Canonical Valuation of Mortality-linked Securities

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- Mortality-linked Securities are becoming more popular.

- Pricing these securities is not straightforward.

- Reasons:
  - Incomplete market.
  - A lack of liquidly traded longevity indexes or securities.
  - A replicating hedge cannot be formed.

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- Let $F_P(x)$ be the d.f. for a future lifetime r.v. under $P$.

- Then, under $Q$, the distribution function for the r.v. is

$$F_Q(x) = \Phi(\Phi^{-1}(F_P(x)) + \lambda),$$

where $\Phi$ is the d.f. for a standard normal r.v., and $\lambda$ is the market price of risk.
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  1. Lin and Cox (2005): calibrate $\lambda$ to market quotes of immediate annuities.

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Step (1): Define a mortality model in $P$ measure.

E.g., Cairns, Blake and Dowd (2006) model:

$$\ln \frac{q_{x,t}}{1 - q_{x,t}} = A_1(t) + A_2(t)(x + t),$$

$$A(t + 1) = A(t) + \mu + CZ(t + 1),$$

where

- $A(t) = (A_1(t), A_2(t))'$,
- $\mu$ is a constant $2 \times 1$ vector,
- $C$ is a constant $2 \times 2$ upper triangular matrix,
- $Z(t)$ is a 2-dimensional standard normal random variable.
Risk-neutral dynamics of death/survival rates

Step (2): Adjust the drift term to obtain a model in $Q$ measure:

$$A(t+1) = A(t) + \tilde{\mu} + C\tilde{Z}(t+1),$$

where

- $\tilde{\mu} = \mu - C\lambda$,
- $\tilde{Z}(t+1)$ is a standard 2-dim. normal r.v. under the $Q$-measure,
- $\lambda = (\lambda_1, \lambda_2)'$ is a vector of market prices of risk.

Cairns, Blake and Dowd (2006) obtain $\lambda$ by calibrating to the price of the BNP/EIB longevity bond.
Risk-neutral dynamics of death/survival rates

- Problem (1): Parameter risk.
  - Even if the process is correct, parameters may be wrong.
  - Can be quantified by MCMC.

- Problem (2): Model risk.
  - The process itself may be incorrect.
  - May be reduced by considering a less stringent mortality model.
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Introduction
The Theory of Canonical Valuation
Non-Parametric Mortality Forecasting
An Equivalent Martingale Measure
An Illustration

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‘Canonical valuation’ (Stutzer, 1996) as an alternative pricing method.

Advantages:

- Largely non-parametric – reducing parameter and model risk.
- Useful even if only a few market prices are available.

Our objective: to develop a framework for pricing mortality-linked securities using canonical valuation.
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The Principle

- Assume there are \( m \) distinct primary securities.

- Each has a time-zero price of \( F_i \) and a random discounted payoff of \( f_i(\omega) \).

- Let \( Q \) is the set of all equivalent martingale measures.

- We require, for any \( Q \) in \( Q \),

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E^Q[f_i(\omega)] = F_i, \quad i = 1, 2, \ldots, m. \tag{1}
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The Kullback-Leibler (1951) information criterion (KLIC):

\[ D(Q, P) = \mathbb{E}^P \left[ \frac{dQ}{dP} \ln \frac{dQ}{dP} \right] \]

We should choose an equivalent martingale measure \( Q_0 \) that minimizes the criterion, i.e.,

\[ Q_0 = \arg \min_{Q \in \mathcal{Q}} D(Q, P), \]

subject to the constraints in equation (1).
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Statistical Justifications

- $D(Q, P)$ represents the information gained by moving from $P$ to $Q$.

- From a Bayesian viewpoint, we may regard $P$ as the prior distribution.

- Given $m$ market prices, we can update the prior by incorporating the information contained in equation (1).

- No information other than equation (1) should be incorporated.
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A Geometric Interpretation

the set of all measures equivalent to $P$

the set of all equivalent martingale measures

$P$

$Q_0$
Rittelli (2000) proved that maximizing the expected exponential utility is equivalent to minimizing the KLIC.

The result also holds true in a multi-period setting.

It implies linkages to the Esscher transform (Gerber and Shiu, 1994).
Expected Utility Hypothesis

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Generate $N$ equally probable scenarios by the bootstrap.

The p.f. for $\omega$ under $P$

$$\Pr(\omega = \omega_j) = \pi_j = \frac{1}{N}, \quad j = 1, 2, \ldots, N.$$  

Let $\pi_j^*, j = 1, 2, \ldots, N$, be the p.f. of $\omega$ under $Q$.

We require

$$\sum_{j=1}^{N} f_i(\omega_j)\pi_j^* = F_i, \quad i = 1, 2, \ldots, m. \quad (2)$$
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- We can rewrite KLIC as

\[ \sum_{j=1}^{N} \pi_j^* \ln \left( \frac{\pi_j^*}{\pi_j} \right). \]

- To find the canonical measure \( Q_0 \), we solve

\[ Q_0 = \arg \min_{\pi_j^*} \sum_{j=1}^{N} \pi_j^* \ln \left( \frac{\pi_j^*}{\pi_j} \right), \]

subject to \( \sum_{j=1}^{N} \pi_j^* = 1 \) and equation (2).
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The Challenge

- An empirical distribution of the mortality-linked security’s payoff is needed.

- Generate from a time-series of past mortality rates or values of a longevity index.

- The data involve two dimensions: age and time.

- Potential dependency over both dimensions.
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Age Dependency

- Mortality rates at different ages are correlated with one another.

- Wills and Sherris (2008) point out that it is a critical factor in pricing mortality-linked securities.

- We consider mortality rates at different ages jointly by treating them as a vector.

- That is, we treat the data as a multivariate time-series of:
  
  \[ m_t = (m_{65,t}, m_{66,t}, \ldots, m_{90,t})' \]
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Time Dependency

Central death rates at representative ages.
We require the time-series to be weakly stationary.

$m(x, t)$ has a clear downward trend, suggesting it is not weakly stationary.

To solve this problem, we consider the transformation of

$$r_{x,t} = \frac{m_{x,t+1}}{m_{x,t}}.$$

This may be interpreted as a one-year mortality reduction factor.

We observe no systematic change in $r_{x,t}$ over time.
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Mortality reduction factors at representative ages.
Time Dependency

Simplified sample cross-correlation matrices constructed from $r_{x,t}$ at ages: 70, 75, 80, 85, and 90.
The naïve bootstrap will lose the serial dependency in the data.

We use the block bootstrap method (Carlstein, 1986; Künsch, 1989) to retain serial dependency.

The sample CCMs indicate the cross-correlations taper off as the lag increases.

Blocks of observations that are separated far enough will be (approximately) uncorrelated.
The Block Bootstrap

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The Block Bootstrap

- We have 55 vectors of $r_t$ (1950 – 2004).

- Assuming a block size of 5, we have 51 blocks:
  $(r_{1950}, r_{1951}, r_{1952}, r_{1953}, r_{1954}), (r_{1951}, r_{1952}, r_{1953}, r_{1954}, r_{1955}), ...,$

- The optimal block size is not always evident.

- Hall et al. (1995) recommend a block size of $n^{1/5}$.

- We use a block size of 2 ($55^{1/5} = 2.23 \approx 2$).
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Forecasts of Survival Probabilities

Empirical distributions of the survival probabilities for the cohort aged 65 in year 2005, on the basis of 46, 56, and 66 years of data.
Comparing with Model-Based Methods

<table>
<thead>
<tr>
<th></th>
<th>Non-parametric</th>
<th>Lee-Carter</th>
<th>Cairns, Blake and Dowd</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10p_{65}$</td>
<td>0.7790</td>
<td>0.7755</td>
<td>0.7814</td>
</tr>
<tr>
<td>$15p_{65}$</td>
<td>0.6048</td>
<td>0.6011</td>
<td>0.6135</td>
</tr>
<tr>
<td>$20p_{65}$</td>
<td>0.3999</td>
<td>0.3995</td>
<td>0.4132</td>
</tr>
<tr>
<td>$25p_{65}$</td>
<td>0.2080</td>
<td>0.2039</td>
<td>0.2146</td>
</tr>
</tbody>
</table>

Central estimates of the survival probabilities for the cohort aged 65 in year 2005, on the basis of the non-parametric bootstrap, the Lee-Carter model and the Cairns, Blake and Dowd model.
The BNP/EIB Longevity Bond

- We use the BNP/EIB bond for the price constraint.

- It is 25-year amortising bond, which pays $50I(t)$, for $t = 1, \ldots, 25$.

- $I(t)$ is defined as:

  \[ I(t) = I(t-1)(1 - m_{64+t,2002+t}), \quad t = 1, 2, \ldots, 25, \]

  where

  - $I(0) = 1$,
  - $m_{x,t}$ is the crude central death rate for the E&W male population at age $x$ and in year $t$. 
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  - $m_{x,t}$ is the crude central death rate for the E&W male population at age $x$ and in year $t$. 
We use the BNP/EIB bond for the price constraint.

It is 25-year amortising bond, which pays $50I(t)$, for $t = 1, \ldots, 25$.

$I(t)$ is defined as:

$$I(t) = I(t - 1)(1 - m_{64+t,2002+t}), \quad t = 1, 2, \ldots, 25,$$

where

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- $m_{x,t}$ is the crude central death rate for the E&W male population at age $x$ and in year $t$. 
The BNP/EIB Longevity Bond

- The issue price was determined by discounting at LIBOR minus 35 basis points the anticipated coupon payments.

- The time-0 value of the bond is £561.
The BNP/EIB Longevity Bond

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- The time-0 value of the bond is £561.
Step (1)

1. Generate a number, say $N$, of equally probable mortality scenarios.

2. From each scenario, calculate the longevity index $I(t)$ at $t = 1, 2, \ldots, 25$.

3. The time-0 value of the BNP/EIB bond in the $j$th scenario is

$$v(\omega_j) = 50 \times \sum_{t=1}^{25} B(0, t)I(t, \omega_j),$$

where $I(t, \omega_j)$ be the index value at time $t$ in the $j$th scenario, and $B(0, t)$ is the time-0 price of a risk-free zero-coupon bond that pays £1 at time $t$. 
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Real World Probability Measure, $\pi_j$
Step (2)

- Let $\pi^*_j$ be the probability associated with $\nu(\omega_j)$ under $Q$.

- We require $\sum_{j=1}^{N} \nu(\omega_j)\pi^*_j = 561$ and $\sum_{j=1}^{N} \pi^*_j = 1$.

- We minimize the KLIC as follows:

$$L = \sum_{j=1}^{N} \pi^*_j \ln \pi^*_j - \lambda_0 \left( \sum_{j=1}^{N} \pi^*_j - 1 \right) - \lambda_1 \sum_{j=1}^{N} (\nu(\omega_j)\pi^*_j - 561).$$
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Step (2)

- Let $\pi_j^*$ be the probability associated with $v(\omega_j)$ under $Q$.

- We require $\sum_{j=1}^{N} v(\omega_j) \pi_j^* = 561$ and $\sum_{j=1}^{N} \pi_j^* = 1$.

- We minimize the KLIC as follows:

$$L = \sum_{j=1}^{N} \pi_j^* \ln \pi_j^* - \lambda_0 \left( \sum_{j=1}^{N} \pi_j^* - 1 \right) - \lambda_1 \sum_{j=1}^{N} \left( v(\omega_j) \pi_j^* - 561 \right).$$
Let $\tilde{\pi}_j^*$, $j = 1, 2, \ldots, N$, be the solution.

We have

$$\tilde{\pi}_j^* = \frac{\exp(\lambda_1 v(\omega_j))}{\sum_{j=1}^{N} \exp(\lambda_1 v(\omega_j))}, \quad j = 1, 2, \ldots, N.$$  

$$\lambda_1 = \arg\min_{\gamma} \sum_{j=1}^{N} \exp(\gamma(v(\omega_j) - 561)).$$
Step (2), Continued

- Let $\tilde{\pi}_j^*$, $j = 1, 2, \ldots, N$, be the solution.

- We have

$$
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$$

$$
\lambda_1 = \arg \min_{\gamma} \sum_{j=1}^{N} \exp(\gamma(v(\omega_j) - 561)).
$$
The Canonical Measure, $\pi_j^*$
Incorporating More Prices

- What if more market prices are available?

- The method can be extended to incorporate additional primary securities.

- Assume the \(i\)th security has a time-0 price of \(V_i\) and a discounted payoff of \(v_i(\omega_j)\) in the \(j\)th scenario.

- To price \(m\) securities correctly, we require

\[
\sum_{j=1}^{N} v_i(\omega_j)\pi^*_j = V_i, \quad i = 1, 2, \ldots, m.
\] (3)
Incorporating More Prices

- We minimize the KLIC subject to the \( m \) constraints and 
  \[ \sum_{j=1}^{N} \pi_j^* = 1. \]

- It can be shown that the resulting canonical measure \( \tilde{\pi}_j^* \), \( j = 1, 2, \ldots, N \) is
  \[ \tilde{\pi}_j^* = \frac{\exp(\sum_{i=1}^{m} \lambda_i v(\omega_j))}{\sum_{j=1}^{N} \exp(\sum_{i=1}^{m} \lambda_i v(\omega_j))}, \quad j = 1, 2, \ldots, N, \]
  where \( \vec{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_m)' \) can be expressed as
  \[ \vec{\lambda} = \arg\min_{\gamma_1, \ldots, \gamma_m} \sum_{j=1}^{N} \exp \left( \sum_{i=1}^{m} \gamma_i(v_i(\omega_j) - V_i) \right). \]
With One Primary Security, $m = 1$

The canonical measure $Q_0$ when $m = 1$. 

the set of all equivalent martingale measures

the set of all measures equivalent to $P$
With Two Primary Securities, $m = 2$

The canonical measure $Q_0$ when $m = 2$. 

- the set of all measures equivalent to $P$
- the set of all equivalent martingale measures
With Infinitely Many Primary Securities, $m \to \infty$

The canonical measure $Q_0$ when $m \to \infty$. 

the set of all equivalent martingale measures

the set of all measures equivalent to $P$
Pricing Vanilla Survivor Swaps

- We consider vanilla survivor swaps with a fixed proportional premium $\theta$ and a fixed maturity $T$.

- At $t = 1, 2, \ldots, T$, the fixed-payer pays a preset amount of $(1 + \theta)K(t)$.

- The fixed-reciever pays a random amount of $S(t)$, which is linked to the realized survival function of the reference population.

- The reference population is the same as that of the BNP/EIB longevity bond.
We set

\[ S(t) = S(t - 1)(1 - q_{64+t,2002+t}), \quad t = 1, 2, \ldots, T, \]

where \( S(0) = 1 \), and \( q_{x,t} \) is the realized death probability.

We set \( K(t) \) to the projected survival function for the reference population, on the basis of GAD’s projection.

\( K(t) \) for declines over time.
The Calculated Swap Premium

![Chart showing the calculated swap premium over different maturities for different years (1940-2002, 1950-2002, 1960-2002). The chart displays a downward trend as maturity increases.]

- Swap premium values range from -0.0140 to 0.0000.
- Maturities range from 1 to 25 years.

The chart illustrates the calculated swap premium for different maturities and years, showing a clear decrease as the maturity increases.
Comparing with Other Pricing Methods

![Graph showing various premium curves for different maturity periods. The x-axis represents maturity, ranging from 1 to 25, and the y-axis represents the swap premium, ranging from 0.0000 to -0.0140. The graph includes curves labeled as Wang transform, Canonical valuation, and Two factor model.]
Conclusions

- The pricing framework is reasonably robust relative to the amount of data used.

- It avoids model risk and parameter risk.

- Additional prices can be incorporated into the canonical measure easily.

- Due to its non-parametric nature, our framework can be applied to reference populations with limited volume of data available.
Q&A