Analytical Computation of Risk Measures for Variable Annuity Guaranteed Benefits

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Joint work with Hans W. Volkmer, University of Wisconsin-Milwaukee
Variable annuity guaranteed benefits

Commonly used risk measures

Analytical solutions to risk measures
Variable annuity guaranteed benefits

Commonly used risk measures

Analytical solutions to risk measures
Product design

- Arguably the most complex equity-based guarantee available to individual investors;
- Policyholders make contributions into subaccounts;
- For simplicity, we assume the dynamics of equity prices is driven by geometric Brownian motion.

\[ S_t = S_0 \exp(\mu t + \sigma B_t), \quad t \geq 0. \]
Product design

- The value of each account varies with the performance of the particular fund in which it invests:

\[ F_t = F_0 \frac{S_t}{S_0} e^{-mt}, \quad 0 \leq t \leq T, \]

where \( m \) is the annualized rate of total fees and charges.

- Without any guarantee, equity participation involves no risk to the variable annuity writer, who merely acts as a steward of the policyholders' funds.
In order to compete with mutual funds, nearly all major variable annuity writers start to offer various types of investment guarantees, which transfer certain financial risks to the insurers. (Liabilities)

- Guaranteed Minimum Maturity Benefit (GMMB)
- Guaranteed Minimum Death Benefit (GMDB)
- Guaranteed Minimum Withdrawal Benefit (GMWB)
- Guaranteed Minimum Accumulation Benefit (GMAB)
- Guaranteed Minimum Surrender Benefit (GMSB)
- Guaranteed Minimum Income Benefit (GMIB)

Nick-named as the GMxB series.
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Guaranteed Minimum Maturity Benefit (GMMB)

- Policyholder receives the greater of a minimum guarantee and account value. The GMMB writer is liable for the difference between the guarantee and account value, should the former exceeds the latter.
M&E charge, rider charge are made on a daily basis as a certain percentage of subaccount values. (Incomes)

\[ M_t = m_x F_t, \quad 0 \leq t \leq T, \]

where \( m_x \) is the charge allocated to fund GMxB rider. \((m_x < m)\)
Guaranteed Minimum Maturity Benefit (GMMB)

- Gross liability: \((T)\) is the maturity date

\[ e^{-rT}(G - F_T) + I(\tau_x > T), \]

where \(\tau_x\) is the future lifetime of policyholder aged \(x\) at issue. *(put-option-like payoff)*

- Net liability = Gross liability - Margin offset income

\[ L_0 = e^{-rT}(G - F_T) + I(\tau_x > T) - \int_0^{T \land \tau_x} e^{-rs} M_s ds. \]

*(exotic option?)* Mostly negative and rarely positive.

\(F\) investment fund value; \(G\) guaranteed benefit; \(T\) maturity date; \(r\) risk-free interest rate; \(M\) margin offset (fees).
Variable annuity guaranteed benefits

Commonly used risk measures

Analytical solutions to risk measures
Commonly used risk measures

- Quantile risk measure (Value-at-Risk)\
  \[ V_\alpha := \inf \{ y : \mathbb{P}[L_0 \leq y] \geq \alpha \}. \]
  The minimum capital required to ensure that there is sufficient fund to cover future liability with the probability of at least \( \alpha \).

- Conditional tail expectation (Expected Shortfall)\
  \[ \text{CTE}_\alpha := \mathbb{E}[L_0 | L_0 > V_\alpha]. \]
  Capital required to cover the liabilities exceeding the quantile measure with the probability of at most \( 1 - \alpha \).
Difficulties with Monte Carlo simulations

- 2008 SOA Report on Economic Capital of Life Insurance Companies

“76% of the respondents in the survey whose companies have more than $10 billion of annual revenue use a form of stochastic approach. In contrast, only 27% of the respondents whose companies have less than $1 billion use a form of stochastic approach.”

“Even the most resourceful companies are often forced to find a balance between accuracy, efficiency and timeliness of delivery.”
Variable annuity guaranteed benefits

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Analytical solutions to risk measures
  Applications of computational finance techniques
  Numerics: Simulations versus analytical Calculation
  Guaranteed minimum withdrawal benefit
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Numerics: Simulations versus analytical Calculation

Guaranteed minimum withdrawal benefit
GMMB net liability versus Asian option

- **Asian put option:**
  \[
  e^{-rT} \left( K - \frac{1}{T} \int_0^T S_t dt \right) = e^{-rT} \left( K - \frac{1}{T} \int_0^T S_0 e^{\mu t + \sigma B_t} dt \right)
  \]
  
  *K* - strike price, *S*ₜ - spot price at *t*, *T* - maturity

- **GMMB net liability:** (when \( \tau_x > T \))
  \[
  e^{-rT} \left( G - F_T \right)_+ - m e \int_0^T e^{-rt} F_t dt
  \]
  \[
  = e^{-rT} \left( G - F_0 e^{(\mu - m)T + \sigma B_T} \right)_+ - m e \int_0^T e^{-rt} F_0 e^{(\mu - m)t + \sigma B_t} dt
  \]

  *G* - minimum guarantee, *F*ₜ - account value at *t*, *T* - maturity, *r* - yield rate on assets, *m* - total fee
Exponential functionals of Brownian motion

- Pricing of Asian options by Yor (1993) led to the study of the joint distribution of the GBM and its integral;
- Linetsky (2004) produced spectral expansion of transition density function of the integral of the GBM;
- Vercer (2002) proposed a numerical PDE approach to price the Asian option;
- Vanduffel et al. (2008) used comonotonic approximation to find analytic bounds;
- We proposed four solution methods for the calculation of risk measures for GMMB.
  1. Numerical integration of Hartman-Watson density;
  2. Numerical inversion of Laplace transform;
  3. Spectral expansion of transition density;
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  1. Numerical integration of Hartman-Watson density;
  2. Numerical inversion of Laplace transform;
  3. Spectral expansion of transition density;
Method 1: Integral of Hartman-Watson density

Example: \( V_\alpha := \inf\{y : P[L_0 \leq y] \geq \alpha\} \).

The quantile risk measure \( V_\alpha \) for the net liability of GMMB is determined by

\[
P(T, V_\alpha) = \sqrt{\frac{2}{\pi^3 \sigma^2 T}} \exp \left( \frac{2\pi^2}{\sigma^2 T} - \frac{\nu^2 \sigma^2 T}{8} \right) \\
\times \int_0^\infty \exp \left( -\frac{2y^2}{\sigma^2 T} \right) \sinh y \sin \left( \frac{4\pi y}{\sigma^2 T} \right) \int_0^{\sqrt{B}} \frac{2\rho^\nu}{1+\rho^2+2\rho \cosh y} \exp \left( -\frac{A(1+\rho^2+2\rho \cosh y)}{2(B-\rho^2)} \right) d\rho dy,
\]

where \( \nu = 2(\mu - m - r)/\sigma^2 \), \( A = 4m_e/\sigma^2 \) and \( B = (e^{-rT} G - V_\alpha)/F_0 \).

▶ Computational difficulties:
  - Overflow problem;
  - Oscillating integrand.

Reminder: \( L_0 = e^{-rT} (G - F_T) + I(\tau_x > T) - \int_0^{T \wedge \tau_x} e^{-rs} m_e F_s ds \).
Method 1: Integral of Hartman-Watson density

Example: \( V_\alpha := \inf \{ y : P[L_0 \leq y] \geq \alpha \} \).

The quantile risk measure \( V_\alpha \) for the net liability of GMMB is determined by

\[
1 - \alpha = TP_x P(T, V_\alpha),
\]

and

\[
P(T, V_\alpha) = \sqrt{\frac{2}{\pi^3 \sigma^2 T}} \exp \left( \frac{2\pi^2}{\sigma^2 T} - \frac{\nu^2 \sigma^2 T}{8} \right)
\times \int_{0}^{\infty} \exp \left( -\frac{2y^2}{\sigma^2 T} \right) \sinh y \sin \left( \frac{4\pi y}{\sigma^2 T} \right) \int_{0}^{\sqrt{B}} \frac{2\rho^\nu}{1+\rho^2+2\rho \cosh y} \exp \left( -\frac{A(1+\rho^2+2\rho \cosh y)}{2(B-\rho^2)} \right) \, d\rho \, dy,
\]

where \( \nu = 2(\mu - m - r)/\sigma^2 \), \( A = 4m_e/\sigma^2 \) and \( B = (e^{-rT} G - V_\alpha)/F_0 \).

- **Computational difficulties:**
  - Overflow problem;
  - Oscillating integrand.

Reminder: \( L_0 = e^{-rT}(G - F_T) + I(\tau_x > T) - \int_{0}^{T \land \tau_x} e^{-rs} m e F_s \, ds \).
Inner integral

(a) $\sigma = 0.2$

(b) $\sigma = 0.8$
Method 3: Spectral expansion of transition density

- Guaranteed Minimum Death Benefit (GMDB)

\[ L_0 = e^{-r\tau_x} (G - F_{\tau_x}) + I(\tau_x < T) - \int_0^{T \wedge \tau_x} e^{-rs} m_d F_s ds. \]

Conditioning on the time of death, the distribution of the net liability is required to be evaluated at multiple time points. Hence, a more efficient algorithm is desired.

- Identity in distribution

\[ e^{-rT} F_T + \int_0^T e^{-rs} M_s \, ds \sim cX_t, \]

with a change of time scale and new constants \( c, \nu, \)

\[ dX_t = [(2\nu + 1)X_t + 1] \, dt + 2X_t \, dB_t. \]

- The right tail distribution of \( L_0 \) is then determined by the distribution of the diffusion process \( X \).
Variable annuity guaranteed benefits

Commonly used risk measures

Analytical solutions to risk measures
Applications of computational finance techniques

Numerics: Simulations versus analytical Calculation

Guaranteed minimum withdrawal benefit
Comparative study: GMMB

- 10-year GMMB with full fund of initial deposit $G/F_0 = 1$;
- Mean and standard deviation of log-returns per annum $\mu = 0.09, \sigma = 0.3$;
- Risk-free discount rate per annum $r = 0.04$;
- M&E charges and rider charges per annum $m = 0.01$;
- GMMB rider charge 35 basis points of account value $m_e = 0.0035$.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Direct integration</th>
<th>Inverse Laplace</th>
<th>Monte Carlo</th>
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<tr>
<td>$V_{95%}/F_0$</td>
<td>28.935%</td>
<td>28.935%</td>
<td>29.111%</td>
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<tr>
<td>Initial value</td>
<td>33%</td>
<td>(28%, 33%)</td>
<td>-</td>
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<td>Time (mins)</td>
<td>3.7916</td>
<td>3.54375</td>
<td>396.224</td>
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<td>$CTE_{95%}/F_0$</td>
<td>40.041%</td>
<td>40.042%</td>
<td>40.029%</td>
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<tr>
<td>Time (mins)</td>
<td>1.9325</td>
<td>0.28775</td>
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Comparative study: GMMB

The same valuation assumptions.

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<tr>
<th>Methods</th>
<th>Integration</th>
<th>Inverse Lap</th>
<th>Spectral</th>
<th>Green</th>
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<tr>
<td>$V_{90%}$</td>
<td>12.55036%</td>
<td>12.55036%</td>
<td>12.55035%</td>
<td>12.55036%</td>
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<tr>
<td>Initial</td>
<td>10%</td>
<td>(12%, 14%)</td>
<td>(12%, 14%)</td>
<td>(12%, 14%)</td>
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<tr>
<td>Time</td>
<td>3.674(mins)</td>
<td>5.032(mins)</td>
<td>51.579(secs)</td>
<td>0.172(secs)</td>
</tr>
<tr>
<td>CTE$_{90%}$</td>
<td>30.29643%</td>
<td>30.29648%</td>
<td>30.29643%</td>
<td>30.29648%</td>
</tr>
<tr>
<td>Time</td>
<td>1.867(mins)</td>
<td>0.285(mins)</td>
<td>2.953(secs)</td>
<td>0(secs)</td>
</tr>
</tbody>
</table>

**Tabelle**: A comparison of computational methods for the GMMB rider
Comparative study: GMDB

- 10-year GMDB with full fund of initial deposit $G/F_0 = 1$;
- Mean and standard deviation of log-returns per annum $\mu = 0.09, \sigma = 0.3$;
- Risk-free discount rate per annum $r = 0.04$;
- M&E charges and rider charges per annum $m = 0.01$;
- Rider charge $m_e = 0.0035$. Roll-up rate $\delta = 0.06$.(heavy tail)

<table>
<thead>
<tr>
<th>Methods</th>
<th>Invse Lap</th>
<th>Spectral</th>
<th>Green</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{90%}/F_0$</td>
<td>2.135314%</td>
<td>2.135314%</td>
<td>2.135314%</td>
</tr>
<tr>
<td>Initial values</td>
<td>(0%, 10%)</td>
<td>(0%, 10%)</td>
<td>(0%, 10%)</td>
</tr>
<tr>
<td>Time</td>
<td>77.784(mins)</td>
<td>27.034(mins)</td>
<td>4.953(secs)</td>
</tr>
<tr>
<td>$CTE_{90%}/F_0$</td>
<td>33.706290%</td>
<td>33.706287%</td>
<td>33.706292%</td>
</tr>
<tr>
<td>Time</td>
<td>11.964(mins)</td>
<td>3.780(mins)</td>
<td>0.828(secs)</td>
</tr>
</tbody>
</table>

Tabelle: A comparison of computational methods for the GMDB rider
Variable annuity guaranteed benefits

Commonly used risk measures

Analytical solutions to risk measures

Applications of computational finance techniques
Numerics: Simulations versus analytical Calculation

Guaranteed minimum withdrawal benefit
Guaranteed Minimum Withdrawal Benefit (GMWB)

- Contains no life insurance component.
- Provides a minimal payout based on the initial purchase payment.
- For example: A policyholder is guaranteed the ability to withdraw $7 per annum per $100 of initial investment until the original $100 has been fully exhausted.
Guaranteed Minimum Withdrawal Benefit (GMWB)
Guaranteed Minimum Withdrawal Benefit (GMWB)
Mathematical Formulation

- The dynamics of VA investment fund is driven by
  \[ dF_t = \left[ (\mu - m)F_t - w \right] dt + \sigma F_t \, dW_t, \quad F_0 = G > 0. \]
- \( \sigma W(t) \sim 2B_{\sigma^2 t/4} \), where \( B \) is an independent B.M.

Then the process \( Y \) satisfies

\[ dY(t) = \left[ 2(\nu+1)Y(t) - 1 \right] dt + 2Y(t) \, dB(t), \quad \text{where} \quad \nu := \frac{2\mu - \sigma^2}{\sigma^2}. \]

- It is easy to show by Itô’s formula that

\[ Y_t = \exp\{2(\nu t + B_t)\} \left( y - \int_0^t \exp\{-2(\nu + B_s)\} \, ds \right). \]
Mathematical Formulation

Pricing from an investor’s point of view (Milevsky and Salisbury (2006))

- Guaranteed income

\[ w \int_{0}^{T} e^{-rt} \, dt = \frac{w}{r} (1 - e^{-rT}). \]

- Investment income

\[ e^{-rT} Y_T I(\tau_0 > T), \quad \tau_0 := \inf\{ t : Y_t < 0 \}. \]

- The fair price \( m_w \) is determined by

\[ y = \frac{w}{r} (1 - e^{-rT}) + \mathbb{E}^Q [e^{-rT} Y_T I(\tau_0 > T)]. \]

\( w \) withdrawal rate; \( r \) risk-free interest rate; \( T \) guaranteed period; \( Y \) fund value; \( \tau_0 \) first time fund is exhausted.
Mathematical Formulation

- Pricing from an insurer’s point of view
  - Incoming cash flow (Assets)
    \[ m_w \int_0^{\tau_0 \wedge T} e^{-rt} Y_t \, dt. \]
  - Outgoing cash flow (Liabilities)
    \[ w \int_{\tau_0}^T e^{-rt} \, dt \cdot I(\tau_0 < T) = \frac{w}{r} (e^{-r\tau_0} - e^{-rT}) I(\tau_0 < T). \]
- The fair price \( m_w \) is determined by
  \[ \mathbb{E} \left[ m_w \int_0^{\tau_0 \wedge T} e^{-rt} Y_t \, dt \right] = \frac{w}{r} \mathbb{E} \left[ (e^{-r\tau_0} - e^{-rT}) I(\tau_0 < T) \right]. \]

\( w \) withdrawal rate; \( r \) risk-free interest rate; \( T \) guaranteed period; \( Y \) fund value; \( m_w \) fees to fund GMWB.
Numerical examples

- Risk-free interest rate $r = 0.05$;
- 100% charges are used to fund the GMWB. ($m = m_w$)

<table>
<thead>
<tr>
<th>$w/G$</th>
<th>$\sigma = 0.2$</th>
<th></th>
<th>$\sigma = 0.3$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Policyholder</td>
<td>Insurer</td>
<td>Policyholder</td>
<td>Insurer</td>
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<tr>
<td>0.05</td>
<td>29</td>
<td>25</td>
<td>77</td>
<td>68</td>
</tr>
<tr>
<td>0.06</td>
<td>41</td>
<td>35</td>
<td>104</td>
<td>90</td>
</tr>
<tr>
<td>0.07</td>
<td>54</td>
<td>45</td>
<td>132</td>
<td>112</td>
</tr>
<tr>
<td>0.08</td>
<td>68</td>
<td>56</td>
<td>162</td>
<td>134</td>
</tr>
<tr>
<td>0.09</td>
<td>82</td>
<td>67</td>
<td>192</td>
<td>156</td>
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</tbody>
</table>

Tabelle: Comparison of fair charges in basis points
Thank you for your attention!