

FAST CHANGE DETECTION ON PROPORTIONAL TWO-POPULATION HAZARD RATES

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LIFE Colloquium
June 24-26th, 2013

joint work with N. El Karoui, S. Loisel and C. Mazza

MOTIVATION

- Mainly concerned with mortality evolution which may change over a projection period
- Mortality profile/level/trend
 - Heterogeneity and other factors
 - Underwriting strategy
 - Adverse selection
- Interest in experienced mortality
 - Deaths can be observed sequentially
- Model risk, parameter uncertainty
 - Long-term projection
- Monitoring and surveillance of mortality dynamics
 - Sequential information on death occurrences
 - Updating mortality assumptions

MATHEMATICAL SETTINGS

We consider a portfolio of insured population:

- Let $N = (N_t)_{t \geq 0}$ be a **counting process** indicating the deaths of policyholders and $\lambda = (\lambda_t)_{t \geq 0}$ its **intensity**.
- The counting process N_t , is **available sequentially** through the filtration $\mathcal{F}_t = \sigma\{N_s, 0 < s \leq t\}$.
- We suppose that the insurance company relies on a **Cox-like** model to project her own experienced mortality:

$$\lambda_t = \underline{\rho} \lambda_t^0,$$

- λ_t^0 is a deterministic **reference intensity** (e.g. *best estimate* national mortality)
- $\underline{\rho}$ is a positive constant

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Model risk/parameter uncertainty: **Change-point**

$$\lambda_t = \mathbf{1}_{\{t < \theta\}} \underline{\rho} \lambda_t^0 + \mathbf{1}_{\{t \geq \theta\}} \bar{\rho} \lambda_t^0.$$

Without loss of generality we can assume that $\underline{\rho} = 1$ and let $\rho = \bar{\rho} > 1$.

PROBABILISTIC FORMULATION

Let \mathbb{P}_θ (resp. $\mathbb{E}_\theta[\cdot]$) be the probability measure (resp. expectation) induced when the change takes place at time θ

Example

- For $\theta = 0$, the process is *out-of-control*
- For $\theta = \infty$, the process is *in-control*

Detect the change-point θ as quick as possible while avoiding false alarms

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OPTIMALITY CRITERIA, LORDEN (1971)-LIKE

- The detection delay $\mathbb{E}_\theta \left[(N_\tau - N_\theta)^+ \mid \mathcal{F}_\theta \right]$
- The frequency of false alarm $\mathbb{E}_\infty [N_\tau]$

OPTIMIZATION PROBLEM

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Find τ^* such that $C(\tau^*) = \inf_{\tau} \left(\sup_{\theta \in [0, \infty]} \text{ess sup } \mathbb{E}_{\theta} \left[(N_{\tau} - N_{\theta})^+ \mid \mathcal{F}_{\theta} \right] \right)$
subject to $\mathbb{E}_{\infty} [N_{\tau}] = \omega$.

ASSUMPTION

1 $N_{\infty} = \infty$ $\mathbb{P}_{\infty}, \mathbb{P}_0$ -a.s.

OPTIMALITY OF THE CUSUM PROCEDURE (1/7)

Let the Radon-Nikodym density of \mathbb{P}_0 with respect to \mathbb{P}_∞ be defined as

$$\frac{d\mathbb{P}_0}{d\mathbb{P}_\infty} \Big|_{\mathcal{F}_t} = \exp U_t,$$

where $U_t = \log(\rho)N_t + (1 - \rho) \int_0^t \lambda_s ds$ is the log-likelihood ratio of \mathbb{P}_∞ with respect to \mathbb{P}_0 .

Let $V(x)$ be the CUSUM process; with head-start $0 \leq x < m$; defined as

$$V_t(x) = U_t - (-x) \wedge \underline{U}_t \tag{1}$$

where \underline{U}_t is the running infimum of U , i.e. $\underline{U}_t = \inf_{s \leq t} U_s$.

Let $\tau_m(x)$ be the first hitting time of $V(x)$ of the barrier m , i.e.

$$\tau_m(x) = \inf\{t \geq 0, V_t(x) \geq m\}.$$

Theorem

If $\mathbb{E}_\infty[N_{\tau_m(0)}] = \omega$ then $\tau_m(0)$ is optimal, i.e. $\inf_\tau C(\tau) = C(\tau_m(0))$

OPTIMALITY OF THE CUSUM PROCEDURE (2/7)

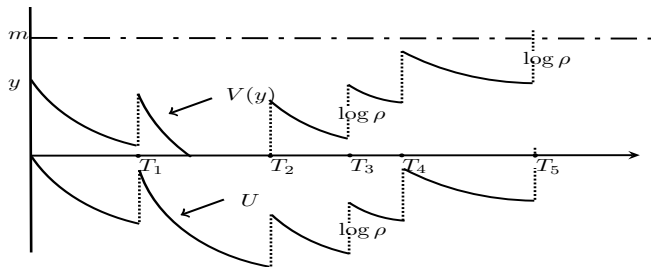
The process $V(x)$ can be rewritten in the following form

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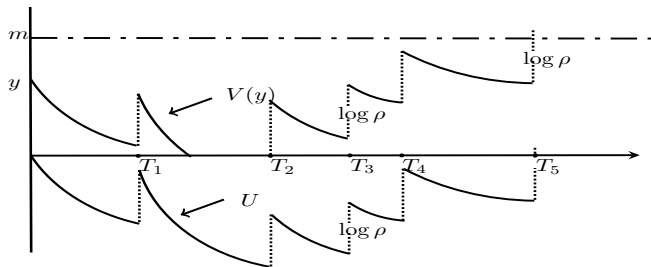
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OPTIMALITY OF THE CUSUM PROCEDURE (2/7)

The process $V(x)$ can be rewritten in the following form

$$V_t(x) = x + U_t + (-U_t - x)^+ = x + \log(\rho)N_t + (1 - \rho) \int_0^t \lambda_s^0 \mathbf{1}_{\{V_s(x) > 0\}} ds$$



OPTIMALITY OF THE CUSUM PROCEDURE (3/7)

CONNECTION TO RUIN THEORY

Observe that

- $m - V_t(x)$ is nothing but the surplus process, with
 - initial wealth $m - x$
 - premium rate $(1 - \rho)\lambda_t$
 - constant size of claims $\log \rho$
- $\tau_m(x)$ is the ruin time

OPTIMALITY OF THE CUSUM PROCEDURE (4/7)

Lemma

$\tau_m(x)$ is \mathbb{P}_0 and \mathbb{P}_∞ -a.s. finite.

Lemma

The detection delay for the stopping time $\tau_m(0)$ is given by

$$\mathbb{E}_\theta[(N_{\tau_m(0)} - N_\theta)^+ | \mathcal{F}_\theta] = \mathbb{E}_0[N_{\tau_m(0)} | x = V_\theta(0)].$$

Thus the worst case detection delay is given by

$$\sup_{\theta \in [0, \theta]} \text{ess sup } \mathbb{E}_\theta[(N_{\tau_m(0)} - N_\theta)^+ | \mathcal{F}_\theta] = \mathbb{E}_0[N_{\tau_m(0)} | x = 0]$$

OPTIMALITY OF THE CUSUM PROCEDURE (5/7)

The performance of the detection only depends on the functions:

$$h_0(x) = \mathbb{E}_0[N_{\tau_m(x)} | V_0 = x] \text{ and } h_\infty(x) = \mathbb{E}_\infty[N_{\tau_m(x)} | V_0 = x].$$

Theorem

- 1 h_0 and h_∞ are decreasing in x and uniformly bounded on $[0, m]$
- 2 h_0 and h_∞ are continuously differentiable and satisfy the following backward delay differential equations (d.d.e.), $0 \leq x < m$,

$$h'_0(x) = \frac{1}{1-\rho} \left(\rho(h_0(x) - h_0(x + \log \rho)) - 1 \right),$$
$$h'_\infty(x) = \frac{1}{1-\rho} \left(h_\infty(x) - h_\infty(x + \log \rho) - 1 \right).$$

OPTIMALITY OF THE CUSUM PROCEDURE (6/7)

Theorem

Let τ be an \mathbb{F} -stopping time, then

$$C(\tau) \geq \rho \frac{\mathbb{E}_\infty \left[\int_0^\tau \exp(V_{t-}) dN_t \right]}{\mathbb{E}_\infty [\exp(V_\tau)]}$$

Proof.

Follows the same reasoning as Moustakides (2004) and Shiryaev (1998). \square

Finally, to show the optimality we need to show that for each stopping τ satisfying the false alarm constraint such that

$\mathbb{E}_\infty [N_\tau] \geq \mathbb{E}_\infty [N_{\tau_m(0)}] = h_\infty(0)$ we have

$$\rho \frac{\mathbb{E}_\infty \left[\int_0^\tau \exp(V_{t-}) dN_t \right]}{\mathbb{E}_\infty [\exp(V_\tau)]} \geq h_0(0)$$

OPTIMALITY OF THE CUSUM PROCEDURE (7/7)

Applying Itô formula yields:

$$\begin{aligned} \mathbb{E}_\infty \left[\rho \int_0^\tau \exp(V_{t-}) dN_t - \exp(V_\tau) h_0(0) \right] &= \mathbb{E}_\infty \left[\rho \underbrace{\left(\mathbb{E}_\infty[N_\tau] - h_\infty(0) \right)}_{\geq 0} \right] \\ &\quad + \mathbb{E}_\infty \left[\left(\rho h_\infty(V_\tau) - \exp(V_\tau) h_0(V_\tau) \right) \right] \end{aligned}$$

Thus to show the optimality we only need the following lemma.

Lemma

For any stopping time τ we have

$$\mathbb{E}_\infty \left[\rho h_\infty(V_\tau) - \exp(V_\tau) h_0(V_\tau) \right] \geq 0$$

DETECTION PROCEDURE – ALGORITHM

- Step 1:** Fix the input parameters: The post-change intensity through the specification of ρ and the false alarm constraint ω .
- Step 2:** Determine the threshold m as the solution of the equation $\mathbb{E}_\infty[N_{\tau_m}] = \omega$.
- Step 3:** For each new observation at time t compute the value of the CUSUM process V given by the iterative relation $V_{t+1} = (V_{t-1} + U_t)^+$.
- Step 4:** Compare the current value of V to the threshold m and stop the procedure once $V_t \geq m$ and sound an alarm. Hence $\tau_m(0) = t$.

DETECTION PROCEDURE – REAL WORLD (1/2)

We consider the CONTINUOUS MORTALITY INVESTIGATION assured lives dataset and England & Wales national population. We split data into two periods:

- We consider the period 1947-1969 as a training period.
- The Cox model is estimated over this period using the MLE.

Hence we monitor sequentially the dataset over the period 1970-2005 and look for changes on the mortality of assured lives.

DETECTION PROCEDURE – REAL WORLD (2/2)

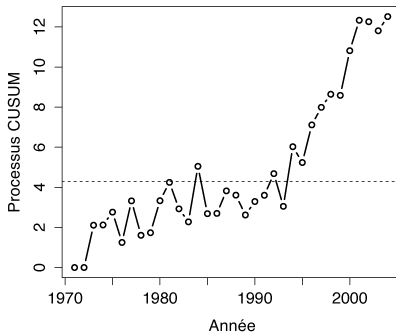
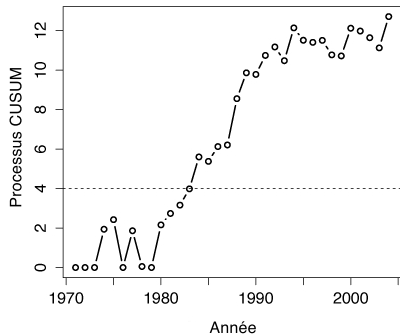


Figure: Detection scheme for age groups 50 – 59 (right) and 80 – 89 (left). The post-change is set to $\rho = 1.15\%$ and the false alarm constraint to $\omega = 100 \times \bar{\lambda}$.

CONCLUSION

- The CUSUM detection rule is optimal in the case of non-homogeneous Poisson process with a modified Lorden criterion

- Applications in non-life insurance
- Further research on possible extensions