PORTFOLIO THEORY AND PENSION FUNDING IN A STOCHASTIC FRAMEWORK

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ABSTRACT

This paper deals with the financing of public pension in a stochastic environment. The aim is to compute the optimal level of mix between pay-as-you-go and funding in a risk management perspective and using portfolio theory argument. Recently, different countries as Sweden or Poland have implemented mix solutions combining pay as you go and funding mechanisms. The purpose of this paper is to check the rationality of such a combination and to find the optimal splitting of the contributions between the two systems.

We introduce different stochastic models where the main processes become random (wage growth, population growth, financial rate of return). We obtain in particular conditions on the parameters in order to justify the diversification and the explicit optimal sharing between pay as you go and funding. We generalize the results in the presence for the funding part of two possible financial assets. In particular we look at the situation of combining a risk free asset, a risky asset and the part in PAYG.

Keywords: Pay as you go, funding, diversification, portfolio theory, utility function

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1. Introduction

Financing of our pension is one of the most challenging macroeconomic issues for the next decades. In a lot of countries, populations are rapidly ageing due to the combined effect of decreasing fertility rates and increasing life expectancy. This implies an increasing dependency ratio, the number of retired people relative to the number of active people in a population. For instance, in Belgium, this ratio was equal to 40.5% in 2000. It should grow to 44.7% in 2010 and even to 68.5% in 2030 according to recent demographic projections.

This demographic evolution will clearly induce major financial problems in the long run for pure unfunded pension schemes. In those systems usually called PAYG (pay-as-you-go), active people finance retired people’s pension. The rate of return of such schemes is then linked to the growth rate of the salary mass. But, according to demographic trends, the growth rate of the mean salary is not enough to compensate the increase of dependency ratio. Therefore, either the charges of contributions or taxes must become higher and higher or pension amounts must be less generous. Another solution is to introduce a more fundamental reform in the public pension system by introducing a part of funding inside the social security. Countries like Chile or Sweden have opted for such a mechanism.

In a pure funding, contributions are invested on the financial market for one cohort’s own future pension. The rate of return of the system mostly depends on the financial return. This mechanism might seem attractive considering future demographic trends in Europe, but it could lead to major problems in case of financial crash as seen recently.

This debate about the relative attractiveness of funding or PAYG is not new but recently diversification policies have been applied combining both techniques. They are no more considered as opposite solutions but as two different tools to diversify the pension investments.

The starting point of this paper is that PAYG and funding schemes are dealing with different risks and have therefore specific advantages and drawbacks. We therefore seek for equilibrium between those two “extreme” pension schemes inside a country’s first pillar, arguing that mixing both systems means diversification between financial and demographic risks. The State would have to determine an optimal mix between those systems, which would be the same for each member of a pension plan. In our model we consider the general case of a pension system in a country, and not from an individual point of view.

This paper proposes some new features compared to the existing literature. Basically there are two important differences. First of all, the idea of blending different pension schemes is not often used in the literature. This method should allow reaching a new and more performing equilibrium. Dutta et al. (1999, 2000) present a mixed model funded and unfunded in a mean variance framework. They find that the mixed systems are better because they enable risk diversification. Van Praag and Cardoso (2003) present a model that explains the mix between funded and unfunded pension systems. Matsen and Thogersen (2004) study the optimal size of PAYG system and the optimal split between funded and unfunded pension savings by means of a theoretical portfolio choice framework. Knel (2010) studies the optimal portfolio combination between funded and unfunded pension systems when people care about relative consumption. Guigou et al (2010) develop a model for evaluating the efficiency of a diversified pension system financed partly by capitalization, measuring the efficiency by the long term sustainability of the system.

Second, we do not assume independence between the demographic and financial risks as it is usually done. We will allow those risks to be correlated and analyze the consequences of this assumption. Three risks are considered: the wage growth, the rate of return of the assets (of the funded system) and the population growth rate. Doing this we will try to answer different questions. Can we obtain a theoretical justification for diversification? What is the optimal level for diversification? How does the correlation between financial and demographic risks impact this level?
The paper is organized as follows. Section 2 introduces the subject with a simple case, the classical deterministic model. We consider a defined contribution pension system: a fixed contribution is applied on the wages. The pension liability generated by these contributions will depend on the chosen funding mechanism (PAYG or funding or both). Our purpose is to compare the level of pension achieved. Section 3 analyzes a two periods overlapping stochastic model with only one risky financial asset. In this section, we examine the optimality of a mixed system. Sometimes diversification is not optimal. If a combined system is proved to be more efficient, we determine the optimal fraction to be invested in each pension scheme, in order to maximize the expected utility of the pension amount received at retirement. Section 4 analyzes a three periods overlapping stochastic model with only one risky financial asset, with two generations of working people and one generation of retired people. Section 5 generalizes these results in the presence for the funding part of two possible financial assets. In particular we look at the situation of combining a risk free asset, a risky asset and the part in PAYG. Finally section 6 concludes the paper with some possible future extensions of the model and some detailed computations and examples have been put in appendix.

2. Classical deterministic model

Let us consider a defined contributions pension plan (DC scheme). The contribution rate is fixed, while the pension that people get depends on the return of the scheme. We will compare the pension amount generated with a PAYG or a funding scheme. In the deterministic model all the variables (wage growth rate, population growth rate and financial rate of return) are constant and supposed to be known in advance. This model will be our starting point for the next sections.

2.1 Notations and assumptions

We will use the following notations:

**Demographics** Let $L(x;t)$ be the number of persons aged $x$ (with $x=x_0,x_1,...,x_\omega$) at time $t$. There are three specific ages for every generation. Between $x_0$ and $x_{r-1}$ the individual is part of the active population and is affiliated to the public pension plan. He gets a salary on which a contribution is perceived. At $x_r$, the individual retires and starts receiving a pension annuity obtained until his death (somewhere before $x_\omega$).

The probability, being alive at age $x$, to survive during $n$ years (up to age $x+n$) is $p(x,n)$. The annual growth of the entrance function is $d$. So we have:

$$L(x_0,t)=L(x_0,t-1).(1+d).$$

**Economics** $\Pi$ is the contribution rate, constant over time and cohorts (DC scheme). $S(t)$ is the mean salary that active people receive at time $t$. It increases at the yearly rate $g$. So we have: $S(t)=S(t-1)(1+g)$. The mean pension at time $t$ is $P(t)$. We define $RR(t)$ the replacement rate at time $t$ as:

$$RR(t)=\frac{P(t)}{S(t)}$$

(1)

It compares the relative level of wages and pensions at one specific time. The rate of return of the financial investments is denoted by $i$. 


2.2 Two period’s model

We start with a simple “two period overlapping generation model”. In this model each generation lives for two periods (working part and pension period) of equal length and \( x_0 = x_{r-1} \). The pension benefits can be computed:

**Funding** The contributions paid by the active generation at time \( t-1 \) are invested for one period and will generate their own pension at time \( t \). The actuarial equilibrium relationship can be written as:

\[
L(x_0, t-1) \pi S(t-1) (1+i) = L(x_r, t) P(t)
\]

(2)

As \( L(x_r, t) = L(x_0, t-1) \rho(x_0, 1) \), the generated benefit will simply given by:

\[
P(t) = \pi S(t-1) \frac{1}{\rho(x_0, 1)} (1+i).
\]

And the replacement rate becomes:

\[
RR(t) = \pi \frac{1}{\rho(x_0, 1)} \frac{1+i}{1+g}
\]

(3)

**Pay-As-You-Go** The contributions paid at time \( t \) by the active generation are used to pay the pension of the retired generation at time \( t \). The equilibrium relation is:

\[
L(x_r, t) \pi S(t) = L(x_r, t) P(t)
\]

(4)

We have \( L(x_r, t) = L(x_r, t-1) (1+d) \) and \( L(x_r, t) = L(x_r, t-1) \rho(x_0, 1) \). The generated benefit is given by \( P(t) = \pi S(t) \frac{1}{\rho(x_0, 1)} (1+d) \). And the replacement rate is:

\[
RR(t) = \pi \frac{1}{\rho(x_0, 1)} (1+d)
\]

(5)

**Comparison and decision rule** We obtain the well known decision rule of Aaron (1966). If \( (1+d) > \frac{(1+i)}{(1+g)} \) then the PAYG must be preferred. In this situation the return of the PAYG system is greater than the “funding return”. If \( (1+d) < \frac{(1+i)}{(1+g)} \), then funding must be preferred.

In particular we note that diversification between the two techniques is never optimal on a two periods deterministic model. It could only be justified in the trivial situation where both systems have the same return.

2.3 Multi periods model

We can easily generalize these rules in a multi-period deterministic model. In this model people are working between age \( x_0 \) and \( x_{r-1} \) and retire at age \( x_r \). We consider the generation who retires in \( t \). Let us compare what we get using PAYG or funding.

**Funding** In order to determine the pension amount \( P(t) \), we will equalize the present value of the past contributions with the present value of the future benefits. Relation (2) becomes:
\[
\sum_{x=x_0}^{x_1} L(x, t - x_r + x) \pi S(t - x_r + x) (1 + i)^{x-x} = P(t) \sum_{x=x_0}^{x_1} L(x, t - x_r + x) \left(\frac{1 + g}{1 + i}\right)^{x-x}
\]

(6)

Doing this we suppose that the pensions are indexed at the rate \( g \) after the retirement. The relation (3) becomes

\[
RR(t) = \frac{P(t)}{S(t)} = \frac{\sum_{x=x_0}^{x_1} p(x, x-x_0) \left(\frac{1 + g}{1 + i}\right)^{x-x_0}}{\sum_{x=x_0}^{x_1} p(x, x-x_0) \left(\frac{1 + g}{1 + i}\right)^{x-x_0}}
\]

(7)

**Pay-as-you-go** At time \( t \), all the contributions are distributed to the retirees. They all get the same pensions. The equilibrium relationship is given by:

\[
\sum_{x=x_0}^{x_1} L(x, t) \pi S(t) = \sum_{x=x_0}^{x_1} L(x, t) P(t)
\]

(8)

and the relation (5) becomes:

\[
RR(t) = \frac{P(t)}{S(t)} = \frac{\sum_{x=x_0}^{x_1} p(x, x-x_0) \left(\frac{1}{1 + d}\right)^{x-x_0}}{\sum_{x=x_0}^{x_1} p(x, x-x_0) \left(\frac{1}{1 + d}\right)^{x-x_0}}
\]

(9)

**Comparison and decision rule** It is easy to show that the function \( f(\cdot) \) is non-decreasing:

\[
f(\xi) = \frac{\sum_{x=x_0}^{x_1} p(x, x-x_0) \left(\frac{1}{\xi}\right)^{x-x_0}}{\sum_{x=x_0}^{x_1} p(x, x-x_0)(\xi)^{x-x_0}}
\]

So the comparison between the two replacement rates induces the same conclusion as in the two period model of section 2.2. If \( 1 + d > \frac{1 + i}{1 + g} \) PAYG must be preferred. And if \( 1 + d < \frac{1 + i}{1 + g} \), funding must be preferred.

2.4 Conclusion

In a deterministic framework the choice between PAYG and funding only depends on the returns of the financial and demographic variables. Diversification is never a good option. It could only be considered if both schemes have identical returns but has then no added value!

But Aaron’s rule is only verified in a deterministic model. When the parameters become random, such conclusions cannot automatically be drawn. This is the object of the next sections.

3. Stochastic model on two periods with one funding asset

In this section we analyze the impact of stochastic assumptions on the previous conclusions.
From now on, the stochastic processes \(i_t, g_t, d_t\) are defined on a probability space \((\Omega, F, P)\). The survival probabilities will not be considered in this paper as random.

Our framework is the overlapping generation set-up with two generations existing at any time, the one working (aged \(x_0\)), and the one retiring (aged \(x_r\)), with \(x_r = x_0 + 1\).

### 3.1 Portfolio modelization

We start with a combined system. A fraction of the contributions of the working generations is invested in a funding system, in order to finance a part of the generation’s own future pension. The rest of the contribution is “given” at cohorts attaining retirement age through a PAYG mechanism. We aim at determining the optimal part which should be invested in the funding system in order to maximize the utility of the pension benefit received at retirement time.

Let us consider the cohort aged \(x_0\) at time \(t-1\).

At time \((t-1)\) this cohort pays a contribution amount: \(L(x_0,t-1)\pi S(t-1)\). Of this, a fraction \(a\) is invested in the contribution system, the rest, \(1-a\), being “invested” in the PAYG system (being paid to the persons retiring). We suppose that this fraction \(a\) is constant across generations and time. At time \(t\) the cohort is aged \(x_r\). As pension amount they receive:

- a “funded part”, the money invested in the pension fund and its return after one year:
  \[aL(x_0,t-1)\pi S(t-1)(1+i_t)\]

- a “non funded” part, fraction \((1-a)\) of the contribution of the cohorts aged \(x_0\) at time \(t\):
  \[(1-a)L(x_0,t)\pi S(t)\]

The mass of pensions for the cohort aged \(x_r\) at time \(t\) is given by:

\[L(x_0,t-1)\pi S(t-1)\left[a(1+i_t) + (1-a)(1+d_t)(1+g_t)\right]\]

At this moment the cohorts counts \(L(x_r,t)\) and the mean salary is \(S(t)\), which means that the replacement rate of the system for one surviving person of this cohort is:

\[RR(t) = \frac{\pi}{p(x_r,\lambda)} \left[ \frac{1+i_t}{1+g_t} + (1-a)(1+d_t) \right] \]

Let us define:

\[f_t = \frac{1+i_t}{1+g_t} - 1\]

(11)

It represents the rate of return of the investment deflated by the rate of increase of the mean salary. If the inflation index should only be driven by the evolution of the wages, this process represents the real rate of financial return (the excess/shortage of return over the wage growth). Later we will refer to this as the “financial process” to make the difference with the pure “demographic process”, \(d_t\).

As in this section we consider a two generations set up, only the values of the process at time \(t\) matter. Therefore, to lighten the formulas, we will not index the stochastic processes by the time.

And in order to simplify the notations even more we will write: \(I = 1+i\), \(G = 1+g\) and \(D = 1+d\). We define:

\[Z = a(1+f) + (1-a)(1+d) = a \cdot F + (1-a) \cdot D\]

(12)
Z, our “pension portfolio”, is the process that we want to maximize. Doing so, we consider as said before in the assumptions that the longevity risk \( \rho(\chi,1) \) is fixed and independent of \( f \) and \( d \). We aim at determining the optimal value for \( a \).

### 3.2 Portfolio maximization

#### 3.2.1 Framework and efficient border

We consider a mean variance framework. We therefore define the different efficient solutions as the mean-variance couples which maximize the expectation, given a certain degree of risk, or minimize the risk, for a specific level of expectation.

The maximization problem can be expressed as follows: for a certain level of expectation \( (\mu) \) of our process \( Z \), we seek to minimize its variance.

\[
P = \min_{a} \left( \frac{1}{2} \cdot \text{Var}[Z] \right) \quad \text{subject to} \quad E[Z] = \mu
\]

Using standard portfolio tools, we can easily determine the mean variance curve’s equation, i.e. the curve which contains all the optimal risk-return couples. The variance of \( Z \) becomes (see appendix 1):

\[
\text{Var}[Z] = \left( \frac{1}{E[F-D]} \right)^2 \left\{ \text{Var}[F-D] \cdot (E[Z])^2 - 2H \cdot E[Z] + \text{Var}[F \cdot E(D) - D \cdot E(F)] \right\}
\]

with
\[
H = \text{E}[D] \cdot \text{Var}[F] + \text{E}[F] \cdot \text{Var}[D] - \text{Cov}[F,D] \cdot \text{E}[F] - \text{Cov}[F,D] \cdot \text{E}[D]
\]

and where \( \text{E}[.] \) is the expectation function, \( \text{Var}[.] \) is the variance function and \( \text{Cov}[.] \) the covariance function.

The minimum of \( \text{Var}[Z] \) is attained when \( E[Z] = \frac{H}{\text{Var}[F-D]} \), i.e. when:

\[
a^* = \frac{\text{Var}[D] - \text{Cov}[F,D]}{\text{Var}[F-D]}
\]

At this point we find:

\[
\text{Var}[Z] = \text{Var}[F] \cdot \text{Var}[D] - \left( \text{Cov}[F,D] \right)^2
\]

The efficient part of the curve is the concave part of this parabola, i.e. the one where, to get a higher return, one needs to accept a higher risk. This concave part of the curve represents all the solutions that the State could choose, at one moment, knowing the financial and the demographical conditions.

#### 3.2.2 Choice of a utility function and optimal funding.

Once the efficient border has been defined, the choice of a specific combination on this curve, \( a^{\text{SOL}} \), depends on the individual preferences (i.e. on the State preference for the population). It can be
determined by the choice of a utility function. The following quadratic utility function will be used: 

\[ U[Z] = E[Z] - \frac{\gamma}{2} Var[Z], \]

where \( \gamma > 0 \) is the risk aversion coefficient. Then we find (see appendix 2):

\[
\frac{\delta U[Z]}{\delta a} = 0 \iff a^{\text{sol}} = \frac{\text{Var}[D] - \text{Cov}[F,D]}{\text{Var}[F-D]} + \frac{1}{\gamma} \cdot \frac{E[F] - E[D]}{\text{Var}[F] + \text{Var}[D]} = a^{*} + \frac{1}{\gamma} \cdot \Delta \quad (18)
\]

where:

\[
\Delta = \frac{\text{E}(F-D)}{\text{Var}(F-D)} \quad (19)
\]

Note that if \( F \) and \( D \) have similar expectations, the optimal portfolio is the one that minimize the variance (\( a^{\text{sol}} = a^{*} \)).

### 3.3 Analysis of the results

#### 3.3.1 Impact of a correlation between demography and finance

Most papers in relation with this topic suppose that the financial and the demographic processes are independent. Let us relax this hypothesis and analyze how the level of correlation between the financial and demographic risk does impact the fraction \( a \).

In case of independence, the solution is given by:

\[
a^{\text{sol}}_{\text{ind}} = \left( \frac{\text{Var}(D)}{\text{VAR}[F]+\text{Var}[D]} \right) + \frac{1}{\gamma} \cdot \frac{E[F] - E[D]}{\text{VAR}[F]+\text{Var}[D]} \quad (20)
\]

Does the fraction \( a \) increase or decrease when we introduce a possible link between risks?

Let us only take into consideration the case where \( F \) is more risky and therefore offers a higher return. We observe that, compared to the independence situation, the first part of relation (18) equal to \( a^{*} \), decreases if the link the risks is positive (\( \text{Cov}[F,D] > 0 \)). The second part of (18) equal to \( \Delta \), increases when the covariance is positive. So we cannot easily predict the variation of \( a^{\text{sol}} \) when introducing a correlation.

But we can find a kind of “rule” referring to the level of risk aversion. Let us rewrite \( a^{\text{sol}} \) in this way:

\[
a^{\text{sol}} = \frac{1}{2} + \frac{1}{\gamma} \cdot \frac{(E[F] - E[D]) - \frac{1}{2}(\text{Var}[F] - \text{Var}[D])}{\text{Var}[F] + \text{Var}[D] - 2\text{Cov}[F,D]} \quad (21)
\]

And let us note \( \gamma^{*} = \frac{E[F] - E[D]}{\text{Var}[F] - \text{Var}[D]} \). We find that \( a^{\text{sol}} \) will be higher than \( a^{\text{sol}}_{\text{ind}} \) in two cases:

- the covariance is positive and the risk aversion coefficient is lower than \( \gamma^{*} \);
- the covariance is negative and the risk aversion coefficient is higher than \( \gamma^{*} \).
So, there is an impact when we take into account a possible correlation between the risks. But the impact is not always positive or always negative. It depends on two things: the sign of the correlation and the level of the risk aversion.

3.3.2 Diversification effect between PAYG and funding

Let us note first that our framework is not the usual portfolio theory context where short selling is admitted. Here we cannot borrow from the financial market to invest more in PAYG or the other way round. So $a$ cannot be less than 0 or more than 1. Therefore the optimal solution will be of the form:

$$a^{opt} = \left( \min \left\{ a^{sol}, 1 \right\} \right)^+$$

Can we predict when the $a^{sol}$ will be out of bounds (i.e. the cases where we will have to constraint \(a\)) and where diversification disappears? We know that $a^{sol} = a^* + \frac{1}{\gamma} \Delta$. The first term $a^* = \frac{\text{Var}[D] - \text{Cov}[F, D]}{\text{Var}[F - D]}$, is linked to the correlation between the two risks. In the case of independence between the financial and the demographic risk, it is guaranteed that $a^*$ belongs to \([0,1]\). If the financial and demographic risks are correlated negatively we can also easily verify that $0 \leq a^* \leq 1$. But when the correlation between the processes is positive, we cannot predict anything: $a^*$ could get below 0 or over 1.

The second term, $\frac{1}{\gamma} \Delta = \frac{E[F - D]}{\gamma \text{Var}[F - D]}$, will be zero if the two risks have similar expectations. It will be positive in case the financial process has got a better return and better risk (it will be optimal to invest more in the financial asset). This term is first linked to the coefficient of risk aversion (\(\gamma\)): if the risk-aversion is high, one will choose a coefficient $a$ which is very closed to the less risky solution ($a^*$). It is also linked to $\Delta$ which measures the excess return per square unit of risk.

The following table summarizes all the possible situations:

<table>
<thead>
<tr>
<th>Case</th>
<th>$E[F] &gt; E[D] \leftrightarrow \Delta &gt; 0$</th>
<th>$E[F] &gt; E[D] \leftrightarrow \Delta &lt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a^* &gt; 0$</td>
<td>$[1] a^{OPT} \in [0,1]$</td>
<td>$[4] a^{OPT} = 0$</td>
</tr>
<tr>
<td>$a^* &gt; 1$</td>
<td>$[2] a^{OPT} = 1$</td>
<td>$[5] a^{OPT} = [0,1]$</td>
</tr>
<tr>
<td>$a^* \in [0,1]$</td>
<td>$[3] a^{OPT} \in [a^*,1]$</td>
<td>$[6] a^{OPT} \in [0,a^*]$</td>
</tr>
</tbody>
</table>

We observe six different situations. If we only consider the cases where the financial process is a higher return/higher risk process, we only look at the first column. In cells [2], diversification is not worth and only funding is the best solution. This case is actually an “arbitrage case” because in this situation the process $F$ has a higher return than $D$ but it is less risky. In cases [1], every strategy can be a possible response. The last case [3] lays somewhere between those two extremes. In this case $a^*$ does for sure belong to \([0,1]\). But as $\Delta$ is positive, people will be willing to increase the fraction to be invested in the funding scheme, depending on their risk aversion.
3.4 Particular models

So far, we have not specified any stochastic process for the growth of the population, the growth of the wages and the interest rate. In this section we propose various stochastic models for the financial and demographic risks (geometric Brownian motions and mean reverting processes) and use then the formulas seen before in order to obtain explicit forms of the optimal strategy.

3.4.1 Geometric Brownian motion model

In this section, we provide and analyze the results if the processes follow geometric Brownian motions:

\[ dX = \mu X \, dt + \sigma X \, dW(t) \]

with \( x = d, f \) and \( X = D, F \).

That kind of geometric Brownian motion is often used for financial processes. For the demographic process it has the interesting property that it cannot fall below zero.

Let us stay general as possible and consider that the two processes \((W_d, W_f)\) are correlated:

\[ d f = \rho \, dW_d(t) \, dW_f(t) \]

We find then:

\[
\begin{align*}
E[F] &= e^{\mu_f} \\
VAR[F] &= e^{2\mu_f} \left( e^{\sigma_f^2} - 1 \right) \\
E[D] &= e^{\mu_d} \\
VAR[D] &= e^{2\mu_d} \left( e^{\sigma_d^2} - 1 \right) \\
Cov[F,D] &= e^{\mu_f + \mu_d} \left( e^{\sigma_f^2 - 1} \right)
\end{align*}
\]

with: \( \sigma_{fd} = \rho \sigma_f \sigma_d \)

Having calculated \( E[F], \ VAR[F], \ E[D], \ VAR[D] \) and \( Cov[F,D] \), we can inject their value in the above results. For example

\[
a^{Sol} = \frac{\text{Var}[D] - \text{Cov}[F,D]}{\text{Var}[F-D]} + \frac{1}{\gamma} \frac{E[F-D]}{\text{Var}[F-D]} = a^* + \frac{1}{\gamma} \Delta
\]

with:

\[
a^* = \frac{e^{2\mu_f + \sigma_f^2} \left( e^{\sigma_f^2} - 1 \right) - e^{\mu_f + \mu_d + \frac{1}{2}(\sigma_f^2 + \sigma_d^2)} \left( e^{\sigma_d^2} - 1 \right)}{e^{2\mu_f + \sigma_f^2} \left( e^{\sigma_f^2} - 1 \right) + e^{2\mu_f + \sigma_d^2} \left( e^{\sigma_d^2} - 1 \right) - 2e^{\mu_f + \mu_d + \frac{1}{2}(\sigma_f^2 + \sigma_d^2)} \left( e^{\sigma_d^2} - 1 \right)}
\]

\[
\Delta = \frac{e^{\mu_f + \sigma_f^2} \left( e^{\sigma_f^2} - 1 \right) - e^{\mu_f + \sigma_d^2} \left( e^{\sigma_d^2} - 1 \right)}{e^{2\mu_f + \sigma_f^2} \left( e^{\sigma_f^2} - 1 \right) + e^{2\mu_f + \sigma_d^2} \left( e^{\sigma_d^2} - 1 \right) - 2e^{\mu_f + \mu_d + \frac{1}{2}(\sigma_f^2 + \sigma_d^2)} \left( e^{\sigma_d^2} - 1 \right)}
\]

In case of independence between the financial and the demographic processes \( (\rho = \sigma_{fd} = 0) \), these values become simply:
\[ a^* = \frac{e^{2\mu_j + \sigma_j^2} \cdot \left( e^{\sigma_j^2} - 1 \right)}{e^{2\mu_j + \sigma_j^2} \cdot \left( e^\sigma_j - 1 \right) + e^{2\mu_j + \sigma_j^2} \cdot \left( e^{\sigma_j^2} - 1 \right)} \]

\[ \Delta = \frac{e^{\mu_j + \frac{1}{2}\sigma_j^2} - e^{\mu_j + \frac{1}{2}\sigma_j^2}}{e^{2\mu_j + \sigma_j^2} \cdot \left( e^\sigma_j - 1 \right) + e^{2\mu_j + \sigma_j^2} \cdot \left( e^{\sigma_j^2} - 1 \right)} \]

3. 4. 2 Mean reverting model

In this section we analyze the case where the financial and the demographic processes follow a mean reverting process. For instance, for the financial process we will have:

\[ F(t) = F_0 e^{\int_0^t r(s) ds} \]

with

\[ r(t) = \gamma + Y(t) \]

where \( Y(t) \) is an Ornstein-Ulenbeck process described by the following differential equation:

\[ dY(s) = -\beta_f Y(s) ds + \rho_f dW_f(s) \]

with \( Y(0) = 0 \) and \( \beta_f, \rho_f > 0 \).

The solution is:

\[ Y(t) = \rho_f \int_0^t e^{-\beta_f (t-s)} dW_f(s) \]

The financial process becomes in this model:

\[ F(t) = F_0 \cdot e^{\gamma t} \cdot e^{\int_0^t \rho_f \int_0^s e^{-\beta_f (t-u)} dW_f(u) ds} = F_0 \cdot e^{\gamma t} \cdot \rho_f \int_0^t e^{-\beta_f (t-s)} dW_f(s) \]

Finally the financial return is given by:

\[ F_t = \frac{F(t)}{F(t-1)} = \frac{e^{\gamma t + \rho_f \int_0^t e^{-\beta_f (t-u)} dW_f(u)} - \gamma \cdot (t-1) - \rho_f \int_0^t \frac{1-e^{-\beta_f (t-u)}}{\beta_f} \cdot \frac{\beta_f (t-u)}{\beta_f} \cdot dW_f(u)}{e^{\gamma (t-1) + \rho_f \int_0^{t-1} \frac{1-e^{-\beta_f (t-1-u)}}{\beta_f} \cdot dW_f(u)} - \gamma \cdot (t-1) - \rho_f \int_0^{t-1} \frac{1-e^{-\beta_f (t-1-u)}}{\beta_f} \cdot dW_f(u)} \]

\[ F = e^{\gamma t + \rho_f \int_0^t e^{-\beta_f (t-u)} dW_f(u)} \]

In particular we can see that the variable \( F \) is log normally distributed (exactly as in the geometric Brownian case of section 3.4.1).

For the demographic process, assuming the same kind of dynamic, we get:
\[
D = e^{\gamma_d + \rho_{d} \left[ 1 - e^{-\beta_d (t-u)} \right] - \gamma_s + \rho_{s} \left[ 1 - e^{-\beta_s (t-u)} \right]} \frac{dW_s}{\rho_{s}} - \frac{1}{\rho_{s}} e^{\gamma_s + \rho_{s} \left[ 1 - e^{-\beta_s (t-u)} \right]} \frac{dW_s}{\rho_{s}}
\]  
(25)

We assume also a correlation between the two Brownian motions:

\[
\text{E}(W_t(t), W_s(t)) = \rho_{t,s} \cdot t
\]

Taking into account the log normal environment, we can adapt formulas (22) and (23) to this case. We obtain finally the following results (see appendix 3 for details):

\[
a^* = \frac{e^{2a_i + b_i} \left( e^{b_i^2} - 1 \right) - e^{a_j + a_d + \frac{1}{2} (b_j^2 + b_d^2)} \left( e^{b_j^2} - 1 \right)}{e^{2a_i + b_i} \left( e^{b_i^2} - 1 \right) + e^{2a_j + b_j} \left( e^{b_j^2} - 1 \right) - 2 e^{a_j + a_d + \frac{1}{2} (b_j^2 + b_d^2)} \left( e^{b_j^2} - 1 \right)}
\]

\[
\Delta = \frac{e^{a_i + b_i} \left( e^{b_i^2} - 1 \right) - e^{a_i + b_d} \left( e^{b_d^2} - 1 \right)}{e^{a_i + b_i} \left( e^{b_i^2} - 1 \right) + e^{a_i + b_d} \left( e^{b_d^2} - 1 \right) - 2 e^{a_i + b_d} \left( e^{b_i^2} - 1 \right)}
\]

(26)

(27)

with:

\[
a_i = \gamma_i
\]

\[
b_i = \frac{\rho_{i}^2}{2 \beta_i} \left[ 2 \beta_i - 2 e^{2 \beta_i} - e^{2 \beta_i (t-1)} + 2 e^{\beta_i} + 2 e^{\beta_i (t-1)} \right]
\]

\[
a_d = \gamma_d
\]

\[
b_d = \frac{\rho_{d}^2}{2 \beta_d} \left[ 2 \beta_d - 2 e^{2 \beta_d} + 2 e^{2 \beta_d (t-1)} \right]
\]

\[
b_{j,d} = \frac{\rho_{j} \rho_{d}}{\beta_j \beta_d} \left[ \frac{1 + 2}{\beta_j + \beta_d} \frac{e^{-(\beta_j + \beta_d) t}}{\beta_j + \beta_d} + \frac{e^{-\beta_j t}}{\beta_j} + \frac{e^{-\beta_j (t-1)}}{\beta_j} + \frac{e^{-\beta_d t}}{\beta_d} + \frac{e^{-\beta_d (t-1)}}{\beta_d} \right]
\]

4. Generalization in a model with three periods of time

In this section we generalize the stochastic case in an overlapping generation model with more than just two periods.

The stochastic processes \( i, g, d \) are defined still on a probability space \( (\Omega, F, P) \).

Our framework is the overlapping generation set-up with three generations existing at any time, two generations working: young workers (aged \( x_0 \)) and old workers (aged \( x_0 + 1 \)), and one retired generation (aged \( x_r = x_0 + 2 \)). We still use the following notations:

\( a \): fraction invested in funding

\( 1 - a \): fraction invested in PAYG

We will denote the survival probability by:

\[
p(x,s,t) = \frac{L(x+s,t+s)}{L(x,t)}
\]

(probability being alive at time \( t \) at age \( x \) to be still alive after \( s \) years). As in section 3, these probabilities are assumed to be known.
Let us consider the cohort aged \( x_0 \) at time \( t - 2 \).

At time \((t-2)\) this cohort pays a contribution mass: \( L(x_0, t-2) \pi S(t-2) \).

At time \((t-1)\) this cohort pays a contribution mass: \( L(x_0+1, t-1) \pi S(t-1) \).

At time \( t \) the cohort is aged \( x_r \). As pension amount they receive:

- a funded part, the money invested in the pension fund and its return:

\[
\left(1 + i_{-1}\right) \left(1 + i_r\right) + p(x_0, 1, t-2)(1 + g_{r-1})(1 + i_r) + \left(1 - a\right) \left(1 + g_{r-1}\right) \left(1 + g_r\right) \left(1 + d_{r-1}\right) \left(1 + d_r\right) + p(x_0, 1, t-1)
\]

The mass of pensions for the cohort aged \( x_r \) at time \( t \) is given by:

\[
L(x_0, t-2) \pi S(t-2) \left(a \left(1 + i_r\right) \left[(1 + i_{-1}) + p(x_0, 1, t-2)(1 + g_{r-1})\right] + \left(1 - a\right) \left(1 + g_{r-1}\right) \left(1 + g_r\right) \left(1 + d_{r-1}\right) \left(1 + d_r\right) + p(x_0, 1, t-1)\right)
\]

At this moment the cohorts counts \( L(x, t) \) and the mean salary is \( S(t) \), which means that the replacement rate of the system for one surviving person at this cohort is:

\[
RR(t) = \frac{\pi}{p(x_0, 1, t-1)} \left[a \left(1 + i_r\right) \left(1 + g_{r-1}\right) \left(1 + g_r\right) \left(1 + d_{r-1}\right) \left(1 + d_r\right) + p(x_0, 1, t-1)\right]
\]

Where: \( p(x_0, 2, t-2) = p(x_0, 1, t-2) \cdot p(x_0+1, 1, t-1) \)

\( p(x_0, 1, t-1) = p(x_0, 1, t-2) \cdot (1 + h_{r-1}) \) (h representing the longevity effect)

\[
RR(t) = \frac{\pi}{p(x_0, 1, t-1)} \left[a \left(1 + i_r\right) \left(1 + g_{r-1}\right) \left(1 + g_r\right) \left(1 + d_{r-1}\right) \left(1 + d_r\right) + p(x_0, 1, t-1)\right] + \left(1 - a\right) \left(1 + d_{r-1}\right) \left(1 + d_r\right)
\]

Let us define: \( f_i = \frac{1 + i_r}{1 + g_r} - 1 \). It represents the rate of return of the investment deflated by the rate of increase of the mean salary. If the inflation index should only be driven by the evolution of the wages, this process represents the real rate of financial return. Later we will refer to this as the “financial process” to make the difference with the pure “demographic process”, \( d_i \).

\[
RR(t) = \frac{\pi}{p(x_0+1, 1, t-1)} \left[a \left(1 + f_i\right) \left(1 + f_{r-1}\right) \left(1 + d_{r-1}\right) \left(1 + d_r\right) + p(x_0, 1, t-1)\right] + \left(1 - a\right) \left(1 + d_{r-1}\right) \left(1 + d_r\right)
\]

In order to simplify the notation we write:

\( F_i = \left(1 + f_i\right) \), \( D_i = \left(1 + d_i\right) \), \( I_i = \left(1 + i_r\right) \) the replacement ratio becomes:

\[
RR(t) = \frac{\pi}{p(x_0+1, 1, t-1)} \left[a F_i \left(1 + F_{r-1}\right) + (1 - a) D_{r-1} \left(1 + D_r\right) + (1 + h_{r-1})\right]
\]

We define the pension portfolio:
If we define: $X = F_t \left( \frac{F_{t-1}}{p(x_0,1,t-2)} + 1 \right)$ and $Y = D_t \left( \frac{D_t}{p(x_0,1,t-2)} + (1 + h_{t-1}) \right)$ our pension portfolio can be written as:

$$Z = aX + (1 - a)Y$$

This portfolio having the same form as (12), after computation of $E[X]$, $Var[X]$, $E[Y]$, $Var[Y]$ and $Cov[X,Y]$ we can inject their value in the results obtained in the previous section. In particular, the optimal mixing coefficient will be given by (cf. (18)):

$$a^{sol} = \frac{Var[X] - Cov[X,Y] + \frac{1}{\gamma} E[X - Y]}{Var[X - Y]} = a^* + \frac{1}{\gamma} \Delta$$

(31)

Appendix 4 details the computation in a geometric Brownian environment.

5. Generalization in a model with two assets

5.1. General model

In this section we assume that the contributions can be invested in two different assets, (for instance a stock index $F_1$ and a bond index $F_2$):

$$F = \beta \cdot F_1 + (1 - \beta) \cdot F_2$$

(32)

where $\beta$ is the fraction of the financial assets invested in bonds, and $(1 - \beta)$ is the fraction of the financial assets invested in stocks.

Then the portfolio becomes:

$$Z = a \cdot \beta \cdot F_1 + a \cdot (1 - \beta) \cdot F_2 + (1 - a) \cdot D$$

This is can written as:

$$Z = x_i \cdot F_1 + (a - x_i) \cdot F_2 + (1 - a) \cdot D$$

(33)

where:

$$x_i = \text{part of the total contribution in } F_1$$

The two first moments of the portfolio become now:

$$E[Z] = a \cdot \beta \cdot E[F_1] + a(1 - \beta) \cdot E[F_2] + (1 - a) \cdot E[D]$$

$$Var[Z] = a^2 \cdot \beta^2 \cdot Var[F_1] + a^2 (1 - \beta)^2 \cdot Var[F_2] + (1 - a)^2 \cdot Var[D] + 2a^2 \beta(1 - \beta) \cdot Cov[F_1,F_2] + 2a(1 - a) \beta \cdot Cov[F_1,D] + 2a(1 - a)(1 - \beta) \cdot Cov[F_2,D]$$

Our purpose is still to maximize the function:

$$U(Z) = E[Z] - \frac{\gamma}{2} Var[Z]$$

(34)
We have then the following proposition:

**PROPOSITION 5.1.**

The optimal solution of the problem \( \max_{a,x_1} U(Z) \) is given by:

- optimal part in funding:

\[
a = \frac{\text{var}(F_1 - F_2).m_1 - \text{cov}(F_1 - F_2, F_2 - D).m_2}{\text{var}(F_1 - F_2).\var(F_2 - D) - (\text{cov}(F_1 - F_2, F_2 - D))^2}
\]

(35)

- optimal part in the first asset:

\[
\beta = \frac{\text{var}(F_2 - D).m_2 - \text{cov}(F_1 - F_2, F_2 - D).m_1}{a \cdot \text{var}(F_1 - F_2).m_1 - \text{cov}(F_1 - F_2, F_2 - D).m_2}
\]

(36)

where:

\[
m_1 = \text{var}(D) - \text{cov}(F_2, D) + \frac{1}{\gamma} E(F_2 - D)
\]

(37)

\[
m_2 = \text{cov}(F_2 - F_1, D) + \frac{1}{\gamma} E(F_1 - F_2)
\]

Proof:

The function (34) to maximize is explicitly given by:

\[
U(Z) = x_1 EF_1 + (a - x_1).EF_2 + (1 - a).ED - \frac{\gamma}{2} x_1^2 \cdot \text{var} F_1 - \frac{\gamma}{2} (a - x_1)^2 \cdot \text{var} F_2 - \frac{\gamma}{2} (1 - a)^2 \cdot \text{var} D - \gamma(ax_1 - x_1^2) \text{cov}(F_1, F_2) - \gamma(x_1 - ax_1) \text{cov}(F_1, D) - \gamma(a - a^2 - x_1 + ax_1) \text{cov}(F_2, D)
\]

The first order derivatives are given respectively by:

\[
\frac{\partial U}{\partial x_1} = EF_1 - EF_2 - \gamma x_1 \cdot \text{var} F_1 + \gamma(a - x_1) \cdot \text{var} F_2 - \gamma(a - 2x_1) \text{cov}(F_1, F_2)
\]

\[
- \gamma(1 - a) \text{cov}(F_1, D) + \gamma(1 - a) \text{cov}(F_2, D)
\]

\[
\frac{\partial U}{\partial a} = EF_2 - ED - \gamma(a - x_1) \text{var} F_2 + \gamma(1 - a) \text{var} D - \gamma x_1 \text{cov}(F_1, F_2)
\]

\[
+ \gamma x_1 \text{cov}(F_1, D) - \gamma(1 - 2a + x_1) \text{cov}(F_2, D)
\]

Putting these two derivatives equal to zero we obtain the following system for the optimal pair \((x_1, a)\):
\[ x_1 \cdot \text{var}(F_1 - F_2) + a \cdot (\text{var} F_2 + \text{cov}(F_1, F_2) - \text{cov}(F_1 - F_2, D)) = \frac{E(F_1 - F_2)}{\gamma} - \text{cov}(F_1 - F_2, D) \]  \hspace{1cm} (38)

\[ x_1 \cdot (\text{var} F_2 + \text{cov}(F_1, F_2) - \text{cov}(F_1 - F_2, D)) + a \cdot \text{var}(F_2 - D) = \frac{E(F_2 - D)}{\gamma} - \text{cov}(F_2 - D, D) \]  \hspace{1cm} (39)

Solving these two equations gives after straight computation the solutions (35) and (36).

Remark:

The optimal part in funding (35) can also be written as:

\[ a = \frac{-\text{var}(F_1 - F_2) \cdot \text{cov}(F_2 - D, D) + \text{cov}(F_1 - F_2, F_2 - D) \cdot \text{cov}(F_1 - F_2, D)}{\text{var}(F_1 - F_2) \cdot \text{var}(F_2 - D) - (\text{cov}(F_1 - F_2, F_2 - D))^2} + \frac{1}{\gamma} \cdot \frac{\text{var}(F_1 - F_2) \cdot E(F_2 - D) - \text{cov}(F_1 - F_2, F_2 - D) \cdot E(F_1 - F_2)}{\text{var}(F_1 - F_2) \cdot \text{var}(F_2 - D) - (\text{cov}(F_1 - F_2, F_2 - D))^2} \]  \hspace{1cm} (40)

This form can be compared with the corresponding solution with only one asset (see section 3):

\[ a = \frac{-\text{cov}(F - D, D)}{\text{var}(F - D)} + \frac{1}{\gamma} \cdot \frac{E(F - D)}{\text{var}(F - D)} \]  \hspace{1cm} (41)

5.2. Particular cases

Let us consider three particular cases of proposition 5.1 in order to check the presence of an eventual diversification effect between pay as you go and funding:

5.2.1. Two identical assets: \( F_1 = F_2 \):

In this case, equations (38) and (39) generate just one condition for the parameter \( a \):

\[ a \cdot (\text{var}(F_2 - D)) = \frac{E(F_2 - D)}{\gamma} - \text{cov}(F_2 - D, D) \]

which corresponds to the situation with one asset (cf. (41)).

5.2.2. Three independent processes: \( \text{cov}(F_1, F_2) = \text{cov}(F_1, D) = \text{cov}(F_2, D) = 0 \)

The two financial assets are assumed to be independent and the demography is not correlated with finance. Then the optimal part in funding takes the following form:

\[ a = \frac{\text{var} D \cdot (\text{var} F_1 + \text{var} F_2)}{\text{var} F_1 \cdot \text{var} F_2 + \text{var} F_1 \cdot \text{var} D + \text{var} F_2 \cdot \text{var} D} + \frac{1}{\gamma} \cdot \frac{\text{var} F_1 \cdot (E(F_1 - D)) + \text{var} F_2 \cdot (E(F_2 - D))}{\text{var} F_1 \cdot \text{var} F_2 + \text{var} F_1 \cdot \text{var} D + \text{var} F_2 \cdot \text{var} D} \]  \hspace{1cm} (42)

This form can be compared with the independent case with only one asset (see (20)).
\[ a = \frac{\text{var} D}{\text{var} F + \text{var} D} + \frac{1}{\gamma} \frac{\text{E}(F - D)}{\text{var} F + \text{var} D} \]

5.2.3. One risk free asset and one risky asset independent from demography:

In this case, we have:

\[ \text{EF}_2 = R = 1 + r < \text{EF}_1 \]
\[ \text{var} F_2 = 0 \]
\[ \text{cov}(F_1, D) = 0 \]
where \( r \) is a constant risk free rate.

Then relation (42) gives the simple expression:

\[ a = 1 - \frac{1}{\gamma} \frac{\text{ED} - R}{\text{var} D} \]

This means that the optimal part in pay as you go is directly proportional to the risk premium of the demographic risk with respect to the risk free rate:

\[ 1 - a = \frac{1}{\gamma} \frac{\text{ED} - R}{\text{var} D} = \frac{1}{\gamma} \lambda_D \]  
(43)

It is interesting to remark that the optimal spreading between pay as you go and funding is independent from the characteristics of the risky asset which will just have an effect on the sharing inside the funding part between the risk free asset and the risky asset.

The optimal part in the risky asset becomes:

\[ x_i = \frac{1}{\gamma} \frac{\text{EF}_i - R}{\text{var} F_i} = \frac{1}{\gamma} \lambda_{F_i} \]  
(44)

Finally the part to invest in the risk free asset is equal to:

\[ a - x_i = 1 - \frac{1}{\gamma} (\lambda_D + \lambda_{F_i}) \]  
(45)

In particular, we will have in this case a diversification effect between funding and pay as you go if we have the following condition:

\[ 0 < a < 1 \]

The risk aversion coefficient \( \gamma \) being positive, the condition \( a < 1 \) (i.e. positive part to invest in pay as you go) is equivalent to:

\[ \text{ED} > R \]

This relation, meaning that the demographic risk must have a mean return bigger than the risk free
rate, seems natural: if the demographic return had a mean lower than the risk free rate, and this together with a positive variance, investing in pay as you go is surely not optimal.

The second condition \( a > 0 \) implies:

\[
\gamma > \lambda_D
\]

Inside the funding part we can observe:

1°) the part to invest in the risky asset is always positive (no short selling) (cf. (44))

2°) the part to invest in the risk free rate is positive if we have the following condition (cf. (45)):

\[
\gamma > \lambda_D + \lambda_F
\]

We can summarize all these conditions in the following proposition:

**PROPOSITION 5.2.**

In the model with one riskless asset and one risky asset independent from demography, if we assume that the mean return of the risky asset and of the demographic effect are strictly bigger than the risk free rate, then:

(i) there will be diversification between pay as you go and funding if:

\[
\gamma > \lambda_D = \frac{ED - R}{\text{Var}D}
\]

(ii) there will be a positive position in the three processes if:

\[
\gamma > \lambda_D + \lambda_F = \frac{ED - R}{\text{Var}D} + \frac{EF - R}{\text{Var}F}
\]

5.3. Numerical illustrations

We will illustrate here the models of sections 5.2.2 (3 independent processes) and of section 5.2.3 (one riskless asset) in a geometric Brownian motion model (section 3.4.1).

5.3.1. Model with 3 independent processes:

If the three processes follow geometric Brownian motions as the model presented in section 5.2.3 for the case of one financial asset, then we have:

\[
a = \frac{\text{var}D.(\text{var} F_1 + \text{var} F_2)}{\text{var} F_1. \text{var} F_2 + \text{var} F_1. \text{var} D + \text{var} F_2. \text{var} D} \quad \gamma \quad \frac{1}{\text{var} F_1. \text{var} F_2 + \text{var} F_1. \text{var} D + \text{var} F_2. \text{var} D} = a^* + \frac{1}{\gamma} \Delta
\]

with:

\[
a^* = \frac{e^{2\mu_1} \cdot (e^{\sigma_1^2} - 1) \cdot \left[e^{2\mu_2} \cdot (e^{\sigma_2^2} - 1) + e^{2\mu_1} \cdot (e^{\sigma_2^2} - 1)\right]}{e^{2(\mu_1 + \mu_2)} \cdot (e^{\sigma_1^2} - 1) \cdot (e^{\sigma_2^2} - 1) + e^{2(\mu_1 + \mu_2)} \cdot (e^{\sigma_1^2} - 1) \cdot (e^{\sigma_2^2} - 1) + e^{2(\mu_2 + \mu_3)} \cdot (e^{\sigma_2^2} - 1) \cdot (e^{\sigma_3^2} - 1)}
\]

and
\[
\Delta = \frac{e^{2\mu_1} \cdot \left( e^{\sigma_1^2} - 1 \right) \cdot \left[ e^{\mu_2} - e^{\mu_3} \right] + e^{2\mu_1} \cdot \left( e^{\sigma_1^2} - 1 \right) \cdot \left[ e^{\mu_2} - e^{\mu_3} \right] + e^{2\mu_2} \cdot \left( e^{\sigma_1^2} - 1 \right) \cdot \left[ e^{\mu_2} - e^{\mu_3} \right] + e^{2\mu_2} \cdot \left( e^{\sigma_1^2} - 1 \right) \cdot \left[ e^{\mu_2} - e^{\mu_3} \right] + e^{2\mu_3} \cdot \left( e^{\sigma_1^2} - 1 \right) \cdot \left[ e^{\mu_2} - e^{\mu_3} \right]}{e^{2(\mu_1 + \mu_2)} \cdot \left( e^{\sigma_1^2} - 1 \right) + e^{2(\mu_1 + \mu_2)} \cdot \left( e^{\sigma_1^2} - 1 \right) + e^{2(\mu_1 + \mu_2)} \cdot \left( e^{\sigma_1^2} - 1 \right) + e^{2(\mu_1 + \mu_2)} \cdot \left( e^{\sigma_1^2} - 1 \right) + e^{2(\mu_1 + \mu_2)} \cdot \left( e^{\sigma_1^2} - 1 \right)}
\]

The fraction \(1-a\) invested in PAYG is:

\[
1 - a = \frac{\text{var } F_1 + \text{var } F_2}{\text{var } F_1 \cdot \text{var } F_2 \cdot \text{var } D \cdot \text{var } F_2 \cdot \text{var } D} - \frac{1}{\gamma} \cdot \frac{\text{var } F_2 \cdot (E(F_1 - D)) + \text{var } F_1 \cdot (E(F_2 - D))}{\text{var } F_1 \cdot \text{var } F_2 \cdot \text{var } D \cdot \text{var } F_2 \cdot \text{var } D}
\]

\[
1 - a = \frac{e^{2\mu_1} \cdot \left( e^{\sigma_1^2} - 1 \right) + e^{2\mu_2} \cdot \left( e^{\sigma_1^2} - 1 \right)}{e^{2(\mu_1 + \mu_2)} \cdot \left( e^{\sigma_1^2} - 1 \right) + e^{2\mu_1} \cdot \left( e^{\sigma_1^2} - 1 \right) + e^{2\mu_2} \cdot \left( e^{\sigma_1^2} - 1 \right) + e^{2\mu_3} \cdot \left( e^{\sigma_1^2} - 1 \right) + e^{2\mu_1} \cdot \left( e^{\sigma_1^2} - 1 \right)}
\]

The fraction \(x_1\) invested in \(F_1\) is:

\[
x_1 = \frac{\text{var } D \cdot \text{var } F_2}{\text{var } F_1 \cdot \text{var } F_2 \cdot \text{var } D \cdot \text{var } F_1 \cdot \text{var } D} - \frac{1}{\gamma} \cdot \frac{\text{var } F_2 \cdot (E(D - F_1)) + \text{var } D \cdot (E(F_2 - F_1))}{\text{var } F_1 \cdot \text{var } F_2 \cdot \text{var } D \cdot \text{var } F_1 \cdot \text{var } D}
\]

\[
x_1 = \frac{e^{2(\mu_1 + \mu_2)} \cdot \left( e^{\sigma_1^2} - 1 \right) \cdot \left( e^{\sigma_1^2} - 1 \right)}{e^{2(\mu_1 + \mu_2)} \cdot \left( e^{\sigma_1^2} - 1 \right) + e^{2\mu_1} \cdot \left( e^{\sigma_1^2} - 1 \right) + e^{2\mu_2} \cdot \left( e^{\sigma_1^2} - 1 \right) + e^{2\mu_3} \cdot \left( e^{\sigma_1^2} - 1 \right) + e^{2\mu_1} \cdot \left( e^{\sigma_1^2} - 1 \right)}
\]

Finally, the fraction \(x_2\) invested in \(F_2\) is:

\[
x_2 = \frac{\text{var } D \cdot \text{var } F_1}{\text{var } F_1 \cdot \text{var } F_2 \cdot \text{var } D \cdot \text{var } F_1 \cdot \text{var } D} - \frac{1}{\gamma} \cdot \frac{\text{var } F_1 \cdot (E(D - F_2)) + \text{var } D \cdot (E(F_1 - F_2))}{\text{var } F_1 \cdot \text{var } F_2 \cdot \text{var } D \cdot \text{var } F_1 \cdot \text{var } D}
\]

\[
x_2 = \frac{e^{2(\mu_1 + \mu_2)} \cdot \left( e^{\sigma_1^2} - 1 \right) \cdot \left( e^{\sigma_1^2} - 1 \right)}{e^{2(\mu_1 + \mu_2)} \cdot \left( e^{\sigma_1^2} - 1 \right) + e^{2\mu_1} \cdot \left( e^{\sigma_1^2} - 1 \right) + e^{2\mu_2} \cdot \left( e^{\sigma_1^2} - 1 \right) + e^{2\mu_3} \cdot \left( e^{\sigma_1^2} - 1 \right) + e^{2\mu_1} \cdot \left( e^{\sigma_1^2} - 1 \right)}
\]
\[
\frac{-1}{\gamma \cdot e^{2(\mu_1 \gamma \mu_2)}} \cdot \left( e^{\sigma_1^2 - 1} \cdot \left( e^{\mu_1 - e^{\mu_1^2}} + e^{2\mu_1} \cdot \left( e^{\sigma_1^2 - 1} \right) \cdot \left( e^{e^{\mu_1 - e^{\mu_1^2}}} \right) \right) + e^{2(\mu_1 \gamma \mu_2)} \cdot \left( e^{\sigma_1^2 - 1} \right) \cdot \left( e^{e^{\mu_1 - e^{\mu_1^2}}} \right) \right)
\]

\[
\text{return asset 1} = \ 0.05 \\
\text{vol asset 1} = \ 0.17 \\
E(F1) = \ 1.051271096 \\
\text{Var}(F1) = \ 0.032405443
\]

\[
\text{return asset 2} = \ 0.06 \\
\text{vol asset 2} = \ 0.18 \\
E(F2) = \ 1.061836547 \\
\text{Var}(F2) = \ 0.037129142
\]

\[
\text{return demo} = \ 0.04 \\
\text{vol demo} = \ 0.15 \\
E(D) = \ 1.040810774 \\
\text{Var}(D) = \ 0.024650234
\]

5.3.2. Model with one riskless asset:

If the three processes follow geometric Brownian motions as the model presented in section 5.2.3 for the case of one financial asset, then we have:

\[
a_1 = 1 - \frac{1}{\gamma} \cdot \frac{ED - R}{\text{Var} \ D} \\
1 - a_1 = \frac{1}{\gamma} \cdot \frac{ED - R}{\text{Var} \ D}
\]

\[
x_1 = \frac{EF - R}{\text{Var} \ F \cdot \gamma}
\]

The part to invest in the risk free asset is equal to:

\[
x_1 = \frac{-1}{\gamma} \cdot \frac{ED - R}{\text{Var} \ D} - \frac{EF - R}{\gamma \cdot \text{Var} \ F}
\]

\[
x_2 = \frac{-1}{\gamma} \cdot \frac{e^{\mu_1} - e^{\mu_1^2}}{e^{2\mu_1} \cdot \left( e^{\sigma_1^2 - 1} \right)} - \frac{1}{\gamma} \cdot \frac{e^{\mu_1} - e^{\mu_1^2}}{e^{2\mu_1} \cdot \left( e^{\sigma_1^2 - 1} \right)}
\]
Funded and PAYG pension schemes may seem completely different but are in fact quite complementary because they deal with different risks. The recent financial crisis has shown the limit of a pure funded approach and the future well known challenges of demography threaten the sustainability of pure PAYG systems.

In a deterministic framework, Aaron rule applies: the relative returns of both schemes indicate which one to choose. One scheme is always better than the other one and there is no room for a blending of them.

But, in a stochastic framework, diversification is generally useful. We have found the optimal level of diversification in several stochastic models with different demographic structures or different asset possibilities. In this paper we have considered the financial risk, the demographic risk and the inflation risk and we have used a mean variance approach as optimization criterion. Future extension could deal with the longevity risk (present as well in funding as in PAYG) and with more general utility functions, such as CRRA utilities.

References

Appendix 1: equation of the efficient border (relation (14))

\[ Z = a \cdot F + (1-a) \cdot D \]

\[ \text{Var}[Z] = a^2 \cdot \text{Var}[F] + (1-a)^2 \cdot \text{Var}[D] + 2a(1-a) \cdot \text{Cov}[F,D] \]

\[ E[Z] = a \cdot E[F] + (1-a) \cdot E[D] \]

\[ a = \frac{E[Z] - E[D]}{E[F] - E[D]} \]

Substituting into the variance formula we have:

\[ \text{Var}[Z] = \left( \frac{E[Z] - E[D]}{E[F] - E[D]} \right)^2 \cdot \text{Var}[F] + \left( 1 - \frac{E[Z] - E[D]}{E[F] - E[D]} \right)^2 \cdot \text{Var}[D] + \]

\[ + 2 \left( \frac{E[Z] - E[D]}{E[F] - E[D]} - \left( \frac{E[Z] - E[D]}{E[F] - E[D]} \right)^2 \right) \cdot \text{Cov}[F,D] \]


\[ \text{Var}[Z] = \frac{1}{[E[F] - E[D]]^2} \left( [E[Z]^2 \cdot \text{Var}[F] + [E[D]^2 \cdot \text{Var}[F] - 2E[Z]E[D]\text{Var}[F] + \right] \]

\[ + [E[F]^2 \cdot \text{Var}[D] + [E[Z]^2 \cdot \text{Var}[D] - 2E[F]E[Z]\text{Var}[D] + \]


\[ \text{Var}[Z] = \left( \frac{1}{E[F] - E[D]} \right)^2 \left( \text{Var}[F - D] \cdot [E[Z]^2 - 2H \cdot E[Z] + \text{Var}[E] \cdot E[D] - D \cdot E(F)] \right) \]
with \( H = E[D] \cdot \text{Var}[F] + E[F] \cdot \text{Var}[D] - \text{Cov}[F,D] \cdot E[F] - \text{Cov}[F,D] \cdot E[D] \)

Appendix 2 : optimal mix with a quadratic utility function ( relation (18))

\[
U[Z] = E[Z] - \frac{\gamma}{2} \text{Var}[Z]
\]

\[
U[Z] = a \cdot E[F] + (1-a) \cdot E[D] - \frac{\gamma}{2} \left[a^2 \cdot \text{Var}[F] + (1-a)^2 \cdot \text{Var}[D] + 2a(1-a) \text{Cov}[F,D] \right]
\]

\[
U[Z] = aE[F] + E[D] - aE[D] - \frac{\gamma}{2} \left[a^2 \text{Var}[F] + \text{Var}[D] + a^2 \text{Var}[F] - 2a \text{Var}[D] + 2(a-a^2) \text{Cov}[F,D] \right]
\]

\[
\frac{\partial U[Z]}{\partial a} = E[F] - E[D] - \gamma \left[a \text{Var}[F] + \text{aVar}[D] - \text{Var}[D] + (1-2a) \text{Cov}[F,D] \right] = 0
\]

\[
E[F] - E[D] + \gamma \left[ \text{Var}[D] - \text{Cov}[F,D] \right] = \gamma a \left[ \text{Var}[F] + \text{Var}[D] - 2 \text{Cov}[F,D] \right]
\]

\[
a^s_{\text{opt}} = \frac{\text{Var}[D] - \text{Cov}[F,D]}{\text{Var}[F-D]} + \frac{1}{\gamma} \frac{E[F-D]}{\text{Var}[F-D]} = a^* + \frac{1}{\gamma} \cdot \Delta
\]

where:
\[
\Delta = \frac{E(F-D)}{\text{Var}(F-D)}
\]

Appendix 3 : mean reverting model ( relations (26), (27))

\[
F = e^{\int F + \rho_f \left[ \int_0^t \frac{1-e^{-\beta_f (r-u)}}{\beta_f} dW_f(u) - \int_0^{t-1} \frac{1-e^{-\beta_f (r-1-u)}}{\beta_f} dW_f(u) \right]}
\]

\[
F = e^{A_f} \quad \text{\(F_t\) is lognormal}
\]

\[
A_f = \gamma_f + \rho_f \left[ \int_0^t \frac{1-e^{-\beta_f (r-u)}}{\beta_f} dW_f(u) - \int_0^{t-1} \frac{1-e^{-\beta_f (r-1-u)}}{\beta_f} dW_f(u) \right]
\]

\[
E[A_f] = \gamma_f
\]
\[
\text{Var}[A_t] = E \left[ \rho_t \left( \int_0^1 \frac{1-e^{-\beta(t-u)}}{\beta} \, dW_t(u) - \int_0^{t-1} \frac{1-e^{-\beta(t-u)}}{\beta} \, dW_t(u) \right)^2 \right] = \\
= E \rho_t^2 \left[ \int_0^1 \frac{1-e^{-\beta(t-u)}}{\beta} \, dW_t(u) - \int_0^{t-1} \frac{1-e^{-\beta(t-u)}}{\beta} \, dW_t(u) \right]^2 
\]

Applying the Itô isometry we obtain:

\[
= E \rho_t^2 \left[ \int_0^1 \left( \int_0^1 \frac{1-e^{-\beta(t-u)}}{\beta} \, dW_t(u) \right)^2 + \int_0^{t-1} \left( \int_0^{t-1} \frac{1-e^{-\beta(t-u)}}{\beta} \, dW_t(u) \right)^2 + 
- 2 \int_0^1 \left( \int_0^1 \frac{1-e^{-\beta(t-u)}}{\beta} \, dW_t(u) \right) \left( \int_0^{t-1} \left( \int_0^{t-1} \frac{1-e^{-\beta(t-u)}}{\beta} \, dW_t(u) \right) \right) \right] = \\
= E \rho_t^2 \left[ \int_0^1 \left( \int_0^1 \frac{1-e^{-\beta(t-u)}}{\beta} \, dW_t(u) \right)^2 + \int_0^{t-1} \left( \int_0^{t-1} \frac{1-e^{-\beta(t-u)}}{\beta} \, dW_t(u) \right)^2 + 
- 2 \int_0^1 \left( \int_0^1 \frac{1-e^{-\beta(t-u)}}{\beta} \, dW_t(u) \right) \left( \int_0^{t-1} \left( \int_0^{t-1} \frac{1-e^{-\beta(t-u)}}{\beta} \, dW_t(u) \right) \right) \right] = \\
= E \left[ \int_0^{t-1} \left( \int_0^1 \frac{1-e^{-\beta(t-u)}}{\beta} \, dW_t(u) \right) \cdot \int_0^1 \left( \int_0^1 \frac{1-e^{-\beta(t-u)}}{\beta} \, dW_t(u) \right) \right] = 0
\]
\[
\begin{align*}
\rho_i^2 & \left[ \left( u + \frac{e^{-2\beta_1(t-u)}}{2\beta_1 - \beta_1} \right)^t + \left( u + \frac{e^{-2\beta_1(t-1-u)}}{2\beta_1 - \beta_1} \right)^{t-1} - 2 \left( u + \frac{e^{-2\beta_1(t-u)}}{2\beta_1 - \beta_1} \right)^{t-1} \right] = \\
\rho_i^2 & \left[ \left\{ t + \frac{1}{2\beta_1} - \frac{2}{\beta_1} + \frac{e^{2\beta_1 t}}{2\beta_1} + \frac{e^{2\beta_1 t}}{2\beta_1} + t - 1 + \frac{1}{2\beta_1} - \frac{2}{\beta_1} + \frac{e^{2\beta_1 t}}{2\beta_1} \right\} + \\
-2t + 2 \frac{e^{2\beta_1 t}}{2\beta_1} + 2 \frac{e^{2\beta_1 t}}{2\beta_1} + 2 \frac{e^{2\beta_1 t}}{2\beta_1} \right] = \\
\text{Var}[A_i] = \rho_i^2 & \left[ 2\beta_1 - 2e^{-2\beta_1 t} - e^{-2\beta_1 (t-1)} + 2e^{-\beta_1} + 2e^{-\beta_1 (2t-1)} \right] \\
F & = e^{\beta_1} \\
E[A_i] & = a_i \\
\text{Var}[A_i] & = b_i^2 \\
a_i & = \gamma_i \\
b_i & = \rho_i^2 \left[ 2\beta_1 - 2e^{-2\beta_1 t} - e^{-2\beta_1 (t-1)} + 2e^{-\beta_1} + 2e^{-\beta_1 (2t-1)} \right] \\
E[F] & = e^{\beta_1} \\
\text{Var}[F] & = e^{2\beta_1} (e^{\beta_1} - 1) \\
D & = e^{\beta_2} \\
E[A_d] & = a_d \\
\text{Var}[A_d] & = b_d \\
a_d & = \gamma_d \\
b_d & = \rho^2 \left[ 2\beta_2 - 2e^{-2\beta_2 t} - e^{-2\beta_2 (t-1)} + 2e^{-\beta_2} + 2e^{-\beta_2 (2t-1)} \right] \\
E[D] & = e^{\beta_2} \\
\text{Var}[D] & = e^{2\beta_2} (e^{\beta_2} - 1) \\
\text{Cov}[F, D] & = e^{\beta_1 + \beta_2 + \beta_3 + \beta_4} (e^{\beta_5} - 1) \\
b_{f,d} & = E \left[ \rho_1 \left( \int_0^t \frac{1 - e^{-\beta_1 u}}{\beta_1} dW_f(u) \right) - \int_0^t \frac{1 - e^{-\beta_1 (u-1)}}{\beta_1} dW_f(u) \right] \left[ \rho_2 \left( \int_0^t \frac{1 - e^{-\beta_2 u}}{\beta_2} dW_d(u) \right) - \int_0^t \frac{1 - e^{-\beta_2 (u-1)}}{\beta_2} dW_d(u) \right] \\
b_{f,d} & = \rho_1 \rho_2 \left[ \left( \int_0^t \frac{1 - e^{-\beta_1 u}}{\beta_1} dW_f(u) \right) \left( \int_0^t \frac{1 - e^{-\beta_1 (u-1)}}{\beta_1} dW_f(u) \right) - \left( \int_0^t \frac{1 - e^{-\beta_2 u}}{\beta_2} dW_d(u) \right) \left( \int_0^t \frac{1 - e^{-\beta_2 (u-1)}}{\beta_2} dW_d(u) \right) + \\
- \left[ \int_0^t \frac{1 - e^{-\beta_1 u}}{\beta_1} dW_f(u) \right] \left( \int_0^t \frac{1 - e^{-\beta_1 (u-1)}}{\beta_1} dW_f(u) \right) + \left[ \int_0^t \frac{1 - e^{-\beta_2 u}}{\beta_2} dW_d(u) \right] \left( \int_0^t \frac{1 - e^{-\beta_2 (u-1)}}{\beta_2} dW_d(u) \right) \right] \\
26
\[ b_{f,d} = \rho_f \rho_d \cdot E \]

\[
\begin{align*}
\frac{1}{\beta_f} \int_0^1 \left( \frac{1-e^{-\beta_f (t-u)}}{1-e^{-\beta_f (t-u)}} \right) dW_f(u) \\
\frac{1}{\beta_d} \int_0^1 \left( \frac{1-e^{-\beta_d (t-u)}}{1-e^{-\beta_d (t-u)}} \right) dW_d(u) \\
\frac{1}{\beta_f} \int_0^1 \left( \frac{1-e^{-\beta_f (t-u)}}{1-e^{-\beta_f (t-u)}} \right) dW_f(u) \\
\frac{1}{\beta_d} \int_0^1 \left( \frac{1-e^{-\beta_d (t-u)}}{1-e^{-\beta_d (t-u)}} \right) dW_d(u) \\
\frac{1}{\beta_f} \int_0^1 \left( \frac{1-e^{-\beta_f (t-u)}}{1-e^{-\beta_f (t-u)}} \right) dW_f(u) \\
\frac{1}{\beta_d} \int_0^1 \left( \frac{1-e^{-\beta_d (t-u)}}{1-e^{-\beta_d (t-u)}} \right) dW_d(u) \\
\frac{1}{\beta_f} \int_0^1 \left( \frac{1-e^{-\beta_f (t-u)}}{1-e^{-\beta_f (t-u)}} \right) dW_f(u) \\
\frac{1}{\beta_d} \int_0^1 \left( \frac{1-e^{-\beta_d (t-u)}}{1-e^{-\beta_d (t-u)}} \right) dW_d(u) \\
\end{align*}
\]

Applying the Ito isometry we obtain:

\[
\begin{align*}
\frac{1}{\beta_f} \int_0^1 \left( \frac{1-e^{-\beta_f (t-u)}}{1-e^{-\beta_f (t-u)}} \right) \rho_{f,d} d(u) - \int_0^1 \left( \frac{1-e^{-\beta_f (t-u)}}{1-e^{-\beta_f (t-u)}} \right) \rho_{f,d} d(u) \\
\frac{1}{\beta_d} \int_0^1 \left( \frac{1-e^{-\beta_d (t-u)}}{1-e^{-\beta_d (t-u)}} \right) \rho_{f,d} d(u) + \int_0^1 \left( \frac{1-e^{-\beta_d (t-u)}}{1-e^{-\beta_d (t-u)}} \right) \rho_{f,d} d(u) \\
\end{align*}
\]

\[
\begin{align*}
\frac{1}{\beta_f} \int_0^1 \left( \frac{1-e^{-\beta_f (t-u)}}{1-e^{-\beta_f (t-u)}} \right) \beta_f \beta_d d(u) - \int_0^1 \left( \frac{1-e^{-\beta_f (t-u)}}{1-e^{-\beta_f (t-u)}} \right) \beta_f \beta_d d(u) \\
\frac{1}{\beta_d} \int_0^1 \left( \frac{1-e^{-\beta_d (t-u)}}{1-e^{-\beta_d (t-u)}} \right) \beta_f \beta_d d(u) + \int_0^1 \left( \frac{1-e^{-\beta_d (t-u)}}{1-e^{-\beta_d (t-u)}} \right) \beta_f \beta_d d(u) \\
\end{align*}
\]

\[
\begin{align*}
\frac{1}{\beta_f} \int_0^1 \left( \frac{1-e^{-\beta_f (t-u)}}{1-e^{-\beta_f (t-u)}} \right) \frac{e^{-\beta_f (t-u)} - e^{-\beta_f (t-u)}}{\beta_f + \beta_d} d(u) - \int_0^1 \left( \frac{1-e^{-\beta_f (t-u)}}{1-e^{-\beta_f (t-u)}} \right) \frac{e^{-\beta_f (t-u)} - e^{-\beta_f (t-u)}}{\beta_f + \beta_d} d(u) \\
\frac{1}{\beta_d} \int_0^1 \left( \frac{1-e^{-\beta_d (t-u)}}{1-e^{-\beta_d (t-u)}} \right) \frac{e^{-\beta_d (t-u)} - e^{-\beta_d (t-u)}}{\beta_f + \beta_d} d(u) + \int_0^1 \left( \frac{1-e^{-\beta_d (t-u)}}{1-e^{-\beta_d (t-u)}} \right) \frac{e^{-\beta_d (t-u)} - e^{-\beta_d (t-u)}}{\beta_f + \beta_d} d(u) \\
\end{align*}
\]

\[
\begin{align*}
\frac{1}{\beta_f} \int_0^1 \left( \frac{1-e^{-\beta_f (t-u)}}{1-e^{-\beta_f (t-u)}} \right) \frac{e^{-\beta_f (t-u)} - e^{-\beta_f (t-u)}}{\beta_f + \beta_d} d(u) - \int_0^1 \left( \frac{1-e^{-\beta_f (t-u)}}{1-e^{-\beta_f (t-u)}} \right) \frac{e^{-\beta_f (t-u)} - e^{-\beta_f (t-u)}}{\beta_f + \beta_d} d(u) \\
\frac{1}{\beta_d} \int_0^1 \left( \frac{1-e^{-\beta_d (t-u)}}{1-e^{-\beta_d (t-u)}} \right) \frac{e^{-\beta_d (t-u)} - e^{-\beta_d (t-u)}}{\beta_f + \beta_d} d(u) + \int_0^1 \left( \frac{1-e^{-\beta_d (t-u)}}{1-e^{-\beta_d (t-u)}} \right) \frac{e^{-\beta_d (t-u)} - e^{-\beta_d (t-u)}}{\beta_f + \beta_d} d(u) \\
\end{align*}
\]

\[
\begin{align*}
\frac{1}{\beta_f} \int_0^1 \left( \frac{1-e^{-\beta_f (t-u)}}{1-e^{-\beta_f (t-u)}} \right) \frac{e^{-\beta_f (t-u)} - e^{-\beta_f (t-u)}}{\beta_f + \beta_d} d(u) - \int_0^1 \left( \frac{1-e^{-\beta_f (t-u)}}{1-e^{-\beta_f (t-u)}} \right) \frac{e^{-\beta_f (t-u)} - e^{-\beta_f (t-u)}}{\beta_f + \beta_d} d(u) \\
\frac{1}{\beta_d} \int_0^1 \left( \frac{1-e^{-\beta_d (t-u)}}{1-e^{-\beta_d (t-u)}} \right) \frac{e^{-\beta_d (t-u)} - e^{-\beta_d (t-u)}}{\beta_f + \beta_d} d(u) + \int_0^1 \left( \frac{1-e^{-\beta_d (t-u)}}{1-e^{-\beta_d (t-u)}} \right) \frac{e^{-\beta_d (t-u)} - e^{-\beta_d (t-u)}}{\beta_f + \beta_d} d(u) \\
\end{align*}
\]
Appendix 4. Modelling in a geometric Brownian motion environment with 3 periods of time

In the model of section 4 with three periods of time, we assume here as example that the stochastic processes $F_t$ and $D_t$ follow a geometric Brownian motion.

We have:

$$F_{t-1} = \frac{F(t-1)}{F(t-2)} = e^{\left(\mu_t - \frac{1}{2} \sigma_t^2\right) + \sigma_t [W_t(t-1) - W_t(t-2)]}$$

$$D_{t-1} = \frac{D(t-1)}{D(t-2)} = e^{\left(\mu_t - \frac{1}{2} \sigma_t^2\right) + \sigma_t [W_t(t-1) - W_t(t-2)]}$$

$$F_t = \frac{F(t)}{F(t-1)} = e^{\left(\mu_t - \frac{1}{2} \sigma_t^2\right) + \sigma_t [W_t(t) - W_t(t-1)]}$$

$$D_t = \frac{D(t)}{D(t-1)} = e^{\left(\mu_t - \frac{1}{2} \sigma_t^2\right) + \sigma_t [W_t(t) - W_t(t-1)]}$$

Let us stay as general as possible and consider that the two hazard processes $(W_t(t), W_d(t))$ are correlated.

For the financial process $F$ we obtain successively taking into account the properties of the Brownian motions:

$$E[F_t \cdot F_{t-1}] = E[F_t] \cdot E[F_{t-1}]$$

$$E[F_t] = e^{\mu_t}$$

$$VAR[F_t] = e^{2\mu_t} \cdot (e^{\sigma_t^2} - 1)$$

$$E[F_{t-1}] = e^{\mu_t}$$

$$VAR[F_{t-1}] = e^{2\mu_t} \cdot (e^{\sigma_t^2} - 1)$$

It is then easy to compute the following elements:

$$E[X] = e^{\mu_t} \cdot \left(\frac{e^{\mu_t}}{p(x_b,1,t-2)} + 1\right) = e^{2\mu_t} + e^{\mu_t}$$

$$VAR[X] = \frac{e^{2\mu_t}}{p^2(x_b,1,t-2)} \left(e^{\sigma_t^2} - 1\right) + 2e^{\mu_t} \left(e^{\sigma_t^2} - 1\right) + \frac{e^{2\mu_t}}{p(x_b,1,t-2)} \left(2e^{\sigma_t^2} - 1\right)$$

Similarly, for the demographic part $Y$, we get:

$$E[Y] = e^{\mu_d} \cdot \left(\frac{e^{\mu_d}}{p(x_b,1,t-2)} + (1 + h_{t-1})\right) = e^{2\mu_d} + e^{\mu_d} \cdot (1 + h_{t-1})$$

$$VAR[Y] = \frac{1}{p^2(x_b,1,t-2)} \left(e^{\mu_d + 2\sigma_d^2} - e^{\mu_d}\right) + (1 + h_{t-1})^2 \left(e^{\mu_d + \sigma_d^2} - e^{\mu_d}\right) + \frac{2(1 + h_{t-1})}{p(x_b,1,t-2)} \left(e^{\mu_d + \sigma_d^2} - 2e^{\mu_d}\right)$$

For the covariance we obtain:

$$Cov[X,Y] = \frac{e^{2\mu_d + 2\mu_t}}{p^2(x_b,1,t-2)} \left(e^{\mu_d} - 1\right) + \frac{e^{2\mu_d + 2\mu_t}}{p(x_b,1,t-2)} \left(e^{\mu_d} - 1\right) + \frac{e^{\mu_d + \mu_t}}{p(x_b,1,t-2)} \left(1 + h_{t-1}\right) \cdot \left(e^{\mu_d} - 1\right) + \frac{e^{\mu_d + \mu_t}}{p(x_b,1,t-2)} \left(1 + h_{t-1}\right) \cdot \left(e^{\mu_d} - 1\right)$$

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So finally the optimal mixing coefficient becomes:

\[
\hat{a}^{\text{opt}} = \frac{\mathbb{V}[Y] - \text{Cov}[X,Y]}{\mathbb{V}[X-Y]} + \frac{1}{\varepsilon} \mathbb{E}[X-Y] = \hat{a}^* + \frac{1}{\varepsilon} \Delta
\]

With:

\[
\hat{a}^* = \left[ \frac{1}{p^2(x_1, t-2)} \left( \epsilon^{\mu_2} \right) - \left( 1 + h_{t-1} \right)^2 \left( \epsilon^{\mu_2} \right) + \frac{2}{p(x_1, t-2)} \left( \epsilon^{\mu_2} \right) \right] - \left[ \phi^{\mu_4} \right] - \left( 1 + h_{t-1} \right)^2 \left( \epsilon^{\mu_4} \right) + \frac{2}{p(x_1, t-2)} \left( \epsilon^{\mu_4} \right) - \left( 1 + h_{t-1} \right)^2 \left( \epsilon^{\mu_4} \right) + \frac{2}{p(x_1, t-2)} \left( \epsilon^{\mu_4} \right) - \left( 1 + h_{t-1} \right)^2 \left( \epsilon^{\mu_4} \right)
\]

And:

\[
\Delta = \frac{\epsilon^{\mu_1}}{p(x_1, t-2)} + \epsilon^{\mu_2} - \frac{\epsilon^{\mu_4}}{p(x_1, t-2)} - \epsilon^{\mu_2} \left( 1 + h_{t-1} \right)
\]