

# Distortion risk measures, ambiguity aversion and optimal effort

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## Abstract

We consider the class of concave distortion risk measures to study how choice is influenced by the decision-maker's attitude to risk and provide comparative static results. We also assume ambiguity about the probability distribution of the risk and consider a framework à la Klibanoff, Marinacci, and Mukerji (2005) to study the value of information that resolves ambiguity. We show that this value increases with greater ambiguity, with greater ambiguity aversion, and in some cases with greater risk aversion. Finally we examine whether a more risk-averse and a more ambiguity-averse individual will invest in more effort to shift his initial risk distribution to a better target distribution.

## 1 Introduction

This paper is concerned with the ambiguity about the probability distribution that is used to evaluate risk measures. Almost all models used in the theory of risk measures assume that the distributions of risks are perfectly known and implicitly agree that ambiguity does not matter for decisions. We rather assume here that there exists an uncertainty about a parameter of the risk distribution and that this uncertainty may modify decisions of the individuals if they are ambiguity-averse. This paper aims to examine the effect of ambiguity-aversion on the valuation of risk reduction and give comparison with effect of risk aversion.

A risk measure is defined as a mapping from the set of random variables representing the risks to the real numbers. It may be interpreted as the amount of money that should be added as a buffer to a risk so that it becomes acceptable to an internal or external risk controller. It may be used for evaluating capital requirements in order to avoid insolvency, for determining provisions, for calculating insurance premiums... In this paper, the risk  $X$  may be thought as an insurance company's risk related to a particular policy, a particular line-of-business or to the entire insurance portfolio over a specified time horizon. A negative outcome for  $X$  means that a gain has occurred. This means that a decision-maker will always try to act to minimize the risk measure associated to  $X$ .

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We consider the class of distortion risk measures introduced by Wang (1996). This class of risk measures used the concept of distortion function as proposed in Yaari's dual theory of choice (see Yaari (1987)). A distortion function is defined as a non-decreasing function  $g : [0, 1] \rightarrow [0, 1]$  such that  $g(0) = 0$  and  $g(1) = 1$ . The distortion risk measure associated with distortion function  $g$  is defined by

$$\mathbb{H}_g[X] = - \int_{-\infty}^0 (1 - g(\bar{F}_X(x))) dx + \int_0^{\infty} g(\bar{F}_X(x)) dx,$$

where  $X$  is a (random) risk with survival distribution function  $\bar{F}_X$ . Under the assumption that  $g(\bar{F}_X(\cdot))$  is right-continuous,  $\mathbb{H}_g[X]$  may also be interpreted as a distorted expectation of  $X$  evaluated with a distorted survival distribution function  $g(\bar{F}_X(\cdot))$ .

Yaari's dual theory and distortion risk measures have been considered for two decades by several authors to study insurance decisions, such as pricing, reserving, designing optimal (re)insurance contracts, capital setting, risk exchanges... (see among others: Denneberg (1990), Doherty and Eeckhoudt (1995), Wang and Young (1998), Wang (2000), Tsanakas and Desli (2003), Tsanakas and Christofides (2006))

Ellsberg (1961) observed that individuals prefer a lottery with a known probability distribution to a lottery with an unknown distribution. This preference violates the expected utility theory, but can be explained by ambiguity-aversion of the individuals. Among the several approaches to model this aversion that have been provided in the economic literature (see Etner et al. (2012)), the approach developed by Klibanoff et al. (2005) is very useful to study the effect of ambiguity about the risk distribution because it permits to separate the effect of ambiguity-aversion from that of risk-aversion. In their model the ambiguity-aversion is captured by introducing a concave transfer of the expected utility. This approach has been recently used to examine the effect of ambiguity-aversion on insurance demand (see e.g. Alary et al. (2012), Snow (2011)) and it is shown that the demand for self-insurance raises with ambiguity-aversion whereas the demand for self-insurance is less clear.

In this paper we assume that the decision-maker is uncertain about a parameter of the risk distribution in order to examine the effect of ambiguity-aversion on the valuation of risk reduction. Our results are established using a similar recursive model of ambiguity preferences as developed by Klibanoff et al. (2005), but with second-order probability distribution for the unknown parameter.

Although the problem of parameter and model uncertainty in insurance has been raised for several years (see e.g. Cairns (2000) for a general point of view), the impact of parameter uncertainty on risk measures has only been recently studied. Bignozzi and Tsanakas (2013) introduce the notion of residual estimation risk when the risk measures are positive homogeneous, translation invariant and law invariant. The residual estimation risk is defined as the risk measure of the difference between the risk itself and its risk measure when the parameter is replaced by a random estimator. This estimation risk reflects the amount of capital that needs to be subtracted from the difference between the risk and its (random) risk measure such that it becomes acceptable. It is expected that, as the size of the sample that is used to construct the estimator the estimator increases, the residual risk decreases. Our approach differs from Bignozzi and Tsanakas (2013) since we are interested in the measure of aversion to parameter uncertainty rather than only its measure. Moreover we want to understand how this aversion modifies decisions. Finally note we that don't need to know the distribution between the risk and its (random) risk measure.

The remainder of the paper is organized as follows. In Section 2, we explain how the notion of risk aversion can be translated into the class of distortion risk measures in order to permit comparative statics results analysis of greater risk aversion. We present the framework à la Klibanoff,

Marinacci, and Mukerji (2005) to take into account ambiguity about the probability distribution in Section 3. We also introduce characterizations for ambiguity measures in order to study the value of information that resolves ambiguity. We show that this value increases with greater ambiguity, with greater ambiguity-aversion, and in some cases with greater-risk aversion. In Section 4, we consider the case of a decision-maker that is able to invest in effort to reduce risk by shifting the initial risk distribution to a better target distribution. We examine whether a more risk-averse and a more ambiguity-averse individual will invest in more effort to shift his initial risk distribution. Section 5 contains brief conclusions and proofs are postponed to Section 6.

## 2 Comparative statics results for the class of distortion risk measures

In Yaari's dual theory of choice, a decision-maker is expected to act in order to maximize the distorted expectation of his wealth. Here we say that the decision-maker bases his preferences on the distorted expectations hypothesis if he acts in order to minimize the distorted expectation of his risk: if his distortion function is denoted by  $g$ , then he prefers the risk  $X$  over the risk  $Y$  when  $\mathbb{H}_g[X] \leq \mathbb{H}_g[Y]$ . Note that a wealth,  $W$ , may be viewed as a negative risk, and that  $\mathbb{H}_g[-W] = -\mathbb{H}_{g^*}[W]$  where  $g^*(u) = 1 - g(1 - u)$  is the dual distortion operator of  $g$ . Therefore the decision-maker's preferences are equivalent if he uses the distortion function  $g^*$  for calculating the distorted expectation of his wealth.

Several definitions of risk aversion have been proposed for non-expected utility theories of choice under uncertainty, see e.g. Montesano (1990). A first definition is linked with the risk premium, i.e. the difference between the expected value of the prospect under consideration and its certainty equivalent: a decision-maker is said to be risk-averse if he always prefers a certain prospect to a risky prospect with the same expectation. A second defines risk aversion as aversion to a mean preserving spread in the distribution of prospects. These definitions are equivalent to the concavity of the von Neumann-Morgenstern utility function for the expected utility theory. But they differ in our case. The former refers to the condition  $g^*(p) \leq p$ , or equivalently, to  $g(p) \geq p$  (a risk-averse decision-maker systematically overestimates the tail probabilities related to the levels of his risk), while the latter is stronger and refers to the condition that  $g^*$  is convex, or equivalently, that  $g$  is concave (see e.g. Yaari (1987), Röll (1987) and Section 2.6.1.4 of Denuit et al. (2004)).

We rather consider the second definition and assume that the distortion function of an individual always belongs to the class of concave distortion functions. Note that, if  $g$  is a concave distortion function, then  $\mathbb{H}_g$  is also a coherent risk measure (i.e. it satisfies translativity, subadditivity, positive homogeneity and monotonicity, see Artzner et al. (1999)). Finally we will say that a decision-maker with distortion function  $h$  is more risk-averse than a decision-maker with distortion function  $g$ , if there exists a concave distortion function  $f$  such that  $h = f \circ g$  (see e.g. Ross (1981) for equivalent definitions when differentiability may be assumed).

**Example 1** *The class of Normal distortion function has been introduced by Wang (2000) and is characterized by the functions  $g_\gamma(\cdot) = \Phi(\Phi^{-1}(\cdot) + \gamma)$  with  $\gamma > 0$ , where  $\Phi$  is the standard normal cumulative distribution function. These functions are concave and satisfy  $g_{\gamma_1 + \gamma_2} = g_{\gamma_1} \circ g_{\gamma_2} = g_{\gamma_2} \circ g_{\gamma_1}$  for  $\gamma_1, \gamma_2 > 0$ . Hence a decision-maker with distortion function  $g_{\tilde{\gamma}}$  is more risk averse than another decision-maker with distortion function  $g_\gamma$  if  $\tilde{\gamma} > \gamma$ .*

Consider an agent who has to choose between two risks  $X$  and  $Y$ . Would a more risk-averse agent make the same choice? Assume that the first agent has a Normal distortion function  $g_\gamma$

and that his choice has to be made between two Lognormal distributions,  $X \sim \mathcal{LN}(\mu_X, \sigma_X^2)$  and  $Y \sim \mathcal{LN}(\mu_Y, \sigma_Y^2)$ . The risk measure of  $X$  is given by  $\mathbb{H}_{g_\gamma}[X] = \exp(\mu_X + \sigma_X^2/2 + \gamma\sigma_X)$  (see Section 6.8). Then  $\mathbb{H}_{g_\gamma}[X] \leq \mathbb{H}_{g_\gamma}[Y]$  if and only if

$$\mu_X - \mu_Y + \sigma_X^2/2 - \sigma_Y^2/2 \leq \gamma(\sigma_Y - \sigma_X).$$

It follows that a more risk-averse agent with a Normal distortion function  $g_{\tilde{\gamma}}$  such that  $\tilde{\gamma} > \gamma$  will always choose  $X$  rather than  $Y$  if and only if  $(\sigma_Y - \sigma_X) > 0$ .

We now give a more general comparative statics result. It may be discussed and compared to the results given by Hammond (1974), Diamond and Stiglitz (1974), Jewitt (1989) for the expected utility framework. Let us before note that, since  $g$  is concave, the set where  $g'$  fails to exist is countable, and therefore its associated distortion risk measure can be written as

$$\mathbb{H}_g[X] = \int_0^1 g'(1-u) \text{VaR}[X; u] du \quad (2.1)$$

where  $\text{VaR}[X; u] = \inf\{x \in \mathbb{R} : F_X(x) \geq u\}$  is the quantile function of  $X$  and is known as the Value-at-Risk.

**Theorem 1** *Let us consider a decision-maker with concave distortion function  $g$  who prefers  $X$  over  $Y$ . Then the following properties are equivalent:*

- i) A more risk-averse decision-maker with distortion function  $h$  prefers  $X$  over  $Y$ ;*
- ii) for all  $\alpha \in (0, 1)$ ,*

$$\int_\alpha^1 g'(1-u) (\text{VaR}[X; u] - \text{VaR}[Y; u]) du \leq 0.$$

Note that condition ii) of the previous theorem is satisfied for the case of a decision-maker with a Normal distortion function who has to choose among the two risks  $X \sim \mathcal{LN}(\mu_X, \sigma_X^2)$  and  $Y \sim \mathcal{LN}(\mu_Y, \sigma_Y^2)$ , if and only if  $(\sigma_Y - \sigma_X) > 0$  (see Section 6.9). This condition does not depend of  $\gamma$  and is actually more general than needed for a more risk-averse decision-maker with a Normal distortion function.

Let us define recursively the family of functions  $(\text{VaR}^{(n)}[X; \cdot])_{n \geq 1}$  by  $\text{VaR}^{(1)}[X; \alpha] = \text{VaR}[X; \alpha]$ ,  $\alpha \in [0, 1]$ , and, for  $n \geq 1$ ,

$$\text{VaR}^{(n)}[X; \alpha] = \int_0^{1-\alpha} \text{VaR}^{(n-1)}[X; 1-u] du = \int_\alpha^1 \text{VaR}^{(n-1)}[X; u] du, \quad \alpha \in [0, 1].$$

Note that  $\text{VaR}^{(2)}[X; \alpha]$  is related to the Tail-Value-at-Risk in the following way:  $\text{VaR}^{(2)}[X; \alpha] = (1-\alpha)\text{TVaR}[X; \alpha]$ .

We derive from Theorem 1 several sufficient conditions for a risk-averse decision-maker to have the same preferences as a less risk-averse decision-maker.

**Corollary 2** *Let us consider a decision-maker with distortion function  $g$  who prefers the risk  $X$  over the risk  $Y$ . Assume that one of the following condition holds:*

- i)  $\text{VaR}[X; \alpha] \leq \text{VaR}[Y; \alpha]$  for all  $\alpha \in (0, 1)$ ;*
- ii) there exists  $\alpha_0 \in (0, 1)$  such that  $\text{VaR}[X; \alpha] \geq \text{VaR}[Y; \alpha]$  for  $0 < \alpha \leq \alpha_0$  and  $\text{VaR}[X; \alpha] \leq \text{VaR}[Y; \alpha]$  for  $\alpha_0 \leq \alpha < 1$ ;*
- iii)  $g$  is twice differentiable and  $\text{VaR}^{(2)}[X; \alpha] \leq \text{VaR}^{(2)}[Y; \alpha]$  for all  $\alpha \in (0, 1)$ ;*

iv)  $g$  is twice differentiable and there exists  $\alpha_0 \in (0, 1)$  such that  $VaR^{(2)}[X; \alpha_0] = VaR^{(2)}[Y; \alpha_0]$ , for  $0 < \alpha \leq \alpha_0$ ,  $VaR^{(2)}[X; \alpha] - VaR^{(2)}[Y; \alpha]$  is non-increasing and, for  $\alpha_0 \leq \alpha < 1$ ,  $VaR^{(2)}[X; \alpha] \leq VaR^{(2)}[Y; \alpha]$ .

Then a more risk-averse decision-maker with distortion function  $h$  also prefers  $X$  over  $Y$ .

These sufficient conditions are strongly linked with well-known stochastic order relations:

- The risk  $X$  is said to be smaller than the risk  $Y$  in stochastic dominance (denoted by  $X \preceq_{ST} Y$ ) if  $\bar{F}_X(d) \leq \bar{F}_Y(d)$  for all  $d \in \mathbb{R}$ , or equivalently if  $\mathbb{E}[u(-X)] \geq \mathbb{E}[u(-Y)]$  for all non-decreasing function  $u$  (such that the expectations exist). This is also equivalent to Condition i) of Corollary 2 (see e.g. Section 3.3 in Denuit et al. (2005)).
- The risk  $X$  is said to precede the risk  $Y$  in the stop-loss order (denoted by  $X \preceq_{SL} Y$ ) if  $\mathbb{E}[(X - d)_+] \leq \mathbb{E}[(Y - d)_+]$  for all  $d \in \mathbb{R}$ . It is well-known that this condition is equivalent to one of the following conditions (a)  $\mathbb{E}[u(-X)] \geq \mathbb{E}[u(-Y)]$  for all non-decreasing and concave function  $u$  (such that the expectations exist), (b)  $TVaR[X; \alpha] \leq TVaR[Y; \alpha]$  for all  $\alpha \in (0, 1)$ , (c)  $\mathbb{H}_g[X] \leq \mathbb{H}_g[Y]$  for any concave distortion functions  $g$  (see e.g. Section 3.4 in Denuit et al. (2005)). It follows that Condition ii) of Corollary 2 may appear as too strong, as well as Condition i) since it implies Condition ii).

Finally note that Condition iv) of Corollary 2 looks like Condition (2.13) in Theorem 2 of Jewitt (1989) where  $VaR[X; \cdot]$  is replaced by  $\bar{F}_X(\cdot)$ . Therefore it should be close to a necessary condition if Condition iii) does not hold.

We are now interested in the willingness to pay for reducing risk. Let  $X$  and  $Y$  be two risks such that  $X$  is preferred over  $Y$ , i.e.  $\mathbb{H}_g[X] \leq \mathbb{H}_g[Y]$ . The willingness to pay for reducing risk from  $Y$  to  $X$ ,  $\pi = \pi(Y, X; g) \geq 0$ , is implicitly defined by

$$\mathbb{H}_g[X] + \pi(Y, X; g) = \mathbb{H}_g[Y].$$

We are interested in a condition for a more risk-averse decision-maker to be willing to pay more for an improvement from  $Y$  to  $X$ . Consider again the decision-maker with a Normal distortion function  $g_\gamma$  who chooses  $X \sim \mathcal{LN}(\mu_X, \sigma_X^2)$  rather than  $Y \sim \mathcal{LN}(\mu_Y, \sigma_Y^2)$ . Then  $\pi(Y, X; g_\gamma)$  will increase with  $\gamma$  (i.e. with risk aversion in the class of Normal distortion function) if and only if

$$\frac{\partial \pi(Y, X; g)}{\partial \gamma} = \sigma_Y \mathbb{H}_g[Y] - \sigma_X \mathbb{H}_g[X] > 0. \quad (2.2)$$

If we assume moreover that, any more risk-averse decision-maker with distortion function  $g_{\tilde{\gamma}}$  ( $\tilde{\gamma} > \gamma$ ) will always prefer  $X$  over  $Y$ , which is equivalent to  $\sigma_Y > \sigma_X$ , we see that Condition (2.2) necessarily holds.

The following corollary gives a necessary and sufficient condition in the general case.

**Corollary 3** *Assume a decision-maker with distortion function  $g$  prefers  $X$  over  $Y$ , i.e.  $\pi(Y, X; g) \geq 0$ , and that any more risk-averse decision-maker also prefers  $X$  over  $Y$ . Then the following properties are equivalent:*

i) *A more risk-averse decision-maker with distortion function  $h$  has willingness to pay  $\pi(Y, X; h)$  that is larger than  $\pi(Y, X; g)$ ;*

ii) *for all  $\alpha \in (0, 1)$ ,*

$$\int_{\alpha}^1 g'(1-u) (VaR[X; u] - VaR[Y; u]) du \leq g(1-\alpha) \int_0^1 g'(1-u) (VaR[X; u] - VaR[Y; u]) du. \quad (2.3)$$

Let  $(g_\alpha)_{\alpha \in (0,1)}$  be the class of distortion functions defined by

$$g_\alpha(u) = \begin{cases} g(u)/g(1-\alpha), & u \leq 1-\alpha, \\ 1, & u > 1-\alpha. \end{cases}$$

The distortion function  $g_\alpha$  is the distortion function of a more risk-averse decision-maker than the decision-maker with distortion function  $g$ . Condition (2.3) is then equivalent to the condition that any decision-maker with distortion function  $g_\alpha$  has a larger willingness to pay than the decision-maker with distortion function  $g$

$$\pi(Y, X; g_\alpha) \geq \pi(Y, X; g), \quad \alpha \in (0, 1).$$

We may also note that a sufficient condition for the more risk-averse decision-maker to have a larger willingness to pay is given by Condition ii) of Corollary 2: there exists  $\alpha_0 \in (0, 1)$  such that  $VaR[X; \alpha] \geq VaR[Y; \alpha]$  for  $0 < \alpha \leq \alpha_0$  and  $VaR[X; \alpha] \leq VaR[Y; \alpha]$  for  $\alpha_0 \leq \alpha < 1$ .

Consider the agent with a Normal distortion function  $g_\gamma$  who chooses  $X \sim \mathcal{LN}(\mu_X, \sigma_X^2)$  rather than  $Y \sim \mathcal{LN}(\mu_Y, \sigma_Y^2)$ . Condition (2.3) is equivalent to

$$\mathbb{H}_g[Y] [\Phi(\sigma_Y + \gamma - \Phi^{-1}(\alpha)) - \Phi(\gamma - \Phi^{-1}(\alpha))] > \mathbb{H}_g[X] [\Phi(\sigma_X + \gamma - \Phi^{-1}(\alpha)) - \Phi(\gamma - \Phi^{-1}(\alpha))],$$

for all  $\alpha \in (0, 1)$ . Since a more risk-averse decision-maker with distortion function  $h$  also prefers  $X$  over  $Y$  if and only if  $\sigma_Y > \sigma_X$ , we see that Condition (2.3) always holds for this particular case.

### 3 Distortion risk measures and ambiguity aversion

We assume that the risk  $X$  is a random variable whose probability distribution belongs to a certain family of distributions  $\{F_\theta, \theta \in \mathcal{T}\}$  ( $\theta$  is a parameter for this family), called the parametric model, so that  $F_X = F_{\theta_0}$  for some  $\theta_0 \in \mathcal{T}$ . The value  $\theta_0$  is unknown and is referred to as the true value of the parameter. We denote by  $X_\theta$  a risk whose probability distribution function is  $F_\theta$ . The risk measure of  $X$ ,  $\mathbb{H}_g[X]$ , is then equal to  $\mathbb{H}_g[X_{\theta_0}]$ .

The subjective beliefs about the value of  $\theta_0$  are captured by a random variable  $\Theta$  with density probability function  $\pi$ . This random variable satisfies the following condition ensuring that the decision-maker's ambiguous beliefs are objectively unbiased

$$\mathbb{H}_g[X] = \mathbb{H}_g[X_{\theta_0}] = \mathbb{E}[\mathbb{H}_g[X_\Theta]].$$

This condition ensures that the behavior of an ambiguity-neutral decision maker will be unaffected by the introduction of ambiguity into the choice setting. The same type of assumption is made for the framework of the expected utility theory by Klibanoff et al. (2005) to disentangle the effect of ambiguity-aversion from that of risk-aversion. Note that, if  $Y_\Theta$  is the random variable whose quantile function is defined by

$$VaR[Y_\Theta; \alpha] = \int_{\mathcal{T}} VaR[X_\theta; \alpha] \pi(\theta) d\theta, \quad 0 \leq \alpha \leq 1,$$

then its risk measure equals that of  $X$  since

$$\begin{aligned} \mathbb{E}[\mathbb{H}_g[X_\Theta]] &= \int_{\mathcal{T}} \left[ \int_0^1 g'(1-\alpha) VaR[X|\theta; \alpha] d\alpha \right] \pi(\theta) d\theta \\ &= \int_0^1 g'(1-\alpha) \int_{\mathcal{T}} [VaR[X|\theta; \alpha] \pi(\theta) d\theta] d\alpha \\ &= \mathbb{H}_g[Y_\Theta]. \end{aligned}$$

In the case where the probability distribution functions,  $F_\theta$ , are probability distribution functions of comonotonic risks,  $Y_\Theta$  is an average of these risks over  $\theta$  (see also the discussion in Wang and Young (1998)). For example, suppose that  $\mathcal{T} = \{\theta_{-1}, \theta_0, \theta_1\}$  and that  $\mathbb{P}(\Theta = \theta_{-1}) = p$  and  $\mathbb{P}(\Theta = \theta_1) = q$  with  $0 < p < 1$ ,  $0 < q < 1$ ,  $p + q < 1$ . Assume moreover that there exists a non-decreasing function  $h_{-1}$  (resp.  $h_1$ ) such that  $X_{\theta_{-1}} = h_{-1}(X)$  (resp.  $X_{\theta_1} = h_1(X)$ ). Then

$$\begin{aligned} \text{VaR}[Y_\Theta; \alpha] &= p\text{VaR}[X_{\theta_{-1}}; \alpha] + (1 - p - q)\text{VaR}[X_{\theta_0}; \alpha] + q\text{VaR}[X_{\theta_1}; \alpha] \\ &= (ph_{-1} + (1 - p - q) + ph_1)(\text{VaR}[X; \alpha]) \\ &= \text{VaR}[(ph_{-1} + (1 - p - q) + ph_1)(X); \alpha], \end{aligned}$$

and  $Y_\Theta$  has the same distribution as  $ph_{-1}(X) + (1 - p - q)X + qh_1(X)$ .

The recursive model of ambiguity preferences with second-order probability distribution developed by Klibanoff et al. (2005) is now adapted to the class of distortion risk measures to investigate the value of information that resolves ambiguity or resolves risk. The main element of this model is a non-decreasing and convex transformation function  $\varphi$  that captures the ambiguity preference. The decision-maker's risk measure is defined by

$$\tilde{\mathbb{H}}_{g,\varphi}[X, \Theta] = \varphi^{-1}(\mathbb{E}[\varphi(\mathbb{H}_g[X_\Theta])]) = \varphi^{-1}\left(\int_{\mathcal{T}} \varphi(\mathbb{H}_g[X_\theta]) \pi(\theta) d\theta\right).$$

By Jensen's inequality, we derive that

$$\varphi^{-1}(\mathbb{E}[\varphi(\mathbb{H}_g[X_\Theta])]) \geq \mathbb{E}[\mathbb{H}_g[X_\Theta]] = \mathbb{H}_g[X] = \mathbb{H}_g[Y_\Theta].$$

Therefore  $\tilde{\mathbb{H}}_{g,\varphi}[X, \Theta]$  is always larger than  $\mathbb{H}_g[X]$ , the risk measure that would be used if  $\theta_0$  was known, and the decision maker is ambiguity-averse when  $\varphi$  is convex. If  $\varphi$  is linear,  $\tilde{\mathbb{H}}_{g,\varphi}[X, \Theta] = \mathbb{H}_g[X]$  and the decision maker is ambiguity-neutral.

If  $g_\gamma(\cdot) = \Phi(\Phi^{-1}(\cdot) + \gamma)$  and  $X \sim \mathcal{LN}(\mu_X, \sigma_X^2)$ , we have  $\mathbb{H}_{g_\gamma}[X] = \mathbb{E}[X] \exp(\gamma\sigma_X)$ . Suppose that the mean of  $X$  is ambiguous in the sense that the decision-maker has an imperfect knowledge of it. Let  $\theta_0 = \mathbb{E}[X]$  and assume that  $X_\theta \sim \mathcal{LN}(\log(\theta) - \sigma_X^2/2, \sigma_X^2)$  with  $\theta \in \mathcal{T} = (0, \infty)$ . Note that  $\mathbb{H}_{g_\gamma}[X_\theta] = \theta \exp(\gamma\sigma_X)$ . The ambiguity takes the form of probability distributions for  $\Theta$  and we may assume for example that  $\Theta \sim \text{Gamma}(\theta_0\beta, \beta)$ , with  $\beta > 0$ . Since  $\mathbb{E}[\Theta] = \theta_0$ , we have  $\mathbb{E}[\mathbb{H}_{g_\gamma}[X_\Theta]] = \mathbb{H}_{g_\gamma}[X]$  and since  $\mathbb{V}[\Theta] = \theta_0/\beta$ , ambiguity should decrease when  $\beta$  increases. Assume moreover that  $\varphi(x) = \exp(\alpha x)$  with  $\alpha \geq 0$ , then for  $\beta > \alpha \exp(\gamma\sigma_X)$

$$\tilde{\mathbb{H}}_{g_\gamma,\varphi}[X, \Theta] = -\frac{\beta}{\alpha} \ln\left(1 - \frac{\alpha}{\beta} \exp(\gamma\sigma_X)\right) \theta_0 = -\rho^{-1}(\alpha, \beta, \gamma) \ln(1 - \rho(\alpha, \beta, \gamma)) \mathbb{H}_{g_\gamma}[X],$$

where  $\rho(\alpha, \beta, \gamma) = \alpha\beta^{-1} \exp(\gamma\sigma_X)$ . The ambiguity-averse decision-maker's risk measure,  $\tilde{\mathbb{H}}_{g_\gamma,\varphi}[X, \Theta]$ , decreases with  $\beta$ , but increases with  $\alpha$ . When  $\alpha$  tends to zero, the risk measure tends to the risk measure of the ambiguity-neutral decision maker, i.e.  $\mathbb{H}_{g_\gamma}[X]$ .

Let  $\Theta_1$  and  $\Theta_2$  be two random variables that capture the subjective beliefs about the value  $\theta_0$  and that satisfy the mean-preserving condition

$$\mathbb{E}[\mathbb{H}_g[X_{\Theta_1}]] = \mathbb{E}[\mathbb{H}_g[X_{\Theta_2}]] = \mathbb{H}_g[X_{\theta_0}]. \quad (3.4)$$

We say that the information used for constructing the prior distribution of  $\Theta_2$  "reduces" ambiguity with respect to the information used for constructing the prior distribution of  $\Theta_1$  if

$$\tilde{\mathbb{H}}_{g,\varphi}[X, \Theta_1] \geq \tilde{\mathbb{H}}_{g,\varphi}[X, \Theta_2].$$

Information that reduces ambiguity has a positive value for decision makers who are ambiguity averse. A sufficient condition is given by the condition that  $\mathbb{H}_g[X_{\Theta_1}]$  dominates  $\mathbb{H}_g[X_{\Theta_2}]$  in the stop-loss order,  $\mathbb{H}_g[X_{\Theta_1}] \succeq_{SL} \mathbb{H}_g[X_{\Theta_2}]$ .

**Example 2** A location-scale family is a family of probability distributions parametrized by a location parameter and a (non-negative) scale parameter: if  $X$  is any random variable whose probability distribution belongs to such a family, then  $Y = a + bX$ ,  $b > 0$ , also belongs to this family. Let  $\mu$  be the mean of  $X$  and  $\sigma$  its standard deviation, then

$$VaR[X_{\mu,\sigma}; \alpha] = \mu + \sigma q_\alpha$$

where  $q_\alpha$  is the quantile function of a centered and standard distribution and

$$\mathbb{H}_g[X_{\mu,\sigma}] = \mu + \sigma \int_0^1 g'(1 - \alpha) q_\alpha d\alpha.$$

The ambiguity may then concern  $\mu$  or  $\sigma$ . In this case  $\mathbb{H}_g[X_{\mu_1,\sigma}] \succeq_{SL} \mathbb{H}_g[X_{\mu_2,\sigma}]$  is equivalent to  $\mu_1 \succeq_{SL} \mu_2$  when  $\mu_1$  and  $\mu_2$  are random variables and  $\sigma$  is fixed, or  $\mathbb{H}_g[X_{\mu,\sigma_1}] \succeq_{SL} \mathbb{H}_g[X_{\mu,\sigma_2}]$  is equivalent to  $\sigma_1 \succeq_{SL} \sigma_2$  if  $\int_0^1 g'(1 - \alpha) q_\alpha d\alpha > 0$ ,  $\sigma_1$  and  $\sigma_2$  are random variables and  $\mu$  is fixed.

For a large number of parametric families of distributions, it is possible to order the distribution by using a distortion risk measure.

**Definition 1** We say that  $\{F_\theta, \theta \in \mathcal{T}\}$  is “invariant” with respect to  $\mathbb{H}_g$ , if, for all  $\theta_1, \theta_2 \in \mathcal{T} \subset \mathbb{R}$ , such that  $\theta_1 \geq \theta_2$ , then  $\mathbb{H}_g[X_{\theta_1}] \geq \mathbb{H}_g[X_{\theta_2}]$ . If  $\Theta$  satisfies the mean preserving condition, i.e.  $\mathbb{H}_g[X_{\theta_0}] = \mathbb{E}[\mathbb{H}_g[X_\Theta]]$ , then  $\theta_{\Theta,g,\varphi}^*$ , defined implicitly by

$$\mathbb{H}_g[X_{\theta_{\Theta,g,\varphi}^*}] = \tilde{\mathbb{H}}_{g,\varphi}[X, \Theta],$$

is larger than  $\theta_0$  and is called the cautious parameter associated to  $\Theta$ ,  $g$  and  $\varphi$ .

Note that if, for all  $\alpha \in (0, 1)$  and  $\theta_1, \theta_2 \in \mathcal{T}$ , such that  $\theta_1 \geq \theta_2$ , we have

$$\int_\alpha^1 g'(1 - u) (VaR[X_{\theta_2}; u] - VaR[X_{\theta_1}; u]) du \leq 0,$$

we deduce by Theorem 1 that, for any more risk-averse decision-maker with distortion function  $h$ ,  $\{F_\theta, \theta \in \mathcal{T}\}$  is also invariant with respect to  $\mathbb{H}_h$ .

If  $g_\gamma(\cdot) = \Phi(\Phi^{-1}(\cdot) + \gamma)$  with  $\gamma \geq 0$ ,  $X_\theta \sim \mathcal{LN}(\log(\theta) - \sigma_X^2/2, \sigma_X^2)$  with  $\theta \in \mathcal{T} = (0, \infty)$ ,  $\Theta \sim \text{Gamma}(\theta_0\beta, \beta)$ , with  $\beta > 0$ ,  $\varphi_\alpha(x) = \exp(\alpha x)$  with  $0 < \alpha < \beta \exp(-\gamma\sigma_X)$ , then  $\{F_\theta, \theta \in \mathcal{T}\}$  is invariant with respect to  $\mathbb{H}_{g_\gamma}$  and, for  $X = X_{\theta_0}$ ,

$$\theta_{\Theta,g_\gamma,\varphi_\alpha}^* = -\rho^{-1}(\alpha, \beta, \gamma) \ln(1 - \rho(\alpha, \beta, \gamma)) \theta_0, \quad (3.5)$$

where  $\rho(\alpha, \beta, \gamma) = \alpha\beta^{-1} \exp(\gamma\sigma_X)$ .

If  $\tilde{\mathbb{H}}_{g,\varphi}[X, \Theta_1] \geq \tilde{\mathbb{H}}_{g,\varphi}[X, \Theta_2]$ , the willingness to pay for information that reduces ambiguity from  $\Theta_1$  to  $\Theta_2$ ,  $\omega = \omega(\Theta_1, \Theta_2)$ , is defined implicitly by

$$\tilde{\mathbb{H}}_{g,\varphi}[X, \Theta_1] = \tilde{\mathbb{H}}_{g,\varphi}[X + \omega, \Theta_2].$$

It is the amount of money that is necessary to use to pass from the a priori knowledge  $\Theta_1$  to that of  $\Theta_2$ . The value of information that resolves ambiguity,  $\omega(\Theta, \theta_0)$ , is therefore defined by

$$\omega(\Theta, \theta_0) = \tilde{\mathbb{H}}_{g,\varphi}[X, \Theta] - \mathbb{H}_g[X].$$

We immediately derive the following proposition.



**Proposition 4** *The willingness to pay for information that resolves ambiguity increases with the stop-loss order:*

$$\mathbb{H}_g[X_{\Theta_1}] \succeq_{SL} \mathbb{H}_g[X_{\Theta_2}] \implies \omega(\Theta_1, \theta_0) \geq \omega(\Theta_2, \theta_0).$$

As for the risk-aversion, we define the notion of a greater ambiguity-aversion: a decision-maker with ambiguity function  $\psi$  is said to be more ambiguity-averse than a decision-maker with ambiguity function  $\varphi$ , if there exists an increasing and convex function  $\chi$  such that  $\psi = \chi \circ \varphi$ .

**Theorem 5** *For ambiguity-averse decision makers, the willingness to pay for information that resolves ambiguity increases with greater ambiguity aversion.*

This result is coherent with Theorem 3 in Snow (2012) which studies the value of information in a non-expected utility model of ambiguity with second-order probabilities. The next theorem gives a sufficient condition on the distortion functions of two decision-makers with the same ambiguity function such that the willingness to pay for information that resolves ambiguity is larger for the more risk-averse decision-maker.

Let us consider a decision-maker with distortion function  $g$  and ambiguity function  $\varphi$  and a more risk-averse decision-maker with distortion function  $h$  and ambiguity function  $\varphi$ . Let  $\Theta$  be a random variable that captures the subjective beliefs about the value  $\theta_0$  such that

$$\mathbb{E}[\mathbb{H}_g[X_\Theta]] = \mathbb{H}_g[X_{\theta_0}] \text{ and } \mathbb{E}[\mathbb{H}_h[X_\Theta]] = \mathbb{H}_h[X_{\theta_0}].$$

**Theorem 6** *Assume that  $\{F_\theta, \theta \in \mathcal{T}\}$  is invariant with respect to  $\mathbb{H}_g$  and  $\mathbb{H}_h$ . The willingness to pay for information that resolves ambiguity is larger for the decision-maker with distortion function  $h$  than for the decision-maker with distortion function  $g$  if  $\theta_{\Theta, h, \varphi}^* \geq \theta_{\Theta, g, \varphi}^*$  and, for all  $\alpha \in (0, 1)$ ,*

$$\begin{aligned} & \int_{\alpha}^1 g'(1-u) \left( \text{VaR}[X_{\theta_0}; u] - \text{VaR}[X_{\theta_{\Theta, g, \varphi}^*}; u] \right) du \\ & \leq g(1-\alpha) \int_0^1 g'(1-u) \left( \text{VaR}[X_{\theta_0}; u] - \text{VaR}[X_{\theta_{\Theta, g, \varphi}^*}; u] \right) du. \end{aligned} \quad (3.6)$$

Assume that  $g_\gamma(\cdot) = \Phi(\Phi^{-1}(\cdot) + \gamma)$  and  $X_\theta \sim \mathcal{LN}(\log(\theta) - \sigma_X^2/2, \sigma_X^2)$  with  $\theta \in \mathcal{T} = (0, \infty)$ . If  $X = X_{\theta_0}$ ,  $\Theta \sim \text{Gamma}(\theta_0\beta, \beta)$ , with  $\beta > 0$ , and  $\varphi_\alpha(x) = \exp(\alpha x)$  with  $0 < \alpha < \beta \exp(-\gamma\sigma_X)$ , then we have  $\theta_{\Theta, g\tilde{\gamma}, \varphi_\alpha}^* \geq \theta_{\Theta, g\gamma, \varphi_\alpha}^*$  for  $\tilde{\gamma} \geq \gamma$  by Equation (3.5), and Condition (3.6) is equivalent to

$$(1 + \rho^{-1}(\alpha, \beta, \gamma) \ln(1 - \rho(\alpha, \beta, \gamma))) \exp(\gamma\sigma_X) [\Phi(\sigma_X + \gamma - \Phi^{-1}(\alpha)) - \Phi(\gamma - \Phi^{-1}(\alpha))] \leq 0,$$

where  $\rho(\alpha, \beta, \gamma) = \alpha\beta^{-1} \exp(\gamma\sigma_X)$ . This condition holds since  $0 < \rho(\alpha, \beta, \gamma) < 1$ . We therefore deduce that the willingness to pay for information that resolves ambiguity is larger for the more risk-averse decision-maker.

## 4 Ambiguity aversion and optimal effort

We consider a decision-maker with distortion function  $g$  who faces a risk  $Z$ . Assume that this decision-maker could make an effort  $e \in [0, 1]$  to shift his risk distribution toward a better target distribution of a risk  $Y$  such that  $Y$  is preferred over  $Z$ , i.e.  $\mathbb{H}_g[Y] \leq \mathbb{H}_g[Z]$ . But this effort has a monetary cost characterized by a function  $c$  with  $c(0) = 0$ ,  $c' > 0$  and  $c'' > 0$ . After the investment

in effort  $e$ , the decision maker's final risk is denoted by  $X^{(e)}$  whose quantile function is assumed to be given by

$$VaR[X^{(e)}; \alpha] = eVaR[Y; \alpha] + (1 - e)VaR[Z; \alpha], \quad 0 < \alpha < 1.$$

The objective function of the decision maker is then given by

$$\min_{e \in [0,1]} \left( \mathbb{H}_g[X^{(e)}] + c(e) \right).$$

We assume that the optimal effort  $e_g^*$  belongs to the interval  $(0, 1)$ .

**Theorem 7** *Assume that for all  $\alpha \in (0, 1)$ ,*

$$\int_{\alpha}^1 g'(1 - u) (VaR[Y; u] - VaR[Z; u]) du \leq g(1 - \alpha) \int_0^1 g'(1 - u) (VaR[Y; u] - VaR[Z; u]) du. \quad (4.7)$$

*Then a more risk-averse decision-maker will invest more in effort.*

Note that, if Condition (4.7) holds, then, by Corollary 3, a more risk-averse decision-maker with distortion function  $h$  has a willingness to pay  $\pi(Z, Y; h)$  that is larger than  $\pi(Z, Y; g)$ . This explains why he will invest more in effort than the decision-maker with distortion function  $g$ .

We now impose ambiguity on the target distribution of  $Y$  as in Huang (2012) for the framework of the expected utility. Let  $\underline{Y}$  and  $\bar{Y}$  be two risks such that  $\mathbb{H}_g[\underline{Y}] \leq \mathbb{H}_g[\bar{Y}] \leq \mathbb{H}_g[Z]$ . We assume that  $Y$  is a random variable whose probability distribution belongs to the family of distributions  $\{F_\theta, \theta \in [0, 1]\}$  characterized by the random variables  $Y_\theta$  with probability distribution functions  $F_\theta$  and quantile functions

$$VaR[Y_\theta; \alpha] = (1 - \theta) VaR[\underline{Y}; \alpha] + \theta VaR[\bar{Y}; \alpha], \quad 0 \leq \alpha \leq 1.$$

The true value of the parameter is denoted by  $\theta_0$ . The subjective beliefs about the value of  $\theta_0$  are captured by a random variable  $\Theta$  satisfying  $\mathbb{E}[\Theta] = \theta_0$ .

For a given  $\theta \in [0, 1]$ , the decision-maker's final risk after the investment in effort  $e$  is denoted by  $X_\theta^{(e)}$  and has a quantile function which is given by

$$VaR[X_\theta^{(e)}; \alpha] = e [(1 - \theta) VaR[\underline{Y}; \alpha] + \theta VaR[\bar{Y}; \alpha]] + (1 - e) VaR[Z; \alpha], \quad 0 < \alpha < 1.$$

The objective function of the decision maker is then given by

$$\min_{e \in [0,1]} \tilde{\mathbb{H}}_{g,\varphi}[X^{(e)} + c(e), \Theta] = \min_{e \in [0,1]} \varphi^{-1} \left( \mathbb{E}[\varphi \left( \mathbb{H}_g[X_\Theta^{(e)}] + c(e) \right)] \right).$$

We assume that the optimal effort  $e_\varphi^*$  belongs to the interval  $(0, 1)$ .

**Theorem 8** *A more ambiguity-averse decision-maker will invest less in effort.*

This result shows that risk-aversion and ambiguity-aversion may have different impact on the effort to reduce risk.

## 5 Conclusion

We consider concave distortion risk measure minimizers choosing two risks and examine how choice is influenced by the decision-makers attitude to risk. We provide a necessary and sufficient condition such that, if a decision-maker prefers a risk over another, then a more risk-averse decision-maker will make the same choice. We derive a condition for a more risk-averse decision-maker to be willing to pay more for an improvement in risk measures.

We then assume that the decision-maker is uncertain about a parameter of the risk distribution and we use this last condition to study the effect of ambiguity-aversion on the valuation of risk reduction. We show that the willingness to pay for information that resolves ambiguity is always larger for a more ambiguity-averse decision-maker, but is not necessarily larger for a more risk-averse decision maker. Moreover we prove that, if a decision-maker is able to make an effort to reduce his risk, then a more risk-averse decision-maker will invest more in effort, but a more ambiguity-averse decision-maker will invest less in effort if the ambiguous beliefs concern the distribution of the less risky distribution. This result is of practical interest for loss prevention and safety incentive programs if there exists ambiguity about the target distributions of risks.

## 6 Appendix

### 6.1 Proof of Theorem 1

Suppose i) holds.  $\mathbb{H}_g[X] \leq \mathbb{H}_g[Y]$  is equivalent to

$$\int_0^1 g'(1-u) (VaR[X; u] - VaR[Y; u]) du \leq 0.$$

Since  $g$  is concave and  $g(0) = 0$ , we have, for  $u_0 \in (0, 1)$ ,  $g(u_0) > 0$ . Let us define the concave distortion function  $f_0$  by

$$f_0(u) = \begin{cases} u/g(u_0) & u \leq g(u_0) \\ 1 & u > g(u_0) \end{cases}.$$

Then we have

$$h(u) = \begin{cases} g(u)/g(u_0) & u \leq u_0 \\ 1 & u > u_0 \end{cases}$$

and  $\mathbb{H}_h[X] \leq \mathbb{H}_h[Y]$  is equivalent to  $\int_{1-u_0}^1 g'(1-u) (VaR[X; u] - VaR[Y; u]) du \leq 0$ . Therefore for all  $\alpha \in (0, 1)$ ,  $\int_\alpha^1 g'(1-u) (VaR[X; u] - VaR[Y; u]) du \leq 0$ .

Suppose ii) holds. Let  $h = f \circ g$  where  $f$  is a concave distortion function. Let  $E_g$  (resp.  $E_f$ ) be the set where  $g'$  (resp.  $f'$ ) fails to exist. For  $u \in E_g^c \cap E_f^c$ , we have

$$h'(u) = f'(g(u))g'(u),$$

and therefore

$$\begin{aligned} & \int_0^1 h'(1-u) (VaR[X; u] - VaR[Y; u]) du \\ &= \int_0^1 f'(g(1-u))g'(1-u) (VaR[X; u] - VaR[Y; u]) du. \end{aligned}$$

We now use the same type of arguments as in the proof of Theorem 5.2.1 in Dhaene et al. (2006). We first prove the implication for concave piecewise linear distortion functions  $f$ . Any such distortion function can be written in the form

$$f(x) = \sum_{i=1}^n a_i(\beta_i - \beta_{i+1}) \min(x/a_i),$$

where  $0 = a_0 < a_1 < \dots < a_{n-1} < a_n = 1$ . Further,  $\beta_i$  is the derivative of  $f$  in the interval  $(a_{i-1}, a_i)$  and is non-negative, and  $\beta_{n+1} = 0$ . Because of the concavity of  $f$ , we have that  $\beta_i$  is a decreasing function of  $i$ . And since  $f$  is a distortion function  $f(1) = \sum_{i=1}^n a_i(\beta_i - \beta_{i+1}) = 1$ . Since

$$f'(x) = \sum_{i=1}^n (\beta_i - \beta_{i+1}) I_{\{0 < x < a_i\}},$$

we deduce that

$$\begin{aligned} & \int_0^1 h'(1-u) (VaR[X; u] - VaR[Y; u]) du \\ &= \sum_{i=1}^n (\beta_i - \beta_{i+1}) \int_0^1 g'(1-u) (VaR[X; u] - VaR[Y; u]) I_{\{0 < g(1-u) < a_i\}} du \\ &= \sum_{i=1}^n (\beta_i - \beta_{i+1}) \int_{1-g^{-1}(a_i)}^1 g'(1-u) (VaR[X; u] - VaR[Y; u]) du \leq 0 \end{aligned}$$

and then  $\mathbb{H}_h[X] \leq \mathbb{H}_h[Y]$ . The result is derived for general concave distortion functions  $f$  by using the monotone convergence theorem as in Dhaene et al. (2006).

## 6.2 Proof of Corollary 2

By Theorem 1, it is sufficient to prove that for all  $\alpha \in (0, 1)$ ,

$$d(\alpha) = \int_{\alpha}^1 g'(1-u) (VaR[X; u] - VaR[Y; u]) du \leq 0.$$

Note that

$$d'(\alpha) = -g'(1-\alpha) (VaR[X; \alpha] - VaR[Y; \alpha]), \quad \alpha \in (0, 1), \quad (6.8)$$

and that by integrating by part (when  $g$  is twice differentiable)

$$\begin{aligned}
d(\alpha) &= \int_0^{1-\alpha} g'(u) (VaR[X; 1-u] - VaR[Y; 1-u]) du \\
&= \left[ g'(u) \left( \int_0^u (VaR[X; 1-v] - VaR[Y; 1-v]) dv \right) \right]_0^{1-\alpha} \\
&\quad - \int_0^{1-\alpha} g''(u) \left( \int_0^u (VaR[X; 1-v] - VaR[Y; 1-v]) dv \right) du \\
&= g'(1-\alpha) \left( \int_0^{1-\alpha} (VaR[X; 1-v] - VaR[Y; 1-v]) dv \right) \\
&\quad - \int_0^{1-\alpha} g''(u) \left( \int_0^u (VaR[X; 1-v] - VaR[Y; 1-v]) dv \right) du \\
&= g'(1-\alpha) \left( VaR^{(2)}[X; \alpha] - VaR^{(2)}[Y; \alpha] \right) \\
&\quad - \int_0^{1-\alpha} g''(u) \left( VaR^{(2)}[X; 1-u] - VaR^{(2)}[Y; 1-u] \right) du \\
&= g'(1-\alpha) \left( VaR^{(2)}[X; \alpha] - VaR^{(2)}[Y; \alpha] \right) - \int_\alpha^1 g''(1-u) \left( VaR^{(2)}[X; u] - VaR^{(2)}[Y; u] \right) du
\end{aligned}$$

i) Obvious.

ii) By Equation (6.8),  $d$  is non-increasing for  $0 < \alpha \leq \alpha_0$  and non-decreasing for  $\alpha_0 \leq \alpha < 1$ . Since  $d(0) \leq 0$  and  $d(1) = 0$ ,  $d(\alpha) \leq 0$  for all  $\alpha \in (0, 1)$ .

iii) By Equation (6.9) and the concavity of  $g$ ,  $d(\alpha) \leq 0$  for all  $\alpha \in (0, 1)$ .

iv) By Equation (6.8),  $d$  is non-increasing for  $0 < \alpha \leq \alpha_0$  and by Equation (6.9) and the concavity of  $g$ ,  $d(\alpha) \leq 0$  for all  $\alpha_0 \leq \alpha < 1$ . The result follows.

### 6.3 Proof of Corollary 3

First recall that

$$\pi(Y, X; g) = \mathbb{H}_g[Y] - \mathbb{H}_g[X] = \int_0^1 g'(1-u) (VaR[Y; u] - VaR[X; u]) du.$$

Let  $X_\pi = X + \pi(Y, X; g)$ . We have  $\mathbb{H}_g[X_\pi] = \mathbb{H}_g[Y]$ . By Theorem 1, a more risk-averse decision-maker with distortion function  $h$  prefers  $X_\pi$  to  $Y$ , i.e.

$$\mathbb{H}_h[X_\pi] \leq \mathbb{H}_h[Y] \Leftrightarrow \mathbb{H}_h[X] + \pi(Y, X; g) \leq \mathbb{H}_h[Y] \Leftrightarrow \pi(Y, X; g) \leq \pi(Y, X; h),$$

if and only if, for all  $\alpha \in (0, 1)$ ,

$$\begin{aligned}
&\int_\alpha^1 g'(1-u) (VaR[X_\pi; u] - VaR[Y; u]) du \leq 0 \\
\Leftrightarrow &\int_\alpha^1 g'(1-u) (VaR[X; u] - VaR[Y; u]) du \leq -\pi(Y, X; g) \int_\alpha^1 g'(1-u) du \\
\Leftrightarrow &\int_\alpha^1 g'(1-u) (VaR[X; u] - VaR[Y; u]) du \leq g(1-\alpha) \int_0^1 g'(1-u) (VaR[X; u] - VaR[Y; u]) du.
\end{aligned}$$

## 6.4 Proof of Theorem 5

We have

$$\begin{aligned}
\psi(\mathbb{H}_g[X_{\theta_0}] + \omega(\Theta, \theta_0; \psi)) &= \mathbb{E}[\psi(\mathbb{H}_g[X_{\Theta}])] \\
&= \mathbb{E}[\chi(\varphi(\mathbb{H}_g[X_{\Theta}]))] \\
&\geq \chi(\mathbb{E}[\varphi(\mathbb{H}_g[X_{\Theta}])]) \\
&\geq \chi(\varphi(\mathbb{H}_g[X_{\theta_0}] + \omega(\Theta, \theta_0; \varphi))) \\
&= \psi(\mathbb{H}_g[X_{\theta_0}] + \omega(\Theta, \theta_0; \varphi))
\end{aligned}$$

and then, since  $\psi$  is an increasing function, we deduce that

$$\omega(\Theta, \theta_0; \psi) \geq \omega(\Theta, \theta_0; \varphi).$$

## 6.5 Proof of Theorem 6

Since  $\{F_{\theta}, \theta \in \mathcal{T}\}$  is invariant with respect to  $\mathbb{H}_g$ , the willingness to pay for information that resolves ambiguity of the decision-maker with distortion function  $g$  is

$$\begin{aligned}
\omega(\Theta, \theta_0; g) &= \tilde{\mathbb{H}}_{g, \varphi}[X, \Theta] - \mathbb{H}_g[X_{\theta_0}] = \mathbb{H}_g[X_{\theta_{\Theta, g, \varphi}^*}] - \mathbb{H}_g[X_{\theta_0}] \\
&= \pi(X_{\theta_{\Theta, g, \varphi}^*}, X_{\theta_0}; g),
\end{aligned}$$

and, since  $\{F_{\theta}, \theta \in \mathcal{T}\}$  is invariant with respect to  $\mathbb{H}_h$ , the willingness to pay for information that resolves ambiguity of the decision-maker with distortion function  $h$  is

$$\begin{aligned}
\omega(\Theta, \theta_0; h) &= \tilde{\mathbb{H}}_{h, \varphi}[X, \Theta] - \mathbb{H}_h[X_{\theta_0}] = \mathbb{H}_h[X_{\theta_{\Theta, h, \varphi}^*}] - \mathbb{H}_h[X_{\theta_0}] \\
&= \pi(X_{\theta_{\Theta, h, \varphi}^*}, X_{\theta_0}; h).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\omega(\Theta, \theta_0; h) - \omega(\Theta, \theta_0; g) \\
&= \pi(X_{\theta_{\Theta, h, \varphi}^*}, X_{\theta_0}; h) - \pi(X_{\theta_{\Theta, g, \varphi}^*}, X_{\theta_0}; g) \\
&= \left[ \pi(X_{\theta_{\Theta, h, \varphi}^*}, X_{\theta_0}; h) - \pi(X_{\theta_{\Theta, g, \varphi}^*}, X_{\theta_0}; h) \right] + \left[ \pi(X_{\theta_{\Theta, g, \varphi}^*}, X_{\theta_0}; h) - \pi(X_{\theta_{\Theta, g, \varphi}^*}, X_{\theta_0}; g) \right] \\
&= \left( \mathbb{H}_h[X_{\theta_{\Theta, h, \varphi}^*}] - \mathbb{H}_h[X_{\theta_{\Theta, g, \varphi}^*}] \right) + \left[ \pi(X_{\theta_{\Theta, g, \varphi}^*}, X_{\theta_0}; h) - \pi(X_{\theta_{\Theta, g, \varphi}^*}, X_{\theta_0}; g) \right].
\end{aligned}$$

First we have

$$\mathbb{H}_h[X_{\theta_{\Theta, h, \varphi}^*}] - \mathbb{H}_h[X_{\theta_{\Theta, g, \varphi}^*}] \geq 0 \Leftrightarrow \theta_{\Theta, h, \varphi}^* \geq \theta_{\Theta, g, \varphi}^*.$$

Then by Corollary 3

$$\pi(X_{\theta_{\Theta, g, \varphi}^*}, X_{\theta_0}; h) - \pi(X_{\theta_{\Theta, g, \varphi}^*}, X_{\theta_0}; g) \geq 0,$$

if for, all  $\alpha \in (0, 1)$ ,

$$\begin{aligned}
&\int_{\alpha}^1 g'(1-u) \left( \text{VaR}[X_{\theta_0}; u] - \text{VaR}[X_{\theta_{\Theta, g, \varphi}^*}; u] \right) du \\
&\leq g(1-\alpha) \int_0^1 g'(1-u) \left( \text{VaR}[X_{\theta_0}; u] - \text{VaR}[X_{\theta_{\Theta, g, \varphi}^*}; u] \right) du,
\end{aligned}$$

and the result follows.

## 6.6 Proof of Theorem 7

The objective function of the decision-maker with distortion function  $g$  is given by

$$\min_{e \in [0,1]} F_g(e) = \min_{e \in [0,1]} \left( \mathbb{H}_g[X^{(e)}] + c(e) \right).$$

The first order condition of the objective function is

$$\left. \frac{\partial F_g(e)}{\partial e} \right|_{e=e_g^*} = \mathbb{H}_g[Y] - \mathbb{H}_g[Z] + c'(e_g^*) = 0,$$

and the second order condition is

$$\left. \frac{\partial^2 F(e)}{\partial e^2} \right|_{e=e_g^*} = c''(e_g^*) > 0.$$

Let us now consider a more risk-averse decision-maker with distortion function  $h = f \circ g$  where  $f$  is a concave distortion function  $f$ . His objective function is given by

$$\min_{e \in [0,1]} F_h(e) = \min_{e \in [0,1]} \mathbb{H}_h[X^{(e)}] + c(e)$$

whose first order condition is given by

$$\left. \frac{\partial F_h(e)}{\partial e} \right|_{e=e_h^*} = \mathbb{H}_h[Y] - \mathbb{H}_h[Z] + c'(e_h^*) = 0.$$

But note that

$$\left. \frac{\partial F_h(e)}{\partial e} \right|_{e=e_g^*} = (\mathbb{H}_h[Y] - \mathbb{H}_h[Z]) - (\mathbb{H}_g[Y] - \mathbb{H}_g[Z]) < 0$$

by Corollary 3. Therefore  $e_h^* > e_g^*$  and the result follows.

## 6.7 Proof of Theorem 8

The objective function of the decision-maker is given by

$$\min_{e \in [0,1]} G_\varphi(e) = \min_{e \in [0,1]} \varphi^{-1} \left( \mathbb{E} \left[ \varphi \left( \mathbb{H}_g[X_\Theta^{(e)}] + c(e) \right) \right] \right).$$

The first order condition of the objective function is

$$\left. \frac{\partial G_\varphi(e)}{\partial e} \right|_{e=e_\varphi^*} = \frac{1}{\varphi'(\varphi^{-1}(\mathbb{E}[\varphi^*]))} \mathbb{E} \left[ \varphi' \left( \mathbb{H}_g[X_\Theta^{(e_\varphi^*)}] + c(e_\varphi^*) \right) \left( \left. \frac{\partial \mathbb{H}_g[X_\Theta^{(e)}]}{\partial e} \right|_{e=e_\varphi^*} + c'(e_\varphi^*) \right) \right] = 0,$$

where  $\varphi^* = \varphi \left( \mathbb{H}_g[X_\Theta^{(e_\varphi^*)}] + c(e_\varphi^*) \right)$ , and is equivalent to

$$\mathbb{E} \left[ \varphi' \left( \mathbb{H}_g[X_\Theta^{(e_\varphi^*)}] + c(e_\varphi^*) \right) \left( \left. \frac{\partial \mathbb{H}_g[X_\Theta^{(e)}]}{\partial e} \right|_{e=e_\varphi^*} + c'(e_\varphi^*) \right) \right] = 0, \quad (6.10)$$

since  $\varphi' > 0$ . Note that

$$\frac{\partial \mathbb{H}_g[X_\Theta^{(e)}]}{\partial e} = (1 - \Theta) (\mathbb{H}_g[\underline{Y}] - \mathbb{H}_g[Z]) + \Theta (\mathbb{H}_g[\bar{Y}] - \mathbb{H}_g[Z])$$

is non-positive and does not depend on  $e$ .

The second order condition is given by

$$\begin{aligned} \left. \frac{\partial^2 G_\varphi(e)}{\partial e^2} \right|_{e=e_\varphi^*} &= \frac{1}{\varphi'(\varphi^{-1}(\mathbb{E}[\varphi^*]))} \mathbb{E} \left[ \varphi'' \left( \mathbb{H}_g[X_\Theta^{(e_\varphi^*)}] + c(e_\varphi^*) \right) \left( \left. \frac{\partial \mathbb{H}_g[X_\Theta^{(e)}]}{\partial e} \right|_{e=e_\varphi^*} + c'(e_\varphi^*) \right)^2 \right] \\ &\quad + \frac{1}{\varphi'(\varphi^{-1}(\mathbb{E}[\varphi^*]))} c''(e_\varphi^*) \mathbb{E} \left[ \varphi' \left( \mathbb{H}_g[X_\Theta^{(e_\varphi^*)}] + c(e_\varphi^*) \right) \right] \end{aligned}$$

which is positive since  $\varphi' > 0$ ,  $\varphi'' > 0$  and  $c'' > 0$ .

Let us now consider a more ambiguity-averse decision maker with ambiguity function  $\psi = \chi \circ \varphi$  where  $\chi$  is an increasing and convex function. His objective function is given by

$$\min_{e \in [0,1]} G_\psi(e) = \min_{e \in [0,1]} \psi^{-1} \left( \mathbb{E}[\psi \left( \mathbb{H}_g[X_\Theta^{(e)}] + c(e) \right)] \right).$$

whose first order condition is given by

$$\mathbb{E} \left[ \psi' \left( \mathbb{H}_g[X_\Theta^{(e_\psi^*)}] + c(e_\psi^*) \right) \left( \left. \frac{\partial \mathbb{H}_g[X_\Theta^{(e)}]}{\partial e} \right|_{e=e_\psi^*} + c'(e_\psi^*) \right) \right] = 0$$

or equivalently by

$$\mathbb{E} \left[ \chi' \left( \varphi \left( \mathbb{H}_g[X_\Theta^{(e_\psi^*)}] + c(e_\psi^*) \right) \right) \varphi' \left( \mathbb{H}_g[X_\Theta^{(e_\psi^*)}] + c(e_\psi^*) \right) \left( \left. \frac{\partial \mathbb{H}_g[X_\Theta^{(e)}]}{\partial e} \right|_{e=e_\psi^*} + c'(e_\psi^*) \right) \right] = 0$$

Now note that

$$\frac{\partial \mathbb{H}_g[X_\Theta^{(e)}]}{\partial e} + c'(e) = (1 - \Theta) (\mathbb{H}_g[\underline{Y}] - \mathbb{H}_g[Z]) + \Theta (\mathbb{H}_g[\bar{Y}] - \mathbb{H}_g[Z]) + c'(e).$$

By the first order condition on  $e_\varphi^*$  (6.10), we deduce that there exists  $\theta_b \in (0, 1)$  such that, for  $0 < \theta \leq \theta_b$ ,  $\partial \mathbb{H}_g[X_\Theta^{(e)}] / \partial e \big|_{e=e_\varphi^*} \leq c'(e)$  and, for  $\theta_b < \theta < 1$ ,  $\partial \mathbb{H}_g[X_\Theta^{(e)}] / \partial e \big|_{e=e_\varphi^*} \geq c'(e)$ . It follows that

$$\mathbb{E} \left[ \chi' \left( \varphi \left( \mathbb{H}_g[X_\Theta^{(e_\varphi^*)}] + c(e_\varphi^*) \right) \right) \varphi' \left( \mathbb{H}_g[X_\Theta^{(e_\varphi^*)}] + c(e_\varphi^*) \right) \left( \left. \frac{\partial \mathbb{H}_g[X_\Theta^{(e)}]}{\partial e} \right|_{e=e_\varphi^*} + c'(e_\varphi^*) \right) \right] > 0$$

since  $\chi'' > 0$ . Therefore  $e_\psi^* > e_\varphi^*$  and the result follows.

## 6.8 Risk measures for the class of Normal distortion functions

If  $X \sim \mathcal{LN}(\mu_X, \sigma_X^2)$ , then  $\mathbb{E}[X] = \exp(\mu_X + \sigma_X^2/2)$  and

$$\begin{aligned} VaR[X; \alpha] &= \exp(\mu_X + \sigma_X \Phi^{-1}(\alpha)) \\ VaR^{(2)}[X; \alpha] &= \exp(\mu_X + \sigma_X^2/2) \Phi(\sigma_X - \Phi^{-1}(\alpha)). \end{aligned}$$



By Equation (2.1), we have

$$\begin{aligned}
\mathbb{H}_{g_\gamma}[X] &= \int_0^1 VaR[X; u] g'(1-u) du = \int_0^1 VaR[X; u] \exp(\gamma \Phi^{-1}(u) - \gamma^2/2) du \\
&= \int_0^1 \exp(\mu_X - \gamma^2/2 + (\gamma + \sigma_X) \Phi^{-1}(u)) du = \exp\left(\mu_X - \gamma^2/2 + (\gamma + \sigma_X)^2/2\right) \\
&= \exp(\mu_X + \gamma\sigma_X + \sigma_X^2/2).
\end{aligned}$$

## 6.9 Comparative static results for the class of Normal distortion functions

Let

$$\begin{aligned}
d(\alpha) &= \int_\alpha^1 g'(1-u) (VaR[X; u] - VaR[Y; u]) du \\
&= \exp\left(\tilde{\mu}_X^{(\gamma)}\right) \Phi\left(\tilde{\sigma}_X^{(\gamma)} - \Phi^{-1}(\alpha)\right) - \exp\left(\tilde{\mu}_Y^{(\gamma)}\right) \Phi\left(\tilde{\sigma}_Y^{(\gamma)} - \Phi^{-1}(\alpha)\right).
\end{aligned}$$

where  $\tilde{\mu}_X^{(\gamma)} = \mu_X + \gamma\sigma_X + \sigma_X^2/2$  and  $\tilde{\sigma}_X^{(\gamma)} = \gamma + \sigma_X$  (resp. for  $Y$ ). We have  $d(0) = \mathbb{H}_{g_\gamma}[X] - \mathbb{H}_{g_\gamma}[Y] \leq 0$ ,  $d(1) = 0$  and

$$\begin{aligned}
d'(\alpha) &= -\exp\left(\tilde{\mu}_X^{(\gamma)} + \Phi^{-1}(\alpha) \tilde{\sigma}_X^{(\gamma)} - \left(\tilde{\sigma}_X^{(\gamma)}\right)^2/2\right) + \exp\left(\tilde{\mu}_Y^{(\gamma)} + \Phi^{-1}(\alpha) \tilde{\sigma}_Y^{(\gamma)} - \left(\tilde{\sigma}_Y^{(\gamma)}\right)^2/2\right) \\
&= -\exp\left(\mu_X - \gamma^2/2 + \Phi^{-1}(\alpha) (\gamma + \sigma_X)\right) + \exp\left(\mu_Y - \gamma^2/2 + \Phi^{-1}(\alpha) (\gamma + \sigma_Y)\right)
\end{aligned}$$

It follows that  $d'(\alpha) < 0$  if and only if  $\mu_X - \mu_Y > \Phi^{-1}(\alpha) (\sigma_Y - \sigma_X)$ . Therefore  $d(\alpha) \leq 0$  for all  $\alpha \in [0, 1]$  if and only if  $(\sigma_Y - \sigma_X) > 0$ .

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