ACTUARIAL ANALYSIS OF THE MULTIPLE LIFE ENDOWMENT INSURANCE CONTRACT

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AGENDA

• Modeling issues
  • general life status / single, joint, last-survivor & term
  • two-life status / survival probabilities

• Technical issues
  • premiums / term life / endowment / annuity
  • reserves / state dependent / state independent
  • premium components / state dependent / independent

• Computational issues
  • reduction formulas / term life / insurance / annuity
  • independence assumption / Höffding-Fréchet bounds
Modeling / general life status

• Random future lifetime of a general life status (u)

  consider group of g lives aged x(1), x(2), …, x(g)
  T[k]=T[x(k)] : future lifetime of single life aged x(k)
  T=T[u] : future lifetime of a status (u) on this group
  p(t,u)=P(T[u]>t) : probability of survival to time t>0
  q(t,u)=P(T[u]≤t) : probability of failure to time t>0

• single life status : u with T[u]=T[x] for single life aged x
• joint-life status : u with T[u]=min{T[1],…,T[g]}
• last survivorship : u with T[u]=max{T[1],…,T[g]}
• term certain : u with deterministic T[u]=n an integer
Modeling / two-life status

• Survival probabilities

two lives aged x, y, with random age-at-deaths X, Y
T[x]=X−x, T[y]=Y−y : random future lifetimes of (x), (y)
S(x,y)=P(X>x,Y>y) : joint survival function of (X,Y)

• joint-life status : u=x:y with T[u]=min{T[x],T[y]}
p(t,u)=P(T[u]>t) = S(x+t,y+t) / S(x,y)

• last survivorship : u′=(x:y)′ with T[u′]=max{T[x],T[y]}
p(t,u′)=P(T[u′]>t) = {S(x+t,y) + S(x,y+t) − S(x+t,y+t)} / S(x,y)

• independence : p(t,u) = p(t,x)·p(t,y)

• partial independence : p(t,u) = p(t,u′) – p(t,x) – p(t,y)
Technical / net single premiums

• **n-year term life insurance**
  \[ D(m,u:n) \] : NSP for one unit of benefit payable at the end of the m-thly period of a year after failure of status \( u \)

• **n-year pure endowment**
  \[ E(u:n) \] : NSP payable at survival of status \( u \)

• **n-year endowment**
  \[ A(m,u:n) = D(m,u:n) + E(u:n) \] : NSP for status \( u \)

• **n-year life annuity**
  \[ a(c,u:n) \] : NSP for one unit of benefit per year payable in installments of \( c \) fractional units at beginning of each payment cycle of length \( c \) as long as status \( u \) survives
Technical / level premiums

- net level premium of n-year endowment (NLP)
  \[ \text{NLP}(m,c,u:n) = \frac{A(m,u:n) \cdot \text{SI}}{a(c,u:n)} : \]
  NLP with benefit SI for status u

- level premium of n-year endowment (LP)
  acquisition costs : rate \( \alpha \) of sum insured
  premium proportional operating costs : rate \( \beta v \)
  constant operating costs : fixed costs \( \beta f \)
  benefit proportional operating costs : rate \( \gamma \)
  LP with benefit SI for status u is determined by
  \[ (1-\beta v) \cdot \text{LP}(m,c,u:n) = \text{NLP}(m,c,u:n) + (\alpha/a(c,u:n) + \gamma) + \beta f \]
Technical / reserves

- **random failure time of contract**
  \( K[u] \): curtate future lifetime, \( S[u] = T[u] - K[u] \): fractional time of life in failure year, \( S[m,u] = m \cdot \text{int}(S[u]/m+1) \): fractional portion \( S[u] \) rounded up to next \( m \)-th of a year
  \( T[m,u] = K[u] + S[m,u] \): moment of benefit payment by failure

- **random prospective loss of n-year endowment**
  \( v = 1/(1+i) \): discount factor to technical interest rate \( i \)
  random prospective loss at contract time \( t > 0 \):
  \[ L(t;m,c,u:n) = v^{\min\{T[m,u+t],n-t\}} \cdot SI - NLP(m,c,u:n) \cdot a(c,\min\{T[m,u+t],n-t\}) \]
  with
  \[ a(c,n) = \frac{1-v^n}{d(c)} \]
  \[ d(c) = i(c)/(1+c \cdot i(c)) \]
  \[ i(c) = (1+i)^c - 1/c \]
Technical / reserves

- states at time $t>0$ of couple $(x,y)$ under mortality risk
  \[X(t)=1 \Leftrightarrow (T[m,x]>t, T[m,y]>t) \quad (x \text{ & } y \text{ alive at } t)\]
  \[X(t)=2 \Leftrightarrow (T[m,x]>t, T[m,y] \leq t) \quad (x \text{ alive } \& \text{ y dead at } t)\]
  \[X(t)=3 \Leftrightarrow (T[m,x] \leq t, T[m,y]>t) \quad (x \text{ dead } \& \text{ y alive at } t)\]
  \[X(t)=4 \Leftrightarrow (T[m,x] \leq t, T[m,y] \leq t) \quad (x \text{ & } y \text{ dead at } t)\]

- state dependent mathematical reserves
  mathematical reserve in state $X(t)=i$ at time $t>0 =$ expected value of prospective loss conditional on state:
  \[V(t,i) = E[L(t;m,c,u:n)|X(t)=i]\]

- joint-life status : $V(t,1) \neq 0, V(t,i) = 0$ for $i=2,3,4$
- last survivorship : $V(t,i) \neq 0$ for $i=1,2,3, V(t,4) = 0$
Technical / reserves

• state independent net premium reserve at time $t>0$
  conditional expectation of prospective loss given survival:
  $V(t) = E[L(t;m,c,u:n)|T[m,u]>t] = \sum V(t,i) \cdot P(X(t)=i|T[m,u]>t)$

• state dependent deferred acquisition costs (DAC)
  $VE(t,i) = -\alpha \cdot \{SI - V(t,i)\}, \ i=1,2,...$

• state independent expense reserve
  $VE(t) = -\alpha \cdot \{SI - V(t)\}$

• state dependent actuarial reserve
  $VA(t,i) = V(t,i) + VE(t,i), \ i=1,2,...$

• state independent premium reserve
  $VA(t) = V(t) + VE(t)$
state dependent components (special case \( m=c \))
At the discrete times \( t=k \cdot c, k=0,1,…,n/c–1 \), one has

- saving premium
  \[
  SP(t,i) = v^c \cdot V(t+c,i) - V(t,i), \quad i=1,2,…
  \]

- risk premium
  \[
  RP(t,i) = v^c \cdot q(c,u+t) \cdot \{SI - V(t+c,i)\}, \quad i=1,2,…
  = NLP(c,c,u:n) - SP(t,i)
  \]
  (saving premium + risk premium = net level premium)

- expense premium
  \[
  EP(c,c,u:n) = LP(c,c,u:n) - NLP(c,c,u:n)
  \]
  The expense premium splits into (similar net level premium)
Technical / premium components

- risk component expense premium
  \[ REP(t,i) = \alpha \cdot RP(t,i), \quad i=1,2,... \]
- saving component expense premium
  \[ SEP(t,i) = EP(c,c,u:n) - REP(t,i), \quad i=1,2,... \]
- state independent components (similar decomposition)
  saving premium: \[ SP(t) = v^c \cdot V(t+c) - V(t) \]
  risk premium: \[ RP(t,i) = v^c \cdot q(c,u+t) \cdot \{SI-V(t+c)\} \]
  \[ = NLP(c,c,u:n) - SP(t) \]
  expense premium: \[ EP(c,c,u:n) = LP(c,c,u:n) - NLP(c,c,u:n) \]
  risk component: \[ REP(t) = \alpha \cdot RP(t) \]
  saving component: \[ SEP(t) = EP(c,c,u:n) - REP(t) \]
All technical values related to the multiple life endowment depend solely on functions $A(m,u:n)$ and $a(c,u:n)$. Under the assumption of uniform distribution of deaths (UDD) further reduction to $D(u:n)$ and $E(u:n)$:

- **n-year term life insurance**
  
  - $D(m,u:n) = D(u:n) \cdot i/i(m)$ if $m > 0$
  
  - $D(m,u:n) = D(u:n) \cdot i/\delta$ if $m = 0$, $\delta = \ln{1+i}$

- **n-year endowment**
  
  - $A(m,u:n) = A(u:n) + \{i/i(m) - 1\} \cdot D(u:n)$

- **n-year life annuity**
  
  - $a(c,u:n) = \{1 - A(u:n) - \{i/i(m)-1\} \cdot D(u:n)\}/d(c)$
Computational / independence

- **simplifying assumption**
  tariff book is generated under independent future lifetimes: measure impact of assumption on actuarial calculations

- **Höffding - Fréchet upper bound**
  Fréchet class of bivariate distributions $F(s,t)$ with fixed margins $q(s,x)=P(T[x] \leq s)$ & $q(t,y)=P(T[y] \leq t)$. Consider
  Fréchet upper bound : $FU(s,t) = \min\{q(s,x),q(t,y)\}$
  Bivariate inequality : $F(s,t) \leq FU(s,t)$

- **upper bound for joint-life survival distribution**
  $p_U(t,u)=\min\{p(t,x),p(t,y)\}$
Computational / independence

- upper bound for last survivor survival distribution
  \[ p_U(t,u') = \max\{p(t,x), p(t,y)\} \]

- joint-life survival distribution under independence
  \[ p(t,u) = p(t,x) \cdot p(t,y) \]

- last survivor survival distribution under independence
  \[ p(t,u') = p(t,x) + p(t,y) - p(t,u) \]

- inequalities between survival distributions
  \[ p_I(t,u) \leq p_U(t,u), \quad p_U(t,u') \leq p(t,u') \]

  => random future lifetimes ordered in stochastic order
  => inequalities between NSP and NLP for status (u) & (u')

- maximum deviations for two-lives endowment in Table:
Computational / independence

- technical interest : 2%

- Life Table: Gompertz survival distribution

- Stress testing results:
  1) joint life status over-estimates NSP & NLP
  2) last survivorship status underestimates NSP & NLP

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