Evaluation of Uncertainty Risk of The Limit Life by Brownian-Bridge Mortality Model

Noriaki Yokoo
Agenda

1. Limit of Life
2. The Brownian Bridge
3. The Brownian bridge Mortality Model
4. Lee-Carter Brownian Bridge Model
5. Application of the Japanese mortality
6. Evaluation the longevity risk
7. Summary
Limit of Life

- Life expectancy in Japan is increasing
- There are various opinions for projection of life expectancy
There are various opinions about the boundary of the limiting age in Japan.

Futoshi Ishii (Statistician)
Masaaki Shibuya
Nobutane Hanayama
(Statistician)

Limiting age has no boundary or has the boundary at very old age.
Limit of Life ③

In my head · · · · ·

Which limiting age has boundary or no influence the evaluation the longevity risk.

We should specify the effect of longevity risk by changing limiting age.

I evaluate the longevity risk by using the model which has limiting age for the parameter.

I construct the mortality model by using the Brownian bridge.
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7. Summery
The Brownian Bridge

The Brownian bridge is •••

• The stochastic process
• It attains some point at non-negative time T almost surely

The Brownian bridge from 0 to 0 on [0,T] is defined by

\[ X(t) = W(t) - \frac{t}{T} \cdot W(T) \]

\( W(t) \) : Wiener process on the probability space \((\Omega, F, P)\)
The Brownian Bridge ②  

But ···· \( X(t) \) is not adapted to the filtration \( F_t \)  
(If we use the Brownian bridge, we need future information (sample paths of Wiener process at time \( T \)). But it is impossible· · · ·.)  

We define the stochastic process \( Y(t) \) as follow;  
\[
Y(t) = \begin{cases} 
(T - t) \int_0^t \frac{1}{T-u} \, dW(u) & (0 \leq t \leq T) \\
0 & \text{otherwise}
\end{cases}
\]

- \( Y(t) \) has the same distribution of \( X(t) \)  
- \( Y(t) \) is adapted to the filtration \( F_t \)  

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The Brownian bridge Mortality Model

Let \( L_t(x,s) \) stochastic differential equation for the number of survivors as follow;

\[
dL_t(x,s) = -L_t(x,0) \cdot \mu_{x,t+s} ds + \sigma(x,s) \cdot L_t(x,s) \cdot dY(s)
\]

\( \mu_{x,t} \) : the force of mortality at entry age \( x \), calendar year \( t \)
\( \sigma(x,t) \) : the \( F(t) \) adapted
\( Y(t) \) : the Brownian bridge
By Itô formula, we can solve \( L_t(x, s) \) follow:

\[
L_t(x, s) = L_t(x, 0) \exp \left( \int_0^s \left( \mu_{x+u} - \frac{1}{2} \cdot \sigma(x, u)^2 \right) du + \int_0^s \sigma(x, u)dY(u) \right)
\]

\[
\frac{1}{2} \cdot \sigma(x, u)^2 \text{ in a mortality model is negligible generally. So we can approximate as follow;}
\]

\[
L_t(x, s) = L_t(x, 0) \exp \left( \int_0^s \mu_{x+u} du + \int_0^s \sigma(x, u)dY(u) \right)
\]

the mortality model defined by \( L_t(x, s) \)

the Brownian bridge mortality model (BBM model)
The Brownian bridge Mortality Model

some merits of

\[
L_t(x, s) = L_t(x, 0) \exp \left( \int_0^s \mu_{x+u} \, du + \int_0^s \sigma(x, u) \, dY(u) \right)
\]

Specify the effect by changing limiting age

Apply various mortality model

Relatively easy to use this model
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Lee-Carter Brownian Bridge Model ②

We define Lee-Carter Brownian Bridge Model as follow;

\[
L_t(x,s) = L_t(x,0) \exp \left( \int_0^s \mu_{x+u} \, du + \int_0^s \sigma(x,u) \, dY(u) \right)
\]

\[
\int_0^1 \mu_{x,t+s} \, ds = \alpha_x + \beta_x \cdot \kappa_t
\]

\(\alpha_x\) : log survivor rate of age \(x\)

\(\beta_x\) : parameter of age-specific response

\(\kappa_t\) : parameter of calendar year \(t\)
Why we use Lee-Carter model for drift coefficient?

- Fit Japanese mortality
- Comparability
- Easy model fitting
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### Application of the Japanese mortality ①

#### Assumption of parameter

<table>
<thead>
<tr>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Human Mortality Database</td>
</tr>
<tr>
<td>Age range 0 – 109</td>
</tr>
<tr>
<td>1995-2009 period for male and female total</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_x \quad \beta_x \quad \kappa_t$</td>
</tr>
</tbody>
</table>

- Set parameters by minimizing the following formula:

$$
\left( \sum_{x=0}^{\omega} \sum_{t=1}^{T} \left( \log \left( \frac{L_{t+1}(x,1)}{L_t(x,0)} \right) - \alpha_x - \beta_x \cdot \kappa_t \right) \right)^2
$$

- $\omega$ : limiting age  
- $T$ : measure period

- Assume every age  
- $\sigma(x,s)$ is denoted by

$$
\sigma(x) = \frac{1}{v_x} \sum_{t} \left( \log \left( \frac{L_{t+1}(x,1)}{L_t(x,0)} \right) - \alpha_x - \beta_x \cdot \kappa_t \right)^2
$$

- $v_x$ : number of data
**Application of the Japanese mortality ②**

Compare the mortality by Lee-Carter BBM with that by typical Lee-Carter

AIC (Akaike's information criterion)

<table>
<thead>
<tr>
<th>Mortality model</th>
<th>Lee-Carter BBM</th>
<th>Lee-Carter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum log likelihood</td>
<td>-13.39</td>
<td>-13.16</td>
</tr>
<tr>
<td>AIC</td>
<td>-1,467</td>
<td>-1,442</td>
</tr>
</tbody>
</table>

Japanese mortality calculated by models

<table>
<thead>
<tr>
<th>Age/model</th>
<th>Lee-Carter BBM</th>
<th>Lee-Carter</th>
<th>Mortality in 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.00243</td>
<td>0.00241</td>
<td>0.00233</td>
</tr>
<tr>
<td>10</td>
<td>0.00007</td>
<td>0.00007</td>
<td>0.00008</td>
</tr>
<tr>
<td>50</td>
<td>0.00260</td>
<td>0.00256</td>
<td>0.00254</td>
</tr>
<tr>
<td>60</td>
<td>0.00578</td>
<td>0.00573</td>
<td>0.00572</td>
</tr>
<tr>
<td>70</td>
<td>0.01281</td>
<td>0.01263</td>
<td>0.01177</td>
</tr>
<tr>
<td>80</td>
<td>0.03637</td>
<td>0.03610</td>
<td>0.03652</td>
</tr>
<tr>
<td>90</td>
<td>0.10964</td>
<td>0.10922</td>
<td>0.10709</td>
</tr>
<tr>
<td>100</td>
<td>0.27997</td>
<td>0.27919</td>
<td>0.27980</td>
</tr>
</tbody>
</table>
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Evaluation the longevity risk

Assumption ①

Definition of Longevity risk

- The capital requirement for longevity risk is defined as a result of longevity scenario as follows; (QIS5)
  \[ \text{Life}_{\text{long}} = (\Delta \text{NAV}|\text{longevity shock}) \]
- We define longevity risk as the capital requirement

\[ \Delta \text{NAV} \]

- The change in the net value of assets minus liabilities
- We define the \( \Delta \text{NAV} \) as a liabilities minus that in longevity shock scenario

Longevity shock

- a (permanent) 20 percentage decrease in mortality rates for each age

Solvency II

- 99.5 percentage probability, the unexpected losses on a 1-year time horizon (definition as Solvency II risk)

VaR99.5%
Evaluation the longevity risk ②

<table>
<thead>
<tr>
<th>Assumption ②</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>The number of policy</td>
<td>• 100,000</td>
</tr>
<tr>
<td>contract</td>
<td>• Single premium immediate annuity for life</td>
</tr>
<tr>
<td></td>
<td>• Annuity payment 100,000 yen per 1 year</td>
</tr>
<tr>
<td>other condition</td>
<td>• No other risk exist</td>
</tr>
</tbody>
</table>
**Evaluation the longevity risk**

- We evaluate longevity risk with Lee-Carter Brownian bridge model
- The value of longevity risk in at every entry age (million yen)

<table>
<thead>
<tr>
<th>entry age</th>
<th>limiting age</th>
<th>120</th>
<th>130</th>
<th>140</th>
<th>∞</th>
<th>Liability</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>Solvency II</td>
<td>10,093</td>
<td>10,093</td>
<td>10,093</td>
<td>10,093</td>
<td>205,473</td>
</tr>
<tr>
<td></td>
<td>VaR99.5%</td>
<td>8,204</td>
<td>8,362</td>
<td>8,483</td>
<td>9,652</td>
<td></td>
</tr>
<tr>
<td></td>
<td>VaR99.5%</td>
<td>7,473</td>
<td>7,623</td>
<td>7,734</td>
<td>8,913</td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>Solvency II</td>
<td>12,183</td>
<td>12,183</td>
<td>12,183</td>
<td>12,183</td>
<td>162,917</td>
</tr>
<tr>
<td></td>
<td>VaR99.5%</td>
<td>5,482</td>
<td>5,591</td>
<td>5,667</td>
<td>6,705</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>Solvency II</td>
<td>11,634</td>
<td>11,634</td>
<td>11,634</td>
<td>11,634</td>
<td>106,153</td>
</tr>
<tr>
<td></td>
<td>VaR99.5%</td>
<td>3,416</td>
<td>3,484</td>
<td>3,528</td>
<td>4,459</td>
<td></td>
</tr>
</tbody>
</table>

- Longevity risk evaluated by VaR99.5% is less than that by solvency II
- Longevity risk evaluated by VaR99.5% increases as limiting age increases, but longevity risk evaluated by Solvency II does not increase
Evaluation the longevity risk

- The stress of mortality around 75-90 year old there are many number of death at influence the quantity of longevity risk
- The stress of mortality evaluate by VaR99.5% is change as limiting age changes
- The stress of mortality evaluate by Solvency II is constant regardless of limiting age

The number of death in the 100 thousand policyholders entry age 50

Mortality ratio for Solvency II and that for Lee-Carter Brownian bridge model at entry age 50 limiting age 120
### Evaluation the longevity risk ⑤

- In Solvency II, the change of limiting age causes longevity risk only a little because of the stress of mortality is constant
- In VaR99.5%, 10 year increasing of limiting age brings about 2~3% increase of longevity risk
- The change of limiting age to infinity brings too much increase of longevity risk.

<table>
<thead>
<tr>
<th>entry age</th>
<th>limiting age</th>
<th>longevity risk Limiting age 120</th>
<th>110 → 120</th>
<th>120 → 130</th>
<th>130 → 140</th>
<th>140 → ∞</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>50</strong></td>
<td>Solvency II</td>
<td>10,093</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>VaR99.5%</td>
<td>8,204</td>
<td>216</td>
<td>158</td>
<td>121</td>
<td>1,169</td>
</tr>
<tr>
<td><strong>60</strong></td>
<td>Solvency II</td>
<td>11,465</td>
<td>9</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>VaR99.5%</td>
<td>7,473</td>
<td>216</td>
<td>151</td>
<td>111</td>
<td>1,179</td>
</tr>
<tr>
<td><strong>70</strong></td>
<td>Solvency II</td>
<td>12,183</td>
<td>13</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>VaR99.5%</td>
<td>5,482</td>
<td>169</td>
<td>109</td>
<td>77</td>
<td>1,038</td>
</tr>
<tr>
<td><strong>80</strong></td>
<td>Solvency II</td>
<td>11,634</td>
<td>21</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>VaR99.5%</td>
<td>3,416</td>
<td>119</td>
<td>68</td>
<td>45</td>
<td>931</td>
</tr>
</tbody>
</table>

The amount of changes of longevity risk (million yen)
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Summary

・The longevity risk evaluated by LC-BBM model is less than that by the definition proposed in Solvency Ⅱ (the stress over 75 is too high)

・This model is useful for evaluation the change of longevity risk as limiting age changes

・We hope that this leads to elaboration of the longevity risk evaluation and the development of ERM
Thank you for your fine attention!
I’m sorry ·····, I can’t answer your questions this time, because of my poor English (especially listening). If you have questions about my presentation, please send e-mail to me.

Email : no-yokoo@meijiyasuda.co.jp