How a single-factor CAPM works in a multi-currency world

By R.J. Thomson, Ş. Şahin and T.L. Reddy

Abstract
As shown in A.D. Wilkie’s paper entitled “Why the capital-asset pricing model fails in a multi-currency world” there is no unique single-factor capital-asset pricing model (CAPM) in a multi-currency world. As he points out, the standard CAPM assumes that all investors measure risk and return in the same currency. He shows that, if two investors measure risk and return in different currencies, the standard CAPM cannot describe the pricing of capital assets for both investors.

In this paper a single-factor multi-currency CAPM (SFM-CAPM) is developed. The advantage in using a single-factor model is that it does not treat currency risks as carrying different weight from investment risks; regardless of its source, risk is measured as variance and weighted accordingly. Unlike international CAPMs developed in the literature to date, and unlike the notion of a unique single-factor CAPM across all currencies (the notion rejected by Wilkie), this paper assumes that, for every currency in which investors measure risk, there is a CAPM that is unique to those investors across all the markets in which they invest, but different from the CAPM of investors who measure their risks in other currencies. It also assumes that, regardless of the currency in which they measure risk, all investors have homogeneous expectations and all investors participate in the formation of equilibrium. It develops a theory for multi-currency CAPMs by developing a CAPM for each set of investors that measure their returns in a particular currency. In the development of this theory the meanings of ‘homogeneous expectations’ and of ‘equilibrium’ are reconsidered in the context of a multi-currency world.

The aim of this paper is primarily to give actuaries a way ahead in the use of the single-factor CAPM in a multi-currency world for the purposes of the stochastic modelling of the assets and liabilities of long-term financial institutions such as pension funds, particularly for the purposes of liability-driven investments, and the application of the model has been designed with that intention. However, it is envisaged that the model will also be of interest to other practitioners. The paper’s major original contribution to the literature is its proof that, for a single-factor CAPM to work in a multi-currency world, there is a necessary and sufficient condition. That condition applies to the ex-ante variances and covariances of returns. The estimation of the variance–covariance matrix of returns by constrained maximum-likelihood estimation is discussed. Some difficulties with that approach are explained and an alternative approach, using ordinary least squares, is developed. The theory is applied to two major currencies and two minor currencies, namely the USA dollar, the UK pound, the South African rand and the Turkish lira. Because the application is designed for use by actuaries in the liability-driven modelling of long-term financial institutions, the longest possible range of time periods is used and quarterly intervals are used rather than the relatively short time intervals typically used in the literature. Indications are given of the way in which the findings of this paper will lead to further research for the purposes of such modelling.

Keywords
International CAPM, single-factor multi-currency CAPM (SFM-CAPM)
1. Introduction

As shown by Wilkie (unpublished), there is no unique single-factor capital-asset pricing model (CAPM) in a multi-currency world. As he points out, the standard CAPM assumes that all investors measure risk and return in the same currency. He shows that, if two investors measure risk and return in different currencies, the standard CAPM cannot describe the pricing of capital assets for both investors.

In this paper a single-factor multi-currency CAPM (SFM-CAPM) is developed. The advantage in using a single-factor model is that it does not treat currency risks as carrying different weight from investment risks; regardless of its source, risk is measured by an investor as the variance of the return in which that investor measures risk, and weighted accordingly. Unlike international CAPMs developed in the literature to date, and unlike the notion of a unique single-factor CAPM across all currencies (the notion rejected by Wilkie (op. cit.)), this paper assumes that, for every currency in which investors measure risk, there is a CAPM that is unique to those investors across all the markets in which they invest, but different from the CAPM of investors who measure their risks in other currencies. It also assumes that, regardless of the currency in which they measure risk, all investors have homogeneous expectations and all investors participate in the formation of equilibrium. It develops a theory for multi-currency CAPMs by developing a CAPM for each set of investors that measure their returns in a particular currency. In the development of this theory the meanings of ‘homogeneous expectations’ and of ‘equilibrium’ are reconsidered in the context of a multi-currency world.

The aim of this paper is primarily to give actuaries a way ahead in the use of the single-factor CAPM in a multi-currency world for the purposes of the stochastic modelling of the assets and liabilities of long-term financial institutions such as pension funds, particularly for the purposes of liability-driven investments, and the application of the model has been designed with that intention. However, it is envisaged that the model will also be of interest to other practitioners.

The paper’s major original contribution to the literature is its proof that, for a single-factor CAPM to work in a multi-currency world, there is a necessary and sufficient condition. That condition applies to the ex-ante variances and covariances of returns.

In section 2 the literature on international versions of the CAPM is reviewed. In section 3 a single-factor multi-currency CAPM (SFM-CAPM) is developed and the necessary and sufficient condition for the SFM-CAPM is derived. That condition applies to the ex-ante variances and covariances of returns. The estimation of the variance–covariance matrix of returns by constrained maximum-likelihood estimation is discussed. Difficulties were experienced with the application of this theory. These difficulties are explained and an alternative approach, using ordinary least squares, is
developed. The theory is applied to two major currencies and two minor currencies, namely the USA dollar, the UK pound, the South African rand and the Turkish lira. The data obtained for this purpose and the results of the application are described in section 4. The application illustrated in this paper is designed for use by actuaries in the modelling of the assets and liabilities of long-term financial institutions. To that end, the longest possible range of time periods is used and quarterly intervals are used rather than the relatively short time intervals typically used in the literature. However, the model could be applied to shorter periods and shorter time intervals. Section 5 concludes the paper with a summary of the findings and some indications of the way in which those findings will lead to further research for the purposes of such modelling.

2. Literature review

2.1 The domestic capital-asset pricing model and market segregation

In this paper, in order to distinguish between the CAPM used in a single domestic (segregated) market and the CAPM used in an international (integrated) market, the terms ‘domestic CAPM’ and ‘international CAPM’ are used respectively.

A market in which the price of an asset depends on where it is traded may be referred to as a ‘segregated market’ (Karolyi & Stulz, 2003). If markets are segregated then different domestic capital markets can be considered as independent entities and the international market consists of individual segregated markets (Solnik, 1974b). Jorion & Schwartz (1986) pointed out that complete segregation implies that only domestic factors such as domestic systemic risk should enter the pricing of assets. Stulz (1981) stated that the widespread use, in all countries that have an equity market, of some proxy of the domestic market portfolio to determine how domestic assets are priced can be justified only by an assumption that markets are internationally segregated.

In terms of the traditional CAPM, which uses the domestic market portfolio, and which, as stated above, is here referred to as the ‘domestic CAPM’, the equilibrium expected return of an asset may be expressed as follows:

\[
E\{R_i\} = R_F + \beta_i [E\{R_M\} - R_F];
\]

where \(R_i\), \(R_F\) and \(R_M\) are the returns on security \(i\), on the risk-free asset \(F\) and on the domestic market portfolio \(M\) respectively, and \(\beta_i\) represents the sensitivity of the asset return to market movements; i.e.:

\[
\beta_i = \frac{\sigma_{iM}}{\sigma_{MM}}; \quad \sigma_{iM} = \text{cov}\{R_i, R_M\}; \quad \text{and} \quad \sigma_{MM} = \text{var}\{R_M\}.
\]

One of the problems in applying the CAPM, whether in the domestic market or to the global market, is that, because \textit{ex-post} means and covariances are generally used as \textit{ex-ante} estimates, some anomalies may arise. For example, for some periods in some markets, market risk premiums may be negative. This may be dealt with in various ways. For example, Thomson & Gott (2009) deal with it by imposing a minimum market risk premium in the application of the CAPM.
2.2 The domestic capital-asset pricing model and market integration

Karolyi & Stulz (op. cit.) defined an ‘integrated market’ as a market in which assets have the same price regardless of where they are traded. In such a market investors should earn the same risk-adjusted expected return on similar assets in different domestic markets, which is consistent with the definition of integration by Jorion & Schwartz (op. cit.). The latter argued that, with integration, the world market index should be mean–variance efficient, and as a result, the only priced risk should be systematic risk relative to the world market.

Because the domestic CAPM considers only domestic investment, it has important limitations (Solnik, 1974a). As Solnik (1974a) pointed out, because there is no universal risk-free asset, and because of exchange-rate risk, there is little intuitive reason to expect that the simple risk-pricing relation in the CAPM could be applied at the international level. Since then, certain authors (e.g. Stehle, 1977; Stulz, 1995a) have argued that a domestic CAPM is appropriate only for an asset that is traded in a closed, domestic financial market.

As observed above, Wilkie (unpublished) showed that there is no unique CAPM in a multi-currency world. He showed that, if two investors measured risk and return in different currencies, the standard CAPM could not describe the pricing of capital assets for both investors.

Karolyi & Stulz (op. cit.) argued that there are systematic patterns in ownership of foreign equities that are hard to reconcile with models assuming perfect financial markets (such as the traditional CAPM) and therefore the only way to rationalise these patterns would be to argue that the gains from international diversification are too small to make it worthwhile to hold foreign assets. The inadequacies of the traditional CAPM in an international setting have therefore led to extensive debate and the development of equilibrium models (most of which are variations of the CAPM) to incorporate exchange-rate risk and global market portfolios.

2.3 The international capital-asset pricing model

As pointed out by Ng (2004), the starting point of the international CAPM literature is the observation that purchasing-power parity does not hold. This means that, in real terms, investors who measure their returns in different currencies earn different returns. This contravenes the standard CAPM assumption that investors have homogeneous expectations of returns, and it presents challenges for the aggregation of individual portfolios into a general asset-pricing equation. Wilkie (unpublished) also concluded that, when different currencies exist, the traditional CAPM’s assumption that all investors measure risk in the same currency breaks down. Stulz (1981) argued that, without a model showing how assets are priced in a world in which asset markets are fully integrated, it is impossible to determine whether asset markets are segregated
internationally or not. In the international CAPM (ICAPM) of Solnik (1974a), Sercu (1980) and Stulz (1981), exchange-rate risk is priced by modifying the CAPM. The ICAPM contains risk premia that are based on the covariances of assets with exchange rates. There are different versions of the ICAPM. For the purpose of this paper, to illustrate the application of the single factor CAPM in a multi-currency world, only the single-factor and multi-factor ICAPMs are considered.

The single-factor ICAPM, also referred to as the ‘global CAPM’ (GCAPM), as developed by Solnik (1974a), Grauer, Litzenberger & Stehle (1976), Sercu (op. cit.), Adler & Dumas (1983), Stulz (1981), and others, is expressed as follows:

\[ E(R_i) = R_F + \beta_i^w [E(R_W) - R_F]; \]  
(2)

where \( R_i \), \( R_F \) and \( R_W \) are the nominal returns in domestic currency on security \( i \), on the risk-free asset \( F \) and on the global market portfolio \( W \) respectively. \( \beta_i^w \) represents the sensitivity of the asset return to global market movements; i.e.:

\[ \beta_i^w = \frac{\sigma_{iw}}{\sigma_{WW}}; \]

\[ \sigma_{iw} = \text{cov}(R_i, R_W) \text{; and} \]

\[ \sigma_{WW} = \text{var}(R_W). \]

Thus the GCAPM looks at the world from the perspective of one currency only—the currency in which the investor measures returns on the risk-free asset and on the global market portfolio. The drawback of this approach is that, in general, the implied price of a security to that investor is different from the price to an investor who measures returns in another currency. An exception occurs if strict purchasing-power parity applies and if returns are measured in real terms (or there is no inflation). Under these conditions, equation (2) applies regardless of the currency in which the investor measures returns. Under such circumstances the risk-free rate in each currency is equal to the risk-free rate in every other currency, the mean–variance optimal portfolio for each investor is equal to that of every other investor, regardless of the currency in which they measure returns, and the variances and covariances of returns—and therefore the beta of every asset—is similarly the same for every investor. (cf. e.g. Karolyi & Stulz, 2003)

In an attempt to determine the factors that affect share-price movements across the world, Solnik (1974b) determined the international market structure of asset prices. The resulting model is referred to as the ‘multi-factor ICAPM’. The risk-pricing relation for the multi-factor ICAPM for asset \( i \) may be expressed as follows:

\[ E(R_i) = R_F + \beta_i^w [E(R_W) - R_F] + \gamma_i^1 [E(R_M^1) - R_F^1] + \gamma_i^2 [E(R_M^2) - R_F^2] + \ldots + \gamma_i^c [E(R_M^c) - R_F^c]; \]

(3)

where \( R_i \), \( R_F \), \( R_W \), \( R_M^c \) and \( R_F^c \) are the returns on security \( i \) (in the domestic currency), on the risk-free asset \( F \) (in the domestic currency), on the global market portfolio \( W \), on the market portfolio in currency \( c \) and on the risk-free asset in currency \( c \) respectively. Here \( \beta_i^w \) represents the sensitivity of the asset return to global market movements and \( \gamma_i^1 \) to \( \gamma_i^c \) are the sensitivities of asset \( i \) to the currencies 1 to \( C \) (the number of exchange-rate factors can be as many as the number of currencies other than the numeraire currency); i.e.:

\[ \beta_i^w = \frac{\sigma_{iw}}{\sigma_{WW}}; \]

\[ \sigma_{iw} = \text{cov}(R_i, R_W) \text{; and} \]
\[ \sigma_{WW} = \text{var}(R_W). \]

Under the assumptions of this model, a risk-free domestic asset is not risky for a domestic investor, but, because of currency risks, this same risk-free domestic asset is a risky asset for a foreign investor. The sensitivity parameters included in the ICAPM model capture these risks. Once again, that model looks at the world from the perspective of the currency in which the investor measures returns. Furthermore, there is no reason why, from the point of view of an investor who measures returns in that currency, investments in other currencies should not be diversified across currencies, nor why, for such an investor, the risks of investment in another currency should not be priced consistently with the risks of investment in the domestic currency. As observed above, the advantage in using a single-factor model is that it does not treat currency risks as carrying different weight from investment risks; regardless of its source, risk is measured as variance and weighted accordingly.

Karolyi & Stulz (op. cit.) demonstrated that systemic mistakes are possible when one uses the domestic CAPM and when domestic investors have access to international markets. Other authors have demonstrated pricing errors from using the domestic CAPM in an integrated market such as Dolde et al. (2011), Stulz (1995b), Stulz (1995c) and Koedijk et al. (2002).

It is standard practice in the USA to use a long-term (say 30-year) yield on treasury bills as a proxy for the risk-free rate (Stulz, 1995c). However, Stulz (1995c) uses monthly returns and computes arithmetic means of returns per period over a long sample period during which markets were fairly integrated. Jorion & Schwartz (op. cit.) derive the risk-free rate from the yield on three-month treasury bills and use monthly returns. In principle one should measure the risk-free rate over a time interval up to the assumed time horizon of market participants, or the interval after which portfolio selection will be reconsidered. This time interval should be equal to the unit time interval adopted. It is inevitable that risk-free rates will have varied over the sample period. Unless one applies the CAPM using time series, the choice of a risk-free rate is problematic, even in a domestic CAPM.

It is accepted in this paper that the assumption of strict purchasing-power parity is untenable, so that, even in real terms, the GCAPM is untenable. Furthermore, it is assumed in this paper that exposure to variances of returns arising from currency exposure is no different from exposure to variances of returns arising from domestic sources, so that a multi-factor model is unnecessary.

In this paper, in the light of Wilkie’s (op. cit.) conclusion and the literature reviewed, the authors develop a theory for a single-factor CAPM in a multi-currency world. They do so by specifying a CAPM for each set of investors that measures their returns in a particular currency. They assume that, for every currency in which investors measure risk, there is a unique CAPM across all the markets in which they invest.
3. The necessary and sufficient condition for the single-factor multi-currency CAPM

In this section a single-factor multi-currency CAPM (SFM-CAPM) is developed. Section 3.1 is a preliminary discussion largely devoted to the definitions required for the following sections. In section 3.2 it is shown that, for a single-factor CAPM to work in a multi-currency world, there is a necessary and sufficient condition. That condition applies to the \textit{ex-ante} variances and covariances of returns. The SFM-CAPM is formulated in section 3.3. Difficulties were experienced with the application of constrained maximum-likelihood to the estimation of the parameters of the model. In section 3.4 these difficulties are explained and alternative approaches are considered. One of these approaches, which uses ordinary least squares, is developed in section 3.5.

3.1 Preliminary discussion

Suppose there are $C$ currencies and that, in currency $c$, there is one risk-free asset and $n_c$ risky capital assets have been issued. It is assumed that every investor measures investment returns in one of these currencies. Regardless of the currency in which an investor measures investment returns, the investor may invest in any currency. An ‘asset issued in currency $c$’ is a risky asset issued in that currency or the risk-free asset denominated in that currency. (For an investor who measures returns in another currency, the risk-free asset denominated in currency $c$ is not risk-free; this matter is dealt with in greater detail below.) The ‘return in currency $c$’ on an asset issued in currency $d$ is the force of return (i.e. the ‘log return’) earned on that asset, over a unit interval, measured in currency $c$. Whilst it is customary to measure returns as rates, there are substantial advantages to the use of forces. The implicit assumption of this approach is that assets are continuously rebalanced during the unit interval, so that the weightings of the respective forces remain constant. Exchange rates are measured per unit of currency 1. The increase in the exchange rate of currency $c$ per unit of currency 1 is measured as a force of increase over the unit interval. Returns and increases in exchange rates may be measured in real terms (relative to a price index) or in nominal terms.

We assume that the CAPM applies for investors in each currency. More specifically, we assume that:

1) investors who measure their investment returns in currency $c$ (i.e. ‘currency-$c$ investors’) have indifference curves in mean–variance space, the means and variances being those measured in that currency; and

2) all investors, regardless of the currency in which they measure returns, have homogeneous expectations of the means, variances and covariances of:
   (a) the returns in each currency on assets issued in that currency; and
   (b) increases in the exchange rates of each currency per unit of currency 1.

First we consider returns in currency $c$ on assets issued in that currency. For this purpose we define the following random variables, where, for $c = 1, \ldots, C$, $i = 1, \ldots, n_c$ denotes the risky assets issued in that currency:

- $X_{ci}$ is the return in currency $c$ on risky asset $i$ issued in that currency for $c = 1, \ldots, C; i = 1, \ldots, n_c$; and
- $X_c$ is the increase in the exchange rate of currency $c$ per unit of currency 1 for $c = 2, \ldots, C$. 

We also define the deterministic return on the risk-free asset denominated in currency $c$ as $X_{c0} = x_{c0}$.

We define the following parameters, where, as above, for $c = 1, \ldots, C$, $i = 0$ denotes the risk-free asset denominated in that currency and $i = 1, \ldots, n_c$ denotes the risky assets issued in that currency:

- $\mu_i$ is the expected return in currency $c$ on risky asset $i$ issued in that currency; i.e.: $\mu_i = E \{ X_{ci} \}$;
- $\sigma_{ci,dj}$ is the covariance of the return in currency $c$ on risky asset $i$ issued in that currency with the return in currency $d$ on risky asset $j$ issued in that currency; i.e.: $\sigma_{ci,dj} = \begin{cases} \text{var} \{ X_{ci} \} & \text{for } d = c, j = i; \\ \text{cov} \{ X_{ci}, X_{dj} \} & \text{otherwise}; \end{cases}$
- $\mu_c$ is the expected increase in the exchange rate of currency $c$ per unit of currency 1; i.e.: $\mu_c = E \{ X_c \}$;
- $\sigma_{c,d_i}$ is the covariance of the increase in the exchange rate of currency $c$ per unit of currency 1 with the return in currency $d$ on risky asset $i$ issued in that currency; i.e.: $\sigma_{c,d_i} = \text{cov} \{ X_c, X_{d_i} \}$;
- $\sigma_{c,c}$ is the variance of the increase in the exchange rate of currency $c$ per unit of currency 1; i.e.: $\sigma_{c,c} = \text{var} \{ X_c \}$.

Because investors have homogeneous expectations (assumption (2) above), the means, variances and covariances defined above are the same for all investors, regardless of the currency in which they measure their returns.

In the case of currency 1 the increase in the exchange rate per unit of currency 1 is trivially zero. For that currency we therefore have:

$$\mu_1 = 0;$$

$$\sigma_{1,d_i} = 0;$$ and

$$\sigma_{1,c} = 0.$$  \hspace{1cm} (4) \hspace{1cm} (5) \hspace{1cm} (6)

Also, for the risk-free asset denominated in currency $c$, the return is deterministic, so we have:

$$\sigma_{c0,d_i} = 0.$$  \hspace{1cm} (7)

The variables defined in above relate to returns in a particular currency as measured in that currency and to exchange rates between that currency and currency 1. Now we need to consider the returns to investors who measure their returns in other currencies, for example a currency-$c$ investor. For this purpose we define the following:

- $X_{ci}$ is the return in currency $c$ on asset $i$ issued in currency $d$ for $c,d = 1, \ldots, C$; $i = 0, 1, \ldots, n_d$; i.e.: 

Because we are working with forces of increase in exchange rates, the increases are additive. We may then determine the following:

- $\mu_{di}$ is the expected return in currency $c$ on asset $i$ issued in currency $d$ for $c,d = 1, \ldots, C; \ i = 0,1, \ldots, n_d$; i.e.:
  \begin{equation}
  \mu_{di} = E\{X_{di} + X_d - X_c\} = \mu_d + \mu_c.
  \end{equation}

- $\sigma_{di,ej}$ is the covariance of the return in currency $c$ on asset $i$ issued in currency $d$ with the return in currency $c$ on risky asset $j$ issued in currency $e$; i.e.:
  \begin{equation}
  \sigma_{di,ej} = \text{cov}\{X_{di} + X_d - X_c, X_{ej} + X_e - X_c\} = \text{cov}\{X_{di}, X_{ej}\} + \text{cov}\{X_d, X_{ej}\} - \text{cov}\{X_{di}, X_c\} + \text{cov}\{X_d, X_c\} - \text{cov}\{X_{ej}, X_c\} + \text{var}\{X_c\} - \sigma_{di, ej} = \sigma_{d, e}\text{, for } c \neq d,
  \end{equation}

Let $p_{di}$ denote the value in currency $c$ of investments in asset $i$ issued in currency $d$ held by currency-$c$ investors, per unit of the total value in that currency of the assets held by such investors. The value of $p_{di}$ is unknown; it is estimated through an optimisation process explained below. We now define the portfolio of risky assets held in currency $d$ by a currency-$c$ investor as:

$$
p^c_d = \begin{cases} 
p^c_{d0} 
\vdots 
p^c_{dn} 
\end{cases} \text{ for } d \neq c; 
\begin{cases} 
p^c_{e1} 
\vdots 
p^c_{en} 
\end{cases} \text{ for } d = c.
$$

This vector has $n_d + 1$ components for $d \neq c$ or $n_e$ for $d = c$. This is because, for currency $d \neq c$, the risk-free asset denominated in currency $d$ is included (as $p^c_{d0}$) as a risky asset in this portfolio, whereas for currency $d = c$, the risk-free asset denominated in that currency is not included, as it is not a risky asset. Then we define the market portfolio of a currency-$c$ investor by combining the vectors $p^c_i, \ldots, p^c_c$ of equation (11) into the vector:

$$
p^c = \begin{cases} 
p^c_1 
\vdots 
p^c_c 
\end{cases}.
$$

$$
X^{c}_{di} = X_{di} + X_d - X_c.
$$

(8)
This vector has $h$ unknown components, where $h$ is the number of risky assets in which an investor can invest, viz.:

$$h = \sum_{e=1}^{C} n_e + C - 1.$$  \hfill (13)

We may express $p^c$ as a simple vector of $h$ components renumbered consecutively, viz.:

$$p^c = \begin{pmatrix} p^c_1 \\ \vdots \\ p^c_h \end{pmatrix}. \hfill (14)$$

By definition, the elements of the vector $p^c$ sum to 1; i.e.:

$$\sum_{\tau=1}^{h} p^c_{\tau} = 1.$$ \hfill (15)

Similarly, we define the return on the risky assets held in currency $d$ by a currency-$c$ investor as:

$$X^c_d = \begin{cases} X^c_{d0} \\ \vdots \\ X^c_{d_{nj}} \end{cases} \quad \text{for } d \neq c;$$

$$X^c_{c1} \quad \text{for } d = c;$$

$$X^c_{cn}$$

where $X^c_{di}$ is the return on risky asset $i$ issued in currency $d$ measured in currency $c$ (equation (8)). Then we define the return on the portfolio of risky assets held by a currency-$c$ investor as the $h$-component vector:

$$X^c = \begin{pmatrix} X^c_1 \\ \vdots \\ X^c_h \end{pmatrix} \hfill (17)$$

We similarly define the expected return on the risky assets held in currency $d$ by a currency-$c$ investor as:
where $\mu^c_{d, i}$ is the expected return on risky asset $i$ issued in currency $d$ measured in currency $c$ (equation (9)). Then we define the expected return on the portfolio of risky assets held by a currency-$c$ investor as the $h$-component vector:

$$
\mu^c = \begin{pmatrix} 
\mu^c_1 \\
\vdots \\
\mu^c_h 
\end{pmatrix} = 
\begin{pmatrix} 
\mu_1^c \\
\vdots \\
\mu_h^c 
\end{pmatrix}.
$$

(19)

Also, we define the covariance matrices of the returns on the risky assets held in currency $d$ with those on the risky assets held in currency $e$ by a currency-$c$ investor as:

$$
\Sigma^{c}_{de} = 
\begin{pmatrix} 
\sigma^{c}_{d,0,0} & \cdots & \sigma^{c}_{d,0,n} \\
\vdots & \ddots & \vdots \\
\sigma^{c}_{d,n,0} & \cdots & \sigma^{c}_{d,n,n} 
\end{pmatrix}
$$

for $d \neq c, e \neq c$;

$$
\begin{pmatrix} 
\sigma^{c}_{c,0} & \cdots & \sigma^{c}_{c,n} \\
\vdots & \ddots & \vdots \\
\sigma^{c}_{c,n} & \cdots & \sigma^{c}_{c,n} 
\end{pmatrix}
$$

for $d = c, e \neq c$;

$$
\begin{pmatrix} 
\sigma^{c}_{e,0} & \cdots & \sigma^{c}_{e,n} \\
\vdots & \ddots & \vdots \\
\sigma^{c}_{e,n} & \cdots & \sigma^{c}_{e,n} 
\end{pmatrix}
$$

for $d \neq c, e = c$;

$$
\begin{pmatrix} 
\sigma^{c}_{c,0} & \cdots & \sigma^{c}_{c,n} \\
\vdots & \ddots & \vdots \\
\sigma^{c}_{c,n} & \cdots & \sigma^{c}_{c,n} 
\end{pmatrix}
$$

for $d = c, e = c$;

(20)

where $\sigma^{c}_{d, i, j}$ is the covariance of the returns on risky assets $i$ and $j$ held in currencies $d$ and $e$ respectively by a currency-$c$ investor (equation (10)). Then we define the $h \times h$ covariance matrix of the return on the portfolio of risky assets held by a currency-$c$ investor as:
\[
\Sigma^c = \begin{pmatrix}
\sigma_{11} & \cdots & \sigma_{1h} \\
\vdots & \ddots & \vdots \\
\sigma_{h1} & \cdots & \sigma_{hh}
\end{pmatrix} = \begin{pmatrix}
\Sigma_{11}^c & \cdots & \Sigma_{1p}^c & \cdots & \Sigma_{1c}^c \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\Sigma_{d1}^c & \cdots & \Sigma_{dc}^c & \cdots & \Sigma_{dc}^c \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\Sigma_{C1}^c & \cdots & \Sigma_{Cc}^c & \cdots & \Sigma_{Cc}^c
\end{pmatrix}.
\] (21)

Now, from the definitions in equations (14) and (17), we may express the return on the portfolio of risky assets held by a currency-\(c\)-investor (i.e. on the ‘market portfolio’ of currency-\(c\) investors) as:

\[X_M^c = p^e X^c.\] (22)

(We use the subscript \(M\) to denote that portfolio.) Similarly, from the definitions in equations (14) and (19), we may express the expected return on the portfolio of risky assets held by a currency-\(c\)-investor as:

\[\mu_M^c = E \{X_M^c\} = p^e \mu^c.\] (23)

Also, from equations (14) and (21), we may express the variance of the return on the portfolio of risky assets held by a currency-\(c\)-investor as:

\[\sigma_{M,M}^c = \text{var} \{X_M^c\} = p^e \Sigma^c p^e.\] (24)

In terms of the CAPM, currency-\(c\) investors determine their portfolio of risky assets in quarter \(t\) by maximising

\[k_t = \frac{\hat{\mu}_{M,t} - x_{c0,t}}{\hat{\sigma}_{M,M,t}};\] (25)

where:
- \(\hat{\mu}_{M,t}\) is the ex-ante estimate of the expected return to a currency-\(c\) investor on her/his portfolio during quarter \(t\);
- \(x_{c0,t}\) is the observed risk-free rate in currency \(c\) during quarter \(t\); and
- \(\hat{\sigma}_{M,M,t}\) is the ex-ante estimate of the variance of the return to a currency-\(c\) investor on her/his portfolio during quarter \(t\).

In order to avoid short positions in the market portfolio of currency-\(c\) investors, \(k_t\) is maximised subject to the constraint:

\[p^c_t \geq 0 \text{ for all } \tau \text{ and } c;\]

and, as in equation (15):

\[\sum_{\tau=1}^h p^c_\tau = 1.\]

For the purposes of this paper we assume that, during each period considered, \(\hat{\sigma}_{MM,t}\) is constant quarter by quarter, so that we may denote it \(\hat{\sigma}_{MM}^c\). It would not be appropriate to assume that \(\hat{\mu}_{M,t}\) is constant, since this would allow the market risk premium
\[ \hat{k}_M^c = \hat{\mu}_M^c - x_{c0t} \]

to be negative when \( x_{c0t} \) is large; where:

\( \hat{\mu}_M^c \) is the sample mean of the expected return to a currency-\( c \) investor on her/his portfolio during the period; i.e.:

\[ \hat{\mu}_M^c = \frac{1}{T} \sum_{t=1}^{T} x_{M,t}^c; \]

and

\( x_{M,t}^c \) is the observed return on a currency-\( c \) investor’s market portfolio during quarter \( t \).

This could be avoided by assuming that the estimated market risk premium

\[ \hat{k}_M^c = \hat{\mu}_{M,t}^c - x_{c0t} \]

is constant, by setting:

\[ \hat{\mu}_{c0} = \frac{1}{T} \sum_{t=1}^{T} x_{c0t}; \]

so that:

\[ \hat{k}_M^c = \frac{1}{T} \sum_{t=1}^{T} (x_{M,t}^c - x_{c0t}) = \hat{\mu}_M^c - \hat{\mu}_{c0}. \]  

(27)

As observed in the previous section, if over the sample period chosen the ex-post estimate \( \hat{\mu}_M^c \) happens to be low in relation to \( \hat{\mu}_{c0} \), even equation (27) may give negative (or unacceptably low) values of the market risk premium. It should be recognised that what is required here is an ex-ante estimate of the market risk premium; the ex-post sample value is merely a provisional indication of the ex-ante estimate. A negative value of the market risk premium would imply that the representative market participant is willing to accept a lower expected return on a risky asset than the return on the risk-free asset; i.e. that she/he is risk-seeking. In order to avoid such effects, we could impose a minimum value on the risk premium for each risky asset. However, this would lead to distortions, which are addressed in section 3.5 below. For the purposes of this paper, the incidence of low market risk premia is analysed rather than avoided.

Equation (25) may now be restated as:

\[ k = \frac{\hat{k}_M^c}{\sqrt{\hat{\sigma}_{M,M}^c}} = \frac{\hat{\mu}_M^c - \hat{\mu}_{c0}^c}{\sqrt{\hat{\sigma}_{M,M}^c}}; \]

(28)

where: \( \hat{\mu}_{c0}^c \) is the sample mean of the risk-free rate in currency \( c \) during the period; i.e.:

\[ \hat{\mu}_{c0}^c = \hat{\mu}_{c0}; \]

as defined above, \( \hat{\sigma}_{M,M}^c \) is the constant ex-ante estimate of the variance of the return to a currency-\( c \) investor on her/his market portfolio during quarter \( t \).

The value of \( p^c \) that maximises \( k \) may be determined by quadratic programming. For this purpose the authors used the code ‘solveRquadprog’ in the fPortfolio package in the programming language R.
3.2 Derivation of the necessary and sufficient condition

Since the CAPM applies for investors in each currency (assumption (1)), it follows that, for asset $i$ issued in currency $d$, the expected returns in currencies $c$ and 1 are respectively:

$$\mu_{di}^c = x_{c0} + \frac{\sigma_{di,M}^c}{\sigma_{M,M}^c} (\mu_M^c - x_{c0});$$  \hspace{1cm} (30)

$$\mu_{di}^1 = x_{10} + \frac{\sigma_{di,M}^1}{\sigma_{M,M}^1} (\mu_M^1 - x_{10});$$  \hspace{1cm} (31)

where $\sigma_{di,M}^c$ is the covariance of the return in currency $c$ on asset $i$ issued in currency $d$ with the return in currency $c$ on the market portfolio of a currency-$c$ investor, i.e.:

$$\sigma_{di,M}^c = \text{cov}\{X_{di}^c, X_M^c\} = \sum_{d=1}^C \sum_{i=1}^N \sum_{j=1}^N p_{di}^c \sigma_{dj, ej}^c = \left( p^c \right)^\prime \sigma_{\hat{d}, i}^c;$$  \hspace{1cm} (32)

and $\sigma_{M,M}^c$ is the variance of the return in currency $c$ on the market portfolio of a currency-$c$ investor, i.e.:

$$\sigma_{M,M}^c = \text{var}\{X_M^c, X_M^c\} = \sum_{d=1}^C \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^M p_{di}^c p_{ej}^c \sigma_{dj, ej}^c = \left( p^c \right)^\prime \sigma_{\hat{d}, c}^c;$$  \hspace{1cm} (33)

where:

$$\sigma_{\hat{d}, i}^c = \begin{pmatrix} \sigma_{di,11}^c \\ \vdots \\ \sigma_{di,\hat{c},c}^c \end{pmatrix};$$

$$\sigma_{\hat{d}, c}^c = \begin{pmatrix} \sigma_{11,c}^c \\ \vdots \\ \sigma_{\hat{c},\hat{c},c}^c \end{pmatrix};$$

and

$$p_{c0}^c = 0.$$

In the vectors $\sigma_{\hat{d}, i}^c$ and $\sigma_{\hat{d}, c}^c$ the components $\sigma_{di,c0}^c$ and $\sigma_{c0,M}^c$ are omitted.

Consider the investment return in currency $c$ on asset $i$ issued in currency $d$. From equations (4) and (9) the expected return on this asset is:

$$\mu_{di}^c = \mu_{di} - \mu_c.$$  \hspace{1cm} (34)

Now, substituting equations (30) and (31) into equation (34) we have:

$$x_{c0} + \frac{\sigma_{\hat{d}, M}^c}{\sigma_{M,M}^c} (\mu_M^c - x_{c0}) = x_{10} + \frac{\sigma_{\hat{d}, M}^1}{\sigma_{M,M}^1} (\mu_M^1 - x_{10}) - \mu_c.$$  \hspace{1cm} (35)

Equation (35) holds if and only if:

$$\sigma_{di,M}^c = \sigma_{\hat{d}, c}^c \frac{(\mu_M^c - x_{c0})}{\sigma_{M,M}^c} \left( \frac{\mu_M^1 - x_{10}}{\sigma_{M,M}^1} \sigma_{\hat{d}, M}^1 + x_{10} - x_{c0} - \mu_c \right).$$  \hspace{1cm} (36)
Now from equations (4) and (9), since \( \mu_{t0} = x_{t0} \), the expected return to a currency-\( c \) investor on the risk-free asset in currency 1 is:

\[
\mu_{t0}^c = x_{t0} - \mu_c;
\]

which, under the CAPM, must be:

\[
x_{t0} - \mu_c = x_{t0} + \frac{\sigma^c_{t0,M}}{\sigma^c_{M,M}} (\mu_M^c - x_{t0});
\]

i.e.:

\[
\frac{\sigma^c_{M,M}}{\mu^c_M - x_{c0}}(x_{t0} - x_{t0} - \mu_c) = \sigma^c_{t0,M}.
\]

On substituting equation (38) into equation (36) we have:

\[
\sigma^c_{d,M} = \frac{\sigma^c_{M,M} (\mu^c_M - x_{t0})}{\sigma^c_{M,M}} (\sigma^c_{d,M} + \sigma^c_{t0,M}).
\]

Now

\[
\sigma^c_{t0,M} = \text{cov} \{x_{t0} - X_c, X_M^c\}
\]

\[
= - \text{cov} \{X_c, X_M^c\}
\]

\[
= - \sigma^c_{c,M},
\]

where:

\[
\sigma^c_{c,M} = \sum_{j=1}^{n} \sum_{j=0}^{c} p_{j}^e \sigma_{c,j} = (p^c)' \sigma_c; \text{ and}
\]

\[
\sigma_c = \begin{pmatrix} \sigma_{c,11} \\ \vdots \\ \sigma_{c,Cn} \end{pmatrix}.
\]

In the vector \( \sigma_c \) the component \( \sigma_{c,0} \) is omitted. Equation (39) (and therefore equation (36) may therefore be expressed as:

\[
\sigma^c_{d,M} + \sigma^c_{c,M} = a_c \sigma_{d,M}^1;
\]

where:

\[
a_c = \frac{\sigma^c_{M,M} (\mu^c_M - x_{t0})}{\sigma^c_{M,M}}.
\]

Given equations (4) and (9), and hence (34) and (37), it follows that the CAPM, as expressed in equations (30), (31) and (38) holds if and only if equation (41) holds. This is because:

\[
(35) \Leftrightarrow (36) \Leftrightarrow (39) \Leftrightarrow (41).
\]

Equation (41) is thus both a sufficient condition and a necessary condition for the SFM-CAPM to apply in a multi-currency world. In the estimation of the parameters of the model, equation (41), must be used as a constraint.
In theory, the variance and covariance parameters $\sigma_{ci,di}$, $\sigma_{c,di}$ and $\sigma_{c,c}$, may be found by constrained maximum-likelihood estimation, that is by maximising the likelihood of the observed values subject to the constraint of equation (41). In practice this is difficult. Problems relating to this approach are explained below.

3.3 Formulation of the SFM-CAPM

Equation (41) may also be expressed as:

$$
\left(\sigma_{di,M}^c + \sigma_{c,M}^c\right) \frac{\mu_M^c - x_c^0}{\sigma_{M,M}^c} = \sigma_{di,M}^c \frac{\mu_M^1 - x_1^0}{\sigma_{M,M}^1}. \tag{43}
$$

In other words, for all $c$:

$$
\left(\sigma_{di,M}^c + \sigma_{c,M}^c\right) \frac{\mu_M^c - x_c^0}{\sigma_{M,M}^c} = \kappa_{di}. \tag{44}
$$

In the case of $c = 1$, from equations (5) and (40), $\sigma_{i,M}^1 = 0$. We may refer to $\kappa_{di}$ as the risk premium for asset $i$ in currency $d$ adjusted for exchange-rate risk. The value of $\kappa_{di}$ will differ according to which currency is chosen as currency 1. However, the equality of the adjusted risk premia will be unaffected by that choice. Given that choice, the adjustment is the same for all $(d, i)$.

Equation (44) applies ex ante. It will not necessarily apply ex post. Because equation (26) is expressed in terms of the ex-post mean $\mu_{c0}$ of quarterly risk-free rates defined above and not in terms of the risk-free rate $x_c$ for a particular quarter, which is known at the start of that quarter, the optimal portfolios, and the variables in equation (44) that arise from that optimisation, will not generally result in the equality shown in that equation.

The equality of the adjusted risk premium $\kappa_{di}$ for all investors may be heuristically explained as comprising two factors: the normal beta (i.e. the GCAPM beta) for the currency-$c$ investor’s return on asset $i$ in currency $d$, i.e.:

$$
\beta_{di}^c = \frac{\sigma_{di,M}^c}{\sigma_{M,M}^c}, \tag{45}
$$

and an additional beta for that investor’s exchange-rate risk relative to currency 1, i.e.:

$$
\beta_{c}^c = \frac{\sigma_{c,M}^c}{\sigma_{M,M}^c}. \tag{46}
$$

The use of exchange-rate risk relative to currency 1 instead of currency $d$ is counter-intuitive. Although the value of beta may be analysed as comprising two factors, both factors relate to the price of covariance in currency $c$, so that in effect we are dealing with one factor. Equation (44) may therefore be expressed as:

$$
\beta_{di,c}^c \left(\mu_M^c - x_c^0\right) = \kappa_{di}, \tag{47}
$$

where:

$$
\beta_{di,c}^c = \beta_{di}^c + \beta_{c}^c. \tag{48}
$$
\( \beta_{d_i,c} \) specifies the single factor. The SFM-CAPM may therefore be expressed as:

\[
E\{X_{d_i}^c\} = x_{c,0} + \beta_{d_i,c}^c (\mu_M^c - x_{c,0}); \tag{49}
\]

i.e.:

\[
\mu_{d_i}^c = x_{c,0} + \kappa_{d_i}; \tag{50}
\]

where:

\[
\kappa_{d_i} = \beta_{d_i,c}^c (\mu_M^c - x_{c,0}) = \sigma_{d_i,M}^1 \frac{\mu_M^c - x_{c,0}}{\sigma_{MM}^1}. \tag{51}
\]

If strict purchasing-power parity holds then, in real terms (or if there is no inflation), the SFM-CAPM reduces to the GCAPM. In this case:

\[
\sigma_{c,M}^c = 0;
\]

and, for all currencies \( c \) and \( e \):

\[
\beta_{d_i}^c = \beta_{d_i}^e.
\]

Equation (49) then reduces to equation (2), i.e. to the GCAPM. As explained in the previous section, equation (2) then applies regardless of the currency in which the investor measures returns.

3.4 Alternative approaches

As explained above, whilst the determination of the variances and covariances by constrained maximum-likelihood estimation is theoretically appealing, it is problematic to apply in practice. The method is complicated; it requires a lengthy program, and it necessitates an iterative process, which may fail to converge. Furthermore, it involves the inversion of a matrix that would run to over 200 rows and columns for the applications envisaged in this paper. Problems with singular matrices arise, especially for the small number of observations typically available at quarterly intervals in each period during which means, variances and covariances may be reasonably assumed to be constant.

Another problem is that, because of the linear constraint in equation (41), there is no reason why a constrained variance would necessarily be positive. For a large sample variance it is unlikely that the constrained estimate of the variance will be negative, but for a small sample variance the constrained estimate may turn out to be negative. Similar problems may arise with correlation coefficients outside of the range \([-1, 1]\).

For the reasons outlined above, alternative approaches are considered.

It might be argued that equation (43) could be used to estimate \( \mu_M^c \); i.e.:

\[
\mu_M^c = x_{c,0} + \frac{\sigma_{d_i,M}^1 \sigma_{MM}^c}{(\sigma_{d_i,M}^c + \sigma_{e,M}^c) \sigma_{MM}^1} (\mu_M^1 - x_{c,0}); \tag{52}
\]

The problem with this argument is that it presupposes that, whilst \( \mu_M^1 \) is known, \( \mu_M^c \) is unknown for \( c = 2,\ldots, C \). If that is accepted as true then it would not be inappropriate
to use equation (52), but the result would not be independent of the choice of currency 1.

Another approach is to use equation (47) to calculate an observed value of $\kappa_{di}$ for each currency, using ordinary least squares based on the usual sample means, variances and covariances. This is dealt with in section 3.5 below.

3.5 Least-squares estimation of the betas

Let $\hat{\kappa}^c_{di}$ denote the observed value of $\kappa_{di}$ for a currency-$c$ investor. Equation (47) states that:

$$\beta^c_{di,c} \left( \mu^c_M - x_{c,0} \right) = \kappa_{di}.$$  

In order to estimate $\kappa_{di}$ using least squares, we would require:

$$\frac{\partial}{\partial \kappa_{di}} \left[ \sum_{c=1}^C \left\{ \beta^c_{di,c} \left( \mu^c_M - x_{c,0} \right) - \kappa_{di} \right\}^2 \right] = 0;$$

i.e.:

$$-2 \sum_{c=1}^C \left\{ \beta^c_{di,c} \left( \mu^c_M - x_{c,0} \right) - \kappa_{di} \right\} = 0;$$

and hence:

$$\hat{\kappa}^c_{di} = \frac{1}{C} \sum_{c=1}^C \left\{ \hat{\beta}^c_{di,c} \left( \hat{\mu}^c_M - \hat{\mu}_{c,0} \right) \right\}$$

$$(53)$$

where:

$$\hat{\kappa}^c_{di} = \hat{\beta}^c_{di,c} \left( \hat{\mu}^c_M - \hat{\mu}_{c,0} \right).$$

$$(54)$$

In equation (53) the summation is with respect to currencies for which asset $i$ in currency $d$ is in the opportunity set of risky assets. For $i = 0$ this excludes $c = d$ as this is the risk-free asset in that currency. For $i = 0$ the summation is therefore with respect to $C - 1$ currencies and the sum is therefore divided by $C - 1$ to give the estimate.

However, as anticipated in section 3.1, this may result in negative values of $\hat{\kappa}^c_{di}$ and therefore of $\hat{\kappa}^c_{di}$. In order to avoid this, we set:

$$\kappa^c_{di} = \max \left( \hat{\beta}^c_{di,c} \left( \hat{\mu}^c_M - \hat{\mu}_{c,0} \right), 0 \right).$$

$$(55)$$

Equation (53) then becomes a constrained least-squares estimate of $\kappa_{di}$.

We may then estimate $\beta^c_{di,c}$ using equation (47) as:

$$\hat{\beta}^c_{di,c} = \frac{\hat{\kappa}^c_{di}}{\hat{\mu}^c_M - \hat{\mu}_{c,0}}.$$  

$$(56)$$
This method is considerably simpler than that of constrained maximum-likelihood estimation of the variances and covariances. Estimates of $\kappa_{di}$ and $\beta_{di,c}$ were therefore determined by this method.

Because we have redetermined the expected values of $X_{di}^c$ the portfolios $p^c$ are no longer optimal. We therefore require an iterative process to determine the expected values for optimal portfolios. This is done as follows.

Step 1:
For $r = 1$ we determine the GCAPM values $p^{c(1)}$ so as to maximise

$$k = \frac{\hat{\kappa}_M^c}{\sqrt{\hat{\sigma}_{MM}^c}}; \text{(equation (28))}$$

where:

$$\hat{\kappa}_M^c = p^{c(r)} \hat{\kappa}^c;$$

$$\hat{\kappa}^c = \begin{bmatrix} \hat{\kappa}_1^c \\ \vdots \\ \hat{\kappa}_c^c \\ \vdots \\ \hat{\kappa}_C^c \end{bmatrix};$$

$$\hat{\kappa}_d^c = \begin{bmatrix} \hat{\kappa}_{d,0}^c \\ \vdots \\ \hat{\kappa}_{d,c}^c \end{bmatrix} \quad \text{for } d \neq c;$$

$$\hat{\kappa}_{d,c}^c = \hat{\mu}_{di}^c - \hat{\mu}_{c,0}^c \text{ (equation (55));}$$

$$\hat{\mu}_{di}^c = \max \left\{ \frac{1}{T} \sum_{t=1}^{T} x_{di,t}^c, \hat{\mu}_{c,0}^c \right\};$$

$$\hat{\mu}_{c,0}^c = \frac{1}{T} \sum_{t=1}^{T} x_{c,0,t}; \text{ and}$$

$$\hat{\sigma}_{MM}^c = p^{c(r)} \hat{\Sigma}^c p^{c(r)} \text{ (equation (24)).}$$

We also calculate:

$$\hat{\sigma}_{di,M}^c = p^{c(r)} \hat{\sigma}_{di}^c \text{ (equation (32)); and}$$

$$\hat{\beta}_d^c = \frac{\hat{\sigma}_{di,M}^c}{\hat{\sigma}_{MM}^c} \text{ (equation (45)).}$$

Step 2:
For $r = 2$ we determine:
\[ \hat{\mu}_{di}^{(r)} = \hat{\mu}_c + \hat{\kappa}_{di}^{(r-1)}; \]

where:
\[ \hat{\kappa}_{di}^{(r-1)} = \frac{1}{C} \sum_{c=1}^{C} \hat{\kappa}_{di}^{(r-1)} \] (equation (53)).

Step 3:
For \( r = 2 \) we determine updated values \( p^{(r)} \) so as to maximise
\[ k = \frac{\hat{\kappa}_{M}^{(r)}}{\sqrt{\hat{\sigma}_{MM}}}; \]

where \( \hat{\kappa}_{M}^{(r)} \) and \( \hat{\sigma}_{MM} \) are calculated as in step 1, except that \( \hat{\mu}_{di}^{(r)} \) is as calculated in step 2:

Step 4:
For \( r = 2 \) we calculate the values of \( \hat{\kappa}_{di}^{(r)} \) and \( \hat{\kappa}_{di}^{(r)} \) following the same method as in steps 1 and 2 and we calculate:
\[ \hat{\mu}_{M}^{(r)} = p^{(r)} \hat{\mu}^{(r)} \] (equation (23)).

Step 5:
If \( \hat{\mu}_{M}^{(r)} - \hat{\mu}_{M}^{(r-1)} \leq \Delta \) then we stop. Here \( \Delta \) was taken as 0.00001. Otherwise we repeat steps 2 to 5 for \( r = 3, 4, \ldots \).

Step 6:
The final values \( \hat{\mu}^{(r)}, p^{(r)} \) and \( \hat{\mu}_{M}^{(r)} \) are the SFM-CAPM values, which can then be used to determine the SFM-CAPM values of beta as:
\[ \hat{\beta}_{di,c}^{(r)} = \frac{\hat{\kappa}_{di}^{(r)}}{\hat{\mu}_{M}^{(r)} - \hat{\mu}_c} \] (equation (56)).

The SFM-CAPM values \( \hat{\beta}_{di,c}^{(r)} \) obtained from step 6 may be compared with the GCAPM values \( \hat{\beta}_c \) determined in step 1.

4. Application
For illustrative purposes the method outlined in section 3.5 was applied to a selection of currencies. In section 4.1 an overview of the data is given. The results are presented in section 4.2.

4.1 Data
The currencies selected are:
\[ c = \begin{cases} 
1 \text{ for USA dollars;} \\
2 \text{ for UK pounds;} \\
3 \text{ for South African rands; and} \\
4 \text{ for Turkish lira;} 
\end{cases} \] (57)
so that \( C = 4 \). The selection is influenced by the authors’ interests and the data most easily available to them, but the application is intended to be illustrative; care has been taken to specify the data sources and calculations so as to facilitate consistent applications to other currencies.

Quarterly forces of return and of increases in exchange rates were used. The methods were applied both to real returns and to nominal returns. The real returns and increases in exchange rates were calculated as:

\[
\tilde{x}_c(t) = x_c(t) - \theta_c(t); \quad \text{and} \quad \tilde{x}_c(t) = x_c(t) - \theta_c(t)
\]

(58)

(59)

respectively; where:

- \( x_c(t) \) is the return in currency \( c \) on asset \( i \) issued in that currency during quarter \( t \);
- \( x_c(t) \) is the increase in the exchange rate of currency \( c \) per unit of currency \( 1 \) during that quarter; and
- \( \theta_c(t) \) is the force of inflation in currency \( c \) during quarter \( t \).

Because we are working with forces, the relationships in equations (58) and (59) are linear.

The risky assets covered by the application comprised equities, and conventional and index-linked bonds (inflation-protected government bonds) of selected maturities. For nominal returns the risk-free asset was the conventional bond with a maturity of one quarter and for real returns it was the corresponding index-linked bond.

The period covered by the application was from 1975Q2 to 2012Q1. The calculation of \( x_c(t) \), \( x_c(t) \) and \( \theta_c(t) \) from the data available\(^1\) entailed intensive work, some of which relied on assumptions and estimations by the authors. These calculations are dealt with in detail in a note entitled “How a single-factor CAPM works in a multi-currency world: information on the determination of data,” which, together with a spreadsheet file showing the determination of the data required, is available free of charge from the authors. As explained in that note, data are not available for every series throughout the period and in some cases the data were not acceptable for the purposes of this paper. The quarters from which acceptable returns (or, in the case of inflation and exchange rates, acceptable forces of inflation and forces of increase in exchange rates respectively) could be calculated from the data available for the respective series are shown in Table 1.

The type of bond used is also shown in Table 1: ‘coupon’ denotes coupon-paying bonds and ‘ZCBs’ denotes zero-coupon bonds. These are not necessarily the type of bond in issue; where possible, returns were determined for zero-coupon bonds in order to avoid inaccuracies relating to the amount of the coupon on different bonds used for the determination of yield-curve data available. For each series of bonds, two terms to redemption were chosen: the short term of one quarter and a long term

\(^1\) Sources: USA Federal Reserve Bank; Bureau of Labor Statistics, U.S. Department of Labor; Bank of England; INet; Central Bank of the Republic of Turkey; Republic of Turkey Prime Ministry, Undersecretariat of Treasury, Public Finance; Istanbul Stock Exchange
depending on the data available. The column headed ‘long term (qtrs.)’ shows the term to redemption chosen as the long term (in quarters). In Table 1, ‘SA’ refers to South Africa and ‘TR’ to Turkey.

Table 1. Periods for which acceptable returns could be calculated

<table>
<thead>
<tr>
<th>Currency no.</th>
<th>Country</th>
<th>Series Type</th>
<th>Bonds Type</th>
<th>Long Term (qtrs.)</th>
<th>Available From</th>
</tr>
</thead>
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<td>1</td>
<td>USA</td>
<td>conventional bonds</td>
<td>coupon</td>
<td>80</td>
<td>1975Q2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>index-linked bonds</td>
<td>ZCBs</td>
<td>40</td>
<td>2003Q2</td>
</tr>
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<td></td>
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<td></td>
<td></td>
<td>1975Q2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>inflation rates</td>
<td></td>
<td></td>
<td>1975Q2</td>
</tr>
<tr>
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<td>UK</td>
<td>conventional zero-coupon bonds</td>
<td>ZCBs</td>
<td>64</td>
<td>1975Q2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>index-linked bonds</td>
<td>ZCBs</td>
<td>64</td>
<td>1985Q2</td>
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</tr>
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<td>coupon</td>
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<td></td>
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</tr>
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<td></td>
<td>inflation rates</td>
<td></td>
<td></td>
<td>1982Q2</td>
</tr>
</tbody>
</table>

In the light of the information in Table 1 it was decided to use various data sets for nominal returns and various data sets (not necessarily the same) for real returns. These data sets, comprising the periods, and the assets included in them, are shown in Table 2. In that table, ‘e, cb’ means equities and conventional bonds and ‘ilb’ means index-linked bonds.

Some of the periods in Table 2 are too short for the estimation of reliable parameters; they are included for the sake of inclusivity and to indicate how they may affect the results. On the other hand, it must be recognised that means and covariances may change over time, so the use of excessively long periods is inappropriate. However, long periods have been included for the purposes of illustration.

Table 2. Periods used

<table>
<thead>
<tr>
<th>Data set</th>
<th>Period</th>
<th>USA</th>
<th>UK</th>
<th>SA</th>
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<td></td>
<td></td>
<td>e, cb</td>
<td>e, cb</td>
<td>e, cb</td>
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</tr>
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<td>1975Q2</td>
<td>1985Q4</td>
<td>√</td>
<td>√</td>
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</table>
4.2 Results
For nominal and real returns, and for each period listed in Table 2, optimal portfolios were determined as explained in Section 3 and values of beta were calculated as follows:
- the unbiased sample betas based on equation (45) (i.e. the GCAPM betas); and
- the least-squares estimates of the betas based on equation (56).

The results of these calculations are set out in this section.

4.2.1 Optimal portfolios
Tables 4 and 5 show, for each dataset and for each investor currency \( c \), the proportion of the investor’s investments in assets issued in currency \( d \), both for the GCAPM and for the SFM-CAPM. Table 4 shows these results for nominal returns and Table 5 shows them for real returns. In these tables, exposures of less than 50% to the home currency have been highlighted in regular boldface (e.g. 0.24). Exposures of the major currencies of more than 25% to the minor currencies have been highlighted in italicised boldface (e.g. 0.34).
Table 4. Portfolio of each investor currency $c$ in each dataset by currency of issue: nominal returns

<table>
<thead>
<tr>
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<th>$d$</th>
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<th>UK</th>
<th>SA</th>
<th>TR</th>
<th>US</th>
<th>UK</th>
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<td>0.14</td>
<td>0.31</td>
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Table 5. Portfolio of each investor currency \( c \) in each dataset by currency of issue: real returns

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<tr>
<th>Dataset</th>
<th>GCAPM</th>
<th>SFM-CAPM</th>
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<tr>
<td>UK</td>
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</table>

In general the SFM-CAPM portfolios are quite similar to the GCAPM portfolios. For nominal returns, exceptions occur in datasets 3, 6 and 8. For real returns, exceptions occur in datasets 2 and 3.

For nominal returns, exposures of less than 50% to the home currency occur for all currencies other than South Africa. In general, such exposures occur under the SFM-CAPM in the same cases as under the GCAPM. Material exceptions occur in datasets 3 and 8 for USA investors. No such exposures occur for real returns. It should be recognised that the optimisation of the portfolio choice makes no allowance for home bias. It may also be noted that, for recent periods (viz, nominal datasets 4, 5 and 6) this effect is virtually eliminated. This may be at least partly due to the fact that, for those datasets, index-linked bonds were available, so that investors could hedge against inflation risks without investing abroad.

The sheer size of the major currencies makes it impossible that substantial proportions of their investments are in assets of minor currencies. It is for this reason that, as mentioned above, exposures of the major currencies of more than 25% to the minor currencies have been highlighted in italicised boldface. (In practice the exposure limits would be considerably lower than that; these levels have been selected merely to show the extremes.) Again, no such exposures occur for real returns. For nominal returns such exposures occur in most of the datasets. In general, such exposures occur under the SFM-CAPM in the same cases as under the GCAPM. The results of this research must be qualified by the understanding that they represented the results that would have obtained if the major-currency investors could have invested substantial proportions of their assets in minor currencies.

Tables 6 and 7 show, for each dataset and for each investor currency \( c \), the proportion of the investor’s investments in each of the asset types indicated. Table 6 shows these results for nominal returns and Table 7 shows them for real returns. In those tables, ‘s’ denotes the short bond and ‘l’ denotes the long bond. The number
shown in each cell is the proportion of a currency-$c$ investor’s assets held in asset $i$ in currency $d$.

Table 6. Portfolio of each investor currency $c$ in each dataset by asset type: nominal returns

<table>
<thead>
<tr>
<th>Dataset</th>
<th>GCAPM</th>
<th>SFM-CAPM</th>
</tr>
</thead>
<tbody>
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<td>$c$</td>
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<td>2</td>
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<tr>
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</tr>
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<td></td>
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<tr>
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<tr>
<td></td>
<td>SA</td>
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</tr>
<tr>
<td></td>
<td>TR</td>
<td>0.68</td>
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</table>
It may be noted from Tables 6 and 7 that the investment portfolios are quite conservative. It may be seen from Table 6 that, for nominal returns, the exposure to equities exceeds 50% only in dataset 1 (and in that instance only for USA investors). In most cases the exposure to equities is considerably less. Where index-linked bonds are included they attract substantial proportions of investors’ portfolios. Otherwise the bulk of the exposure is to conventional bonds. From Table 7 it may be seen that, for real returns, the exposure to equities never exceeds 25% for any of the datasets.

In comparing the portfolios for real returns with those for nominal returns it should be borne in mind that the periods under consideration are different. The period covered for real returns is from 2003 to 2012, which includes the periods covered by datasets 4, 5 and 6 for nominal returns. Other combinations of datasets are not comparable.

4.2.2 Beta

It was expected that the SFM-CAPM estimates of beta would be approximately equal to the GCAPM estimates, with some shift to represent the effects of the constraint and some noise. An upward shift would indicate that the sample betas are understated and a downward shift would indicate that they are overstated. The shifts indicate the corrections required to the sample betas by the underlying assumptions of the SFM-CAPM.

Figure 1 shows the relationship of the SFM-CAPM estimates of the betas to the GCAPM estimates for nominal returns for the data sets listed in Table 2, as enumerated in the legend. In that figure, every dot shows the values of these betas for a particular asset in a particular currency during a particular period. Where a particular asset does not appear in both the GCAPM and SFM-CAPM portfolios, no betas have been calculated. Also, for some investor currencies in some of the datasets, the expected return on the GCAPM market portfolio is less than the prospective return on the risk-free asset. Whereas it may happen that the ex-post market risk premium is negative, a
negative *ex-ante* market risk premium would imply that market participants are risk-seeking. Whilst this was corrected in the SFM-CAPM, no corresponding correction was made to the GCAPM. These cases are also excluded from the figure.

Figure 1. SFM-CAPM beta versus GCAPM beta: nominal returns

A number of outliers are evident in Figure 1, in particular for dataset 4. Figure 2 shows the relationship of the SFM-CAPM estimates of the betas to the GCAPM betas for nominal returns for dataset 4. In that figure the legend indicates the investor’s currency.
Here it is evident that the outliers relate to South African investors. For these investors, we have, under the GCAPM:

\[ \hat{\mu}_M = 0.0211; \quad \text{and} \quad \hat{\mu}_c = 0.0191; \]

so that the numerator in equation (56), i.e. the market risk premium, is only 0.002, giving large values of the GCAPM beta. In practice it may be expected that ex-ante market risk premiums would be greater than 0.2% a year.

Figure 3 shows the same results as Figure 1, but with dataset 4 excluded, and with consistent scaling. That figure shows that, for small values of beta, the SFM-CAPM beta is quite close to the GCAPM beta. The funnel of doubt widens as the value of beta increases. Particularly large values of GCAPM beta are generally reduced by the SFM-CAPM constraint, but even values less than 1 may require substantial adjustment to conform to the assumptions of the SFM-CAPM.
Figure 3. SFM-CAPM beta versus GCAPM beta: nominal returns excluding dataset 4

Figure 4 shows the relationship of the SFM-CAPM betas to the GCAPM betas for real returns for the data sets listed in Table 2.
In datasets 2 and 4, similar problems to those of the nominal dataset 4 occur, but here the problem currencies are the USA and the UK.

Figure 5 shows the relationship of the SFM-CAPM betas to the GCAPM betas for real returns for datasets 1, 3 and 5 with consistent scaling. For datasets 1 and 5, that figure shows similar effects to those of nominal returns. Dataset 3 shows two substantial reductions in the SFM-CAPM betas relative to the GCAPM betas. These relate to USA investors.
5. Conclusions

5.1 Summary

It is shown above that, for a single-factor CAPM to work in a multi-currency world, there is a necessary and sufficient condition. That condition applies to the \textit{ex-ante} variances and covariances of returns. The resulting SFM-CAPM model developed in this paper may be specified as:

\[
E \{ X_{di}^c \} = x_{c0} + \beta_{di,c}^c \left( E \{ X_M^c \} - x_{c0} \right);
\]

where:

- \( X_{di}^c \) is the return in currency \( c \) on asset \( i \) issued in currency \( d \) for \( c, d = 1, \ldots, C; i = 0,1, \ldots, n_d \);
- \( x_{c0} \) is the return on the risk-free asset denominated in currency \( c \);
- \( X_M^c \) is the return in currency \( c \) on the optimal market portfolio of a currency-\( c \) investor;

\[
\beta_{di,c}^c = \beta_{di}^c + \beta_c^c;
\]

\[
\beta_{di}^c = \frac{\sigma_{di,M}^c}{\sigma_{MM}};
\]

\[
\beta_c^c = \frac{\sigma_{c,M}^c}{\sigma_{MM}};
\]

\[
\sigma_{di,M}^c = \text{cov} \{ X_{di}^c, X_M^c \};
\]

Figure 5. SFM-CAPM beta versus GCAPM beta: real returns, datasets 1, 3 and 5
\[ \sigma_{c,M}^e = \text{var}\{X_c^e\}; \]
\[ \sigma_{c,M} = \text{cov}\{X_c, X_M^c\}; \]

\( X_c \) is the increase in the exchange rate of currency \( c \) relative to an arbitrarily chosen currency 1.

The constraint may be expressed as:
\[ \sigma_{c,M}^e + \sigma_{c,M} = a_c \sigma_{c,MM}^1; \]

where:
\[ a_c = \frac{\sigma_{c,MM}^e (\mu_M^e - x_{10})}{\sigma_{c,MM}^1 (\mu_M^c - x_{c0})}; \]

or alternatively, for all \( c \):
\[ \left( \sigma_{c,M}^e + \sigma_{c,M} \right) K_c = \kappa_{c,M}. \]

It was found that the application of this constraint in the maximum-likelihood estimation of the variance-covariance matrix of returns is difficult. The difficulties with that approach are explained and an alternative approach, using ordinary least squares, is developed.

The least-squares method was applied to the USA dollar, the UK pound, the South African rand and the Turkish lira. Returns were measured both as nominal forces of return and as real forces of return. The data sets used cover a selection of periods ranging from 1975 to 2012 for nominal returns and from 2003 to 2012 for real returns. The assets considered, to the extent that acceptable data were available, included, for each currency, two conventional bonds, two index-linked bonds and a comprehensive index of equities. The two bonds were a short-term (one-quarter) bond and a long-term bond. For nominal returns the short-term conventional bond was taken as the risk-free asset and for real returns the short-term index-linked bond was taken as such.

It was found that, for some datasets of nominal returns, the optimisation process did not produce credible portfolios. In some cases exposures to foreign currencies exceeded 50% and in some cases exposures of major-currency investors to minor currencies exceeded 25%. This is not a product of the SFM-CAPM itself. In fact the portfolios produced by the SFM-CAPM were generally quite similar to those produced by the GCAPM. However, it meant that the results of the application of both models to nominal returns should be qualified. For one dataset the value of the GCAPM market risk premium for South African investors was unrealistically low, giving relationships between GCAPM betas and SFM-CAPM betas that were not realistic. In applying the SFM-CAPM (as in the application of the GCAPM) care must be taken to ensure that ex-post market risk premiums are credible estimates of ex-ante values.

For the datasets used for real returns, the optimisation process produced more credible portfolios. However, for two of the datasets, similar problems arose to those that arose for nominal returns, except that here the problems arose for the major
currencies. For the remaining three datasets the SFM-CAPM produced relationships between GCAPM betas and SFM-CAPM betas that were generally acceptable.

Both for nominal and real returns the portfolios produced both by the GCAPM and by the SFM-CAPM were quite conservative relative to market capitalisation. Exposure to equities was generally low. This suggests that *ex-ante* expected returns on equities were higher than *ex-post* means.

Both for nominal and real returns it was found that, for small values of the GCAPM beta, the SFM-CAPM beta was quite close to the GCAPM beta. The funnel of doubt widens as the value of beta increases. Particularly large values of GCAPM beta are generally reduced by the SFM-CAPM constraint.

Mean–variance analysis in general, and the CAPM in particular, must relate to consumption and therefore to purchasing power. In principle, it would therefore be better to develop stochastic models for actuarial use in real terms, particularly if the liabilities being modelled are effectively so expressed, as in defined-benefit pension funds. The principal problem with the use of real returns is that the history of index-linked bonds—and therefore of risk-free rates—is shorter. Also, there is little trade and no primary market at the short end of the real yield curve, making it difficult to estimate real risk-free returns.

The findings of this paper give adequate grounds for the implementation of the SFM-CAPM. It is preferable to multi-factor models in that it does not treat currency risks as carrying different weight from investment risks; regardless of its source, risk is measured as variance in returns in the investor’s currency and weighted accordingly. It is preferable to the GCAPM in that the implied price of a security to an investor who measures returns in a particular currency is the same as the price to an investor who measures returns in another currency.

The paper suggests that, if this model is to be applied, it would be better to apply it to real returns than to nominal returns. Even for real returns, though, the market risk premiums produced by the estimation procedure used in this paper are not invariably realistic. This, however, is a problem not unique to the SFM-CAPM; other versions of the CAPM suffer equally from this problem. It arises because *ex-post* means are used as measures of *ex-ante* values.

Nevertheless, where the portfolios produced by the optimisation procedure are credible and the market risk premiums are realistic, the results of this paper show that the SFM-CAPM produces reasonable results. High values of GCAPM beta are particularly unreliable, but even values less than 1 may require substantial adjustment to conform to the assumptions of the SFM-CAPM.

5.2 Further research

In practice the stochastic modelling of the assets and liabilities of a long-term financial institution requires the use of time series in which the expected returns on assets and the expected forces of inflation, and perhaps their variances and covariances, may vary over time. This means that the application of the SFM-CAPM to such modelling will necessitate the use of the method in a time series. The variances and covariances, or the parameters governing autoregressive conditional
heteroscedasticity (ARCH) effects, subject to the constraint required for the use of the SFM-CAPM, will need to be estimated in that context. That is the subject of further research. Such research should demonstrate the development of equilibrium models for applications that are specific both to a particular country and to a particular type of financial institution, with allowance for investment in other currencies. Because the SFM-CAPM is an equilibrium model, it is well suited to such applications. The advantages of equilibrium models for such purposes is that they do not assume that the investor can outperform the market on a risk-adjusted basis, thus allowing market-consistency, and that, unlike more general no-arbitrage models, they do not assume complete markets, thus allowing for the fact that the liabilities of a financial institution cannot be replicated in the market.

As Wilkie (op. cit.) points out, the effects of different ways of measuring returns and of different ways of defining the risk-free rate are mathematically analogous to the effect of different currencies in which investors measure returns. Thus, for example, instead of distinguishing investors by the currencies in which they measure returns, we could distinguish them by the mean term of their liabilities (and therefore the term that defines their risk-free rate) or by the extent to which their liabilities are expressed in real or nominal terms. On the other hand, for such applications there is no divergence from purchasing-power parity. The application of the principles explored in this paper to the use of the CAPM for a domestic market with different time horizons and different definitions of the risk-free asset requires further research.

The testing of hypotheses regarding the SFM-CAPM is another field of research. For the purposes of such testing, however, it must be recognised that our tests should, in principle, be based on \textit{ex-ante} expectations, not necessarily \textit{ex-post} estimations of those expectations. The use of the rational-expectations hypothesis to argue that the latter is an unbiased estimate of the former is at best only asymptotically true. For this reason tests of the SFM-CAPM have been avoided in this paper.

In the first place it may be best to estimate investors’ \textit{ex-ante} expected returns in the various asset classes instead of using \textit{ex-post} estimations. This could be done by using actual market capitalisations to derive \textit{ex-ante} expected returns instead of deriving optimal portfolios from \textit{ex-post} sample means.

As far as variances and covariances are concerned, a further difficulty with the testing of the SFM-CAPM will be that, by virtue of the constraint, the estimates of the variances and covariances will be biased. In essence, the constrained estimates are estimates of \textit{ex-ante} variances and covariances, not \textit{ex-post} estimates. This means that there will be no basis for the rejection of hypotheses relating to these estimates. The SFM-CAPM therefore belongs to the world of normative modelling: if the financial institution accepts the assumptions on which it is based then she should accept the model derived from those assumptions. In the short run there is no basis for asserting that the model is descriptively valid. But if and as it becomes accepted, it will become descriptively valid. In the meantime, if it is found that the constrained estimates are significantly different from the unconstrained estimates then there are three possibilities, viz. that:

\begin{itemize}
  \item the rational expectations hypothesis is true and the SFM-CAPM is false; or
  \item the rational expectations hypothesis is false; or
\end{itemize}
there is a type-1 error.

Repeated tests may reduce the probability of a type-1 error, but they will still not reduce the probability that the rational expectations hypothesis is false. The suggestion that normatively valid behaviour will become descriptively valid if decision-makers subscribe to it is an alternative to the rational expectations hypothesis. This tendency will be reinforced by the fear that, if one does not follow normative rationality then it will be more difficult to explain one’s errors than if one does.

Instead of eliminating bias in *ex-ante* assumptions through the processing of *ex-post* information, here the decision-maker eliminates bias by adopting a rational basis for the determination of *ex-ante* assumptions, namely the constrained least-squares estimate proposed in this paper, even though those assumptions may be biased *ex post* (or, to challenge the dominant paradigm, even though the *ex-post* estimates are biased *ex ante*).

The complexities of the constrained maximum-likelihood method and the production of negative estimates of the variances of the returns on certain assets is not acceptable for practical application, nor is the production of correlation coefficients outside the range \([-1, 1]\). Further research is necessary to find ways of overcoming the problems associated with the constrained maximum-likelihood method. In the meantime, however, provided the optimisation process produces credible portfolios and the market risk premia are acceptable, the least-squares estimation method may be used for real returns. Thanks to the relative simplicity of the latter method, it may well prove more popular with practitioners, but that does not obviate the need for further research on the constrained maximum-likelihood method.

Whilst the principal interest of the authors is in the development of stochastic models for actuarial use, the SFM-CAPM clearly has wider application—for example in determining cost of capital. For such applications it is not necessarily envisaged that this model will replace other models, but subject to the results of the further research suggested here, there is no reason why the SFM-CAPM should not take its place alongside other models in informing subjective assessment by practitioners of the expected returns on assets in a multi-currency world.

**References**


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