Internal Models
Agenda

• The concept – required capital
  • Typical issues
  • The calibration-dilemma
  • Typical approaches for internal models
  • Linearisation approaches
  • The Delta-Gamma-approach
  • The numerical error in a full stochastic approach
  • Using the Delta Gamma approximation as control variate
Required Economic Capital is the capital you need to have ensured that in most situations policyholder obligations can be fulfilled.

- **Available Economic Capital**
- **Distribution of the available economic capital in one year’s time**
- **Required Economic Capital** in order to ensure that the market value of liabilities is still available after taking the risk

- Value of liabilities
- **tolerable probability of financial distress**
Conceptually a full internal economic capital model is simple…

- Create a sufficiently large number of scenarios for all relevant risk factors
- Revalue the assets and liabilities for each risk factor scenario at the end of the first year
  - including all options and guarantees
- which in most cases requires a stochastic valuation and thus leads to a nested stochastic calculation

But in most cases this approach is
- Technically not feasible
- Not necessary
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Typical issues

- We focus on actuarial issues here, not on the related issues, like data availability.
- There are many risk factors:
  - Which distribution do I assume?
  - How do I *calibrate* these distributions?
  - How do I reflect extreme events properly?
  - How do I reflect the dependency structure of the risk factors?
- What are the numerical errors involved in a nested stochastic approach?
- How do I determine the net-value at time 1 for each risk-factor scenario?
- How do I reflect fungibility and intra-group dependency?
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Calibration: Does history tell us anything about the future?

The observed history is not a sufficient basis for projecting the future...

...because

- time-series are too short to derive robust input for extreme events
- there will be new risks and developments, not yet observed

One solution:

Actuarial judgement-based extreme-scenarios as basis for a calibration

of e.g. copulae
Two major issues: „fat tails“ and dependency structure

Normal (blue) vs t-Student (red)

Gaussian

Gumbel-copula
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Approaches for internal models observed so far

**Risk-factors**
- Multivariate normal – analytic
- Multivariate normal – stochastic scenarios
- Bootstrapping / historical resampling
- Extreme scenarios
- Copulæ

**Valuation**
- Stress tests and linearity assumption
- Replication portfolio approach
- Full stochastic approach
- Roll-up-approach
The standard approach

- Assumes all risk factors are multivariate normal and
- Exposure is linear in risk factors
- Resulting net value distribution is normal again with known volatility
- All necessary statistics can be derived easily, but
- Too simplistic
A pragmatic approach

- Use Cholesky-decomposition of correlation matrix and normal distribution assumption to produce stochastic scenarios
- Add extreme scenarios
- Determine net value at $t=1$ for each risk factor scenario using replication portfolios
- Captures non-linearity adequately
- Extreme scenarios can reflect tail-dependency and fat tails up to a certain degree
- Allows for non-trivial management actions
- Allows to model group diversification and fungibility
Cholesky-decomposition

- Let $S$ be the covariance matrix of the multivariate distribution considered
- Determine a matrix such that $C^*C^T = S$
  - If $S$ is positive semi-definite (which we should expect) then Cholesky factorisation creates a lower triangular matrix such that $C^*C^T = S$
- If $Z$ is a vector of independent standard normal random variables then $C^*Z$ is a vector of normal random variables with covariance matrix $S$
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Non-linearity is an issue

An example is interest rate convexity: $\text{loss} = (1 + i)^n \neq 1 + i \cdot n$

- Large stress, e.g. QIS: Overestimation of loss around median
- Small stress, e.g. SST: Underestimation of loss in tail

Linearisation using a large shock
Linearisation using a small shock
Linear approaches can show all kind of behaviour

Example

- Loss function given by discounting: $100 \cdot (1 + i)^{-20}$
- $i$ fluctuates normally around 2% with volatility 0.8%
- Example A: exposure to this one risk factor only
- Example B: exposure based on the sum of the exposure of 6 similar but independent risk factors

<table>
<thead>
<tr>
<th>VaR in % of correct VaR</th>
<th>Linearisation using small shock (1% – SST)</th>
<th>Linearisation using large shock (99.5% percentile: 2.06%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>One risk factor</td>
<td>91%</td>
<td>102%</td>
</tr>
<tr>
<td>6 risk factor exposures added</td>
<td>102%</td>
<td>115%</td>
</tr>
</tbody>
</table>

- This is by no means astonishing, looking at the graph
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Delta-Gamma helps

**Idea:** Determine two stresses for each risk factor and fit quadratic function to the three known values

- Typically an up-stress and a down-stress is used
- No cross-risk-factor stresses: “diagonal” approach

**Example**

- As above

<table>
<thead>
<tr>
<th>VaR in % of correct VaR</th>
<th>Quadratic approximation using small shock (1% - SST)</th>
</tr>
</thead>
<tbody>
<tr>
<td>One risk factor</td>
<td>97.76%</td>
</tr>
<tr>
<td>6 risk factor exposure added</td>
<td>98.75%</td>
</tr>
</tbody>
</table>

- But who knows whether in other situations the approximation works as well?
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A full stochastic approach – even with accurate valuation – shows considerable numerical error

- Example as above – one risk factor
- 50 Batches of 1’000 simulations
- Distribution of empiric VaR
- Volatility is 6% of the VaR
Estimating the error in a stochastic approach

• In a nested stochastic approach we have two sources for numerical error
  – the error in determining the value for each risk-factor scenario
    • in fact we determine $\text{VaR}(\text{Accurate}+\text{valuation error})$ instead of $\text{VaR}(\text{Accurate})$
  – and the numerical error in determining (e.g.) the 99.5% VaR
    • for this error there exists a nice non-parametric estimation
    • and a parametric formula
A non-parametric formula for VaR-estimation error

The probability that the correct VaR is higher than the m-th result \(L_m\) is
\[1 - \text{cum\_binom}(m - 1, n, \alpha),\]
where \(\text{cum\_binom}(m, n, \alpha)\) is the cumulative binomial distribution for \(m, n\) and \(\alpha\).

And the probability that the correct VaR is lower than the p-th result \(L_p\) is
\[\text{cum\_binom}(p - 1, n, \alpha).\]

Here \(\text{cum\_binom}(p - 1, n, \alpha)\) is the cumulative binomial distribution for \(p-1\) successes, \(n\) experiments and success-probability \(\alpha\).

<table>
<thead>
<tr>
<th>Confidence-interval (both sides)</th>
<th>(\alpha)</th>
<th>(n)</th>
<th>(p)</th>
<th>(m)</th>
<th>(P(\text{VaR}&lt;L_p))</th>
<th>(P(\text{VaR}&gt;L_m))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.995</td>
<td>100</td>
<td>0.95</td>
<td>98</td>
<td>101</td>
<td>0.014103</td>
<td>0</td>
</tr>
<tr>
<td>0.995</td>
<td>500</td>
<td>0.95</td>
<td>494</td>
<td>501</td>
<td>0.013944</td>
<td>0</td>
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<tr>
<td>0.995</td>
<td>1000</td>
<td>0.95</td>
<td>990</td>
<td>1000</td>
<td>0.013469</td>
<td>0.006654</td>
</tr>
<tr>
<td>0.995</td>
<td>10000</td>
<td>0.95</td>
<td>9936</td>
<td>9964</td>
<td>0.023315</td>
<td>0.023495</td>
</tr>
</tbody>
</table>

Read: the probability that the 9936-smallest scenario is larger than the 99.5% VaR when considering 10'000 scenarios is smaller than 2.3%
A parametric formula for VaR-estimation error

- For a given quantile $\alpha$ define $k:=\text{int}(\alpha \cdot n) + 1$
- The $k$-th result $\bar{L}_k$ is an estimation of the VaR
- The error $L_k - \text{VaR}$ has approximately variance $\frac{\alpha \cdot (1 - \alpha)}{n \cdot f(\text{VaR})^2}$ around 0
- $f$ is the density of the distribution function of $L$
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Enhancing the full stochastic approach with the DeltaGamma-function as control-variate substantially reduces numerical error

We use that:
\[ P(L > VaR) = E(I_{L>VaR}) = E(I_{L>VaR} - I_{\text{DeltaGamma}>VaR'} + I_{\text{DeltaGamma}>VaR'}) = E(I_{L>VaR} - I_{\text{DeltaGamma}>VaR'}) + E(I_{\text{DeltaGamma}>VaR'}) \]

where \( E(I_{\text{DeltaGamma}>VaR'}) \) can be determined with a high degree of accuracy and
\[ E(I_{L>VaR} - I_{\text{DeltaGamma}>VaR'}) \approx \frac{1}{n} \sum_{\text{scenarios}} I_{L(\text{scenario})>VaR} - I_{\text{DeltaGamma}(\text{scenario})>VaR'} \]

is an estimation with lower estimation error as \( I_{L>VaR} - I_{\text{DeltaGamma}>VaR'} \) has lower volatility than \( I_{L>VaR} \)
Enhancing the full stochastic approach with the DeltaGamma-function as control-variate substantially reduces numerical error

- Example as above, Delta-Gamma-approximation as above
- Volatility is 0.85% of the VaR, i.e. 50 times less scenarios with same accuracy
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