\{\max(Lp(i)-\ell,0),S\} \), with \( Lp(i) \) the pool’s cumulative losses up to time \( t(i) \) (similar to limited stop-loss reinsurance in actuarial science)
Pricing model / fair spread

- Let \( s \) be the *fair spread* of CDO tranche per annum
- **default leg** (=losses to maturity): \( DL = \sum D(i) \cdot \{L(i) - L(i-1)\} \)
- **contingent** (=PV of default leg): \( PV(DL) = E[DL] \)
- **premium leg** (=premiums to maturity)
  \[ PL = s \cdot \sum D(i) \cdot \Delta(i) \cdot \{S - L(i)\}, \quad \Delta(i) = t(i) - t(i-1) \]
- **fee** (=PV of premium leg): \( PV(PL) = E[PL] \)
- **market-to-market value** CDO tranche: \( MV = \text{fee} - \text{contingent} \)
- Pricing equation fair spread is \( MV = 0 \) or \( E[PL - DL] = 0 \)
- Valuation problem reduced to computation of \( E[L(i)] \leftrightarrow \) specification of default process for each name & correlation structure of default events
Pricing model / one-factor model

- T(k) : random default time of name k
- q(k,i)=P(T(k)<t(i)) : risk-neutral default probabilities
- dependence structure of default times determined by creditworthiness indices

\[ Y(k) = \sqrt{\rho(k)} \cdot X + \sqrt{1-\rho(k)} \cdot Z(k), \]

with systematic risk factor X, mutually independent idiosyncratic factors Z(k), all independent of X, and correlation factors \( \rho(k) \), through one-factor copula model:

\[ q(k,i) = P( Y(k) < H(k,t(i)) ) \] with

\[ H(k,t(i)) : default threshold \] of name k at time t(i)

Pricing model / one-factor Gaussian

• Assume *standard normal* distributions for $X$ and $Z(k)$:
  (i) $H(k,t(i)) = \Phi^{-1}(q(k,i))$
  (ii) $\text{Cov}[Y(k),Y(j)] = \sqrt{\rho(k)\rho(j)}$
  (iii) $q(k,i) = P(Y(k)<H(k,t(i)) | X=x)$ (conditional default
       $= \Phi\{[\Phi^\top-1(q(k,i)) - x\sqrt{\rho(k)]/\sqrt{1-\rho(k)}} \}$ probabilities

• Model extensions:
  • $X$ random vector => multi-factor copula model
  • $X$, $Z(k)$ Student-t => double-t copula model of Hull & White
Evaluation / conditional tranche loss

• In one-factor Gaussian model one has

\[ E[L(i)] = \int E[L(i) | X=x] \cdot d\Phi(x) \] with

\[ E[L(i) | X=x] = E[L(i)=\min\{\max(Lp(i)-\ell,0),S\} | X=x] \] and

\[ Lp(i) = \sum LGD(k) \cdot I\{ Y(k) < \Phi^{-1}(q(k,i)) \}, \] where the random indicators \( I\{ \cdot \} \) are mutually independent conditional on \( X \)

\Rightarrow \text{reduction to computation of conditional expected cumulative tranche losses} \ E[L(i) | X=x], \ i=1,\ldots,n

conditional on \( X=x \) the characteristic function of \( L_p(i) \) is
\[
\varphi(t) = \prod \varphi(k,t), \quad \varphi(k,t) = \exp\{ \ln[1+c(k)\cdot(e^{it\cdot\text{LGD}(k)}-1)] \}
\]
• J-th order appr. \( \varphi(J,t) \): truncate logarithmic expansion
\[
\ln(1+x) = \sum (-1)^{j+1} \cdot x^j/j \quad \text{at \ J-th term to get finite sums}
\]
\[
\ln \varphi(J,t) = \sum \sum (-1)^{j+1} \cdot [c(k)\cdot(e^{it\cdot\text{LGD}(k)}-1)]^j/j
\]
• J=1: compound Poisson approximation (\( \lambda(1), h(1,y) \))
\[
\ln \varphi(1,t) = \lambda(1)\cdot[\psi(1,t)-1], \quad \psi(1,t) = 1/\lambda(1)\cdot\sum c(k)\cdot e^{it\cdot\text{LGD}(k)}
\]
\[
\lambda(1) = \sum c(k), \quad h(1,y) = 1/\lambda(1)\cdot\sum c(k)\cdot I\{\text{LGD}(k)=y\}, \quad y=1,2,\ldots
\]
• J>1: pseudo compound Poisson approximation
\[
\ln \varphi(J,t) = \lambda(J)\cdot[\psi(J,t)-1], \quad \text{Poisson parameter} \ \lambda(J) \ \text{and}
\]
\[
\text{pseudo severity distribution} \ h(J,y) \ \text{determined as follows:}
\]
pseudo compound Poisson approximations of low order:

**J=2:**
\[
\lambda(2) = \sum c(k) \cdot (1 + \frac{1}{2} \cdot c(k)) \\
h(2, y) = \frac{1}{\lambda(2)} \cdot \left[ \sum c(k)(1+c(k)) \cdot I\{LGD(k)=y\} \right. \\
- \frac{1}{2} \sum c(k)^2 \cdot I\{2\cdot LGD(k)=y\}, \quad y=1,2,\ldots
\]

**J=3:**
\[
\lambda(3) = \sum c(k) \cdot (1 + \frac{1}{2} \cdot c(k) + \frac{1}{3} \cdot c(k)^2) \\
h(3, y) = \frac{1}{\lambda(3)} \cdot \left[ \sum c(k)(1+c(k)+c(k)^2) \cdot I\{LGD(k)=y\} \right. \\
- \sum c(k)^2(\frac{1}{2}+c(k)) \cdot I\{2\cdot LGD(k)=y\} + \frac{1}{3} \sum c(k)^3 \cdot I\{3\cdot LGD(k)=y\} \right],
\]

**J=4:**
\[
\lambda(4) = \sum c(k) \cdot (1 + \frac{1}{2} \cdot c(k) + \frac{1}{3} \cdot c(k)^2 + \frac{1}{4} \cdot c(k)^3) \\
h(4, y) = \frac{1}{\lambda(4)} \cdot \left[ \sum c(k)(1+c(k)+c(k)^2+c(k)^3) \cdot I\{LGD(k)=y\} \right. \\
- \sum c(k)^2(\frac{1}{2}+c(k)+\frac{1}{2} \cdot 3 \cdot c(k)^2) \cdot I\{2\cdot LGD(k)=y\} \\
+ \sum c(k)^3(\frac{1}{3}+c(k)) \cdot I\{3\cdot LGD(k)=y\} - \frac{1}{4} \sum c(k)^4 \cdot I\{4\cdot LGD(k)=y\} \right],
\]
Evaluation / pseudo comp. Poisson appr.

- some mathematical properties:
  - $h(J,y)$ for $J>1$ is not a true probability measure but only a signed measure
  - conditions for a pseudo compound Poisson distribution $(\lambda, h(y))$ to be a true probability distribution identified by Lévy(1937) (see Lukacs(1970), Johnson et al.(1992)):
    a negative value $h(y)<0$ is preceded by a positive value and followed by at least two positive values
  - CDO examples: criterion is fulfilled for $J=3$ but not for $J=2,4$
- solution to moment problem: $J$-th order approximation preserves first $J$ moments (Dhaene et al.(1996))
Evaluation / Panjer recursive algorithm

• probability function $f(z)$, $z=0,1,2,...$ of a pseudo compound Poisson distribution $(\lambda, h(y))$ satisfies **Panjer recursion formula** (Hürlimann(1990)):

$$f(0) = \exp(-\lambda), \quad z \cdot f(z) = \sum y \cdot h(y) \cdot f(z-y), \quad z=1,2,...$$

• numerical **stable algorithm** (Panjer & Wang(1993))

• **convergence** of J-th order approximation in distribution obtained from the **error bound** (Hipp & Michel(1990)):

$$|F(z) - F^J(z)| \leq \exp(\varepsilon) - 1, \quad \varepsilon = \sum \varepsilon(k), \text{ with}$$

$$\varepsilon(k) = [1/(J-1)] \cdot [2c(k)]^{(J-1)}/[1-2c(k)], \quad c(k)<1/2$$
Evaluation / Numerical examples

- **Example 1**: completely homogenous pool
  
m = 100 names, t(i)=i=1,2,3,4,5 dates, LGD(k)=1 all k, risk-neutral default probabilities q(k,i)=q(i)=1−exp(−0.01·i), correlation factors ρ(k)=ρ=0.3, flat interest rates of 5%
  
The cumulative losses L_p(i) are binomially distributed and converge to the Vasicek limiting distribution as m → ∞

- **Table 1**: par spreads for completely homogeneous pool

<table>
<thead>
<tr>
<th>CDO tranche</th>
<th>J=1</th>
<th>J=2</th>
<th>J=3</th>
<th>J=4</th>
<th>exact</th>
<th>Vasicek</th>
</tr>
</thead>
<tbody>
<tr>
<td>mezzanine</td>
<td>6.004%</td>
<td>6.024%</td>
<td>6.024%</td>
<td>6.024%</td>
<td>6.024%</td>
<td>6.488%</td>
</tr>
<tr>
<td>senior</td>
<td>0.271%</td>
<td>0.269%</td>
<td>0.269%</td>
<td>0.269%</td>
<td>0.269%</td>
<td>0.201%</td>
</tr>
</tbody>
</table>
Evaluation / Numerical examples

• **Example 2**: inhomogeneous pool
  5 sub-pools with 20 names each with LGD(k)=k=1,2,3,4,5, t(i)=i=1,2,3,4,5 dates, q(k,i)=1−exp{−(0.005+0.005⋅k)⋅i}, correlation factors ρ(k)=0.25+0.05⋅k, flat interest rates of 5%

• **Table 2**: par spreads for inhomogeneous pool

<table>
<thead>
<tr>
<th>CDO tranches</th>
<th>par spread for different distributions</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>J=1</td>
</tr>
<tr>
<td>equity</td>
<td>25.954%</td>
</tr>
<tr>
<td>mezzanine</td>
<td>11.002%</td>
</tr>
<tr>
<td>senior</td>
<td>3.002%</td>
</tr>
</tbody>
</table>

• **Conclusions**:
  J=3,4 => quasi-exact spreads for synthetic CDO tranches
  J=2  => almost accurate spreads
  J=1  => compound Poisson approximation not reliable