Managing Capital Market and Longevity Risks in a Defined Benefit Pension Plan

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ABSTRACT

This paper proposes a model for a defined benefit pension plan to optimize the mean-variance tradeoff of its funding status while controlling expected total pension cost and managing longevity risk. With this setup, we first investigate the plan’s optimal contribution and asset allocation strategies, given the projection of stochastic asset returns and random mortality evolutions. We then show the sensitivity of the plan’s funding status to mortality improvement. To manage longevity risk, the plan can use either the ground-up hedging strategy or the excess-risk hedging strategy. Our numerical examples demonstrate that the plan transfers more longevity risk with the excess-risk strategy due to its lower total hedging cost and more attractive structure.

Keywords: defined benefit pension plan, funding, asset allocation, contribution, longevity risk hedging.

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Operating defined benefit (DB) pension plans involves risks: no matter what happens in financial markets and how long employees live after retirement, DB plans are responsible for keeping the plans fully funded to provide the employees guaranteed retirement income security. The inherent volatility of the capital market and longevity risk force sponsors of DB pension plans to consider risk management strategies.

DB pension plans can reduce risk by pursuing a strategy with stable contributions (Lee, 1986). In this light, Haberman et al. (2000) investigate an optimization problem with an objective to minimize the variance of periodic contributions. To cope with the risk, the sponsors can also adjust their asset allocation (Black, 1989, Bodie, 1991, Haberman et al., 2000, Colombo and Haberman, 2005, Maurer et al., 2009, Lucas and Zeldes, 2009). A group of researchers is in favor of holding a 100% bond portfolio to minimize pension risk (Black, 1980, Gold and Hudson, 2003). The most prominent example of a company that follows this advice is the Boots, a U.K. pharmaceutical retailer, which shifted all of its pension assets to bonds between the spring of 2000 and July 2001.¹ Yet, given that there exists a tradeoff between contribution and the level of risk a DB scheme is willing to take (Colombo and Haberman, 2005), this practice may raise total pension costs as more contributions are needed to meet retirement benefit liabilities. Total pension cost is all of the costs and penalties associated with normal contributions, supplementary contributions and withdrawals (Maurer et al., 2009). By contrast, some others advocate investing heavily in higher yielding but riskier equities, reasoning that stocks are expected to earn higher returns than bonds over the long haul and help to reduce under-funding over time. Still, there is a downside of this strategy. That is, equity investment may increase funding variation and lead to a higher insolvency risk, i.e., the risk that the plan fails to maintain adequate funding levels to fulfill its obligation. Thus, pension investment strategy should take into account both solvency risk and total pension costs.

To derive the optimal proportion invested in risky assets, Maurer et al. (2009) minimize the variance of plan contributions subject to a total pension cost constraint but they do not explicitly control the plan’s solvency risk. Delong et al. (2008), on the other hand, investigate the optimal investment

¹However, the plan partially shifted back to equity investment in 2005.
strategy by minimizing the funding variation. While they include supplementary contribution in the generalized optimization problem, they do not directly study the effect of total pension cost on pension asset allocation. Josa-Fombellida and Rincón-Zapatero (2004) examine the pension asset allocation with the aim of minimizing solvency risk and contribution rate risk, along the lines of Haberman and Sung (1994), Haberman (1997) and Josa-Fombellida and Rincón-Zapatero (2001). They conclude that the plan will invest more in risky assets (e.g. stocks) when the funding status deteriorates. Bogentoft et al. (2001) model a multi-period problem to adjust contributions and investments at each decision point by minimizing costs of the fund subject to a conditional value at risk constraint on under funding. Based on a pension fund dataset from the Netherlands, they suggest investing, on average, 84% of the pension fund in bonds and the remainder in equities. Josa-Fombellida and Rincón-Zapatero (2004) and Bogentoft et al. (2001) do not directly control total pension cost subject to the plan’s financial constraint, as Delong et al. (2008) do. Nevertheless, considering total pension cost is important since it includes all costs a plan incurs during a period of interest. The budget constraint creates a strong incentive for the plan to specify a target total pension cost.

To extend the previous analysis, not only do we consider a traditional mean-variance problem in the pension asset-liability management setting, but we also impose a constraint to control the expected total pension cost with various stochastic invested assets and dynamic mortality rates. We consider a financial market with a low-risk asset driven by a Brownian motion and each risky asset driven by a Lévy process. The dynamic mortality rates are described by the Lee and Carter (1992) model. Our model is straightforward and easy to use. It allows us to derive a recursive formula for the funding level and use simulated data to realize optimization.

Longevity risk has been recognized as a major threat to pension sponsors. Delong et al. (2008) conclude that pension plans with higher longevity risk are subject to higher risk. To meet solvency requirements, the plans have to invest more in riskier assets in order to accumulate sufficient funds to pay the claim. By controlling the total pension cost, our paper adds new evidence on longevity effects to the literature. Specifically, we analyze how the optimal normal contribution and the proportion invested in risky assets will change with the different levels of mortality improvement.
Our numerical results show that, given a total pension cost constraint, higher life expectancy of pensioners leads to a lower optimal normal contribution rate and more investment in risky assets. In our example when the common risk parameter in the Lee and Carter (1992) model decreases from $-0.20$ to $-0.40$ (a lower common risk parameter implies a higher level of mortality improvement), the funding variation, defined as the expected squared difference between pension assets and liabilities, rises by 78.5%.

To ensure the long-term financial health of the plan, sponsors must find ways to mitigate longevity risk. The topic of longevity risk management of pension plans has attracted a great deal of attention in academia and industry (Lin and Cox, 2005, Blake et al., 2006, Cairns et al., 2006, Cox and Lin, 2007, Sherris and Wills, 2008, Lin and Cox, 2008, Brcic and Brisebois, 2010, Wills and Sherris, 2010). These authors discuss possible longevity risk management solutions involving plan design, annuity purchase and longevity securities. We take this discussion one step further. Instead of just comparing and proposing different longevity hedging tools, we study how much longevity risk a plan should transfer. In particular, we add a longevity hedging decision to our pension optimization problem. Given this setup, we solve for the plan’s optimal hedge ratio.

Specifically, we look for optimal solutions for two longevity risk hedging strategies: the ground-up hedging strategy and the excess-risk hedging strategy. The ground-up hedging strategy transfers a proportion of all future retirement payments. A prominent example of the ground-up strategy is the EIB longevity bond offered in 2004. The excess-risk strategy is to cede the part of longevity risk that exceeds a given level. The 2008 survivor swap between Canada Life and owners of a portfolio of insurance-linked securities as well as other investors is based on the same idea. Compared with the excess-risk hedging strategy, the ground-up strategy is more expensive and capital intensive. Our optimization results show that there exists a negative relation between hedging ratio and longevity hedging cost. Therefore the plan tends to hedge much less with the ground-up hedging strategy. Our results offer a new explanation for the failure of the EIB bond. The approach is unique in that it describes how longevity risk management strategy affects the magnitude of the plan’s hedging costs, which in turn determine its hedging level, whereas earlier research on pension asset-liability optimization does not explicitly recognize longevity risk management.
The paper is organized as follows. Section 2 describes the pension fund optimization model. We provide a numerical example to illustrate how to implement our model for a DB pension plan with a single cohort of employees during the accumulation phase. Then we illustrate how longevity risk changes the optimal normal contribution and asset allocation. To hedge longevity risk, we examine the ground-up longevity hedging strategy in Section 3 and solve for the optimal hedging ratio. The excess-risk strategy has a lower cost than the ground-up hedging strategy. Section 4 shows that the plan tends to hedge more with the excess-risk longevity hedging strategy. The results from Sections 3 and 4 provide important insights into longevity securitization, which is discussed in Section 5. The last section concludes the paper.

2. PENSION FUND OPTIMIZATION

2.1. Basic framework. Here we look for an optimal pension asset allocation and contribution strategy during the accumulation phase for a DB plan. In our setting, all members join the plan at the age of $x_0$ at time 0 and retire at the age of $x$ at time $T$. We are using years as time periods (i.e., period = year) because it is common in practice. One can adjust our model for other periods if needed. Assume the cohort is stable across the entire accumulation phase. That is, every member who withdraws is replaced by a one at the same age.

To be more specific, suppose the plan starts with an accumulated fund $PA_0 = M$ at time 0, which is invested in different assets. The value of the accumulated fund $PA_t$ at time $t$ depends on the amount invested in asset $i$ at time $t-1$, $A_{i,t-1}$, and its return in period $t$, $r_{i,t}$:

$$PA_t = \sum_{i=1}^{n} A_{i,t-1}(1 + r_{i,t}) \quad i = 1, 2, \ldots, n; t = 1, 2, \ldots, T.$$  
(1)

Regulations require the following balance equation to hold:

$$\sum_{i=1}^{n} A_{i,t} = PA_t + C + k \cdot UL_t \quad t = 1, 2, \ldots, T.$$  
(2)

That is, the sum of all investments at $t$, $\sum_{i=1}^{n} A_{i,t}$, equals the accumulated fund $PA_t$ plus a constant normal contribution $C$ (to be determined by optimization in later sections) and a supplemental contribution $k \cdot UL_t$. The unfunded liability, $UL_t$ is defined below. We assume that regulations
allow the plan to amortize the unfunded liability over \( m > 1 \) periods at the plan’s periodic discount rate \( \rho \). This approach is common; see, for example, Maurer et al. (2009)\(^2\). Therefore, the constant \( k \) is the pension amortization factor, \( k = 1/\ddot{a}_m^\varpi \), where

\[
\ddot{a}_m^\varpi = \sum_{i=0}^{m-1} \frac{1}{(1 + \rho)^i}.
\] (3)

The plan’s unfunded liability is the present value of future benefit obligations (or liabilities), \( PBO_t \) minus the current value of assets. That is, the unfunded liability in (2) is

\[
UL_t = PBO_t - PA_t - C
\]

\[
= PBO_t - \left[ \sum_{i=1}^{n} A_{i,t} - (C + k \cdot UL_t) \right] - C
\]

\[
= PBO_t - \sum_{i=1}^{n} A_{i,t} + k \cdot UL_t \quad t = 1, 2, \ldots, T.
\] (4)

The plan’s benefit liability at time \( t \), \( PBO_t \), is defined as the discounted expected value of future benefit obligations,

\[
PBO_t = \frac{Ba(x(T))}{(1 + \rho)^{T-t}} \quad t = 1, 2, \ldots, T.
\] (5)

The constant \( B \) is the promised annual survival payment after the plan participants reach retirement age \( x \) at time \( T \). It depends on the number of pensionable service years accrued until \( T \) and projected salaries before retirement. In terms of the future curtate lifetime \( K(x) \) at age \( x \) the present value of benefits of 1 per year is an annuity for \( K(x) \) years:

\[
a_{K(x)} = \begin{cases} 
v^1 + v^2 + \cdots + v^{K(x)} & \text{if } K(x) \geq 1 \\ 0 & \text{if } K(x) = 0 \end{cases}
\] (6)

where \( v = 1/(1 + r) \) is the discount factor with the discount rate \( r \).

The probability that a plan member age \( x \) at time \( T \) survives to age \( x + s \) at the beginning of year \( T + s \) (and gets a benefit payment) given the mortality table at time \( T \) is denoted \( s\bar{p}_{x,T} \). These are random variables for \( s = 1, 2, \ldots \) which we simulate later. This allows us to compute the

\(^2\)Following Maurer et al. (2009), we assume pension shortfall has the same amortization period as surplus.
conditional expected value of life annuity in (6) as

$$a(x(T)) = E\left[ a \frac{1}{K(x)} \mid \tilde{p}_{x,T}, 2\tilde{p}_{x,T}, \ldots \right]$$

$$= \sum_{s=1}^{\infty} v_s^s s\tilde{p}_{x,T}. \quad (7)$$

That is, the life annuity factor for age $x$ at retirement $T$, $a(x(T))$, is the discounted conditional expected value of payments of 1 per year as long as the retiree survives. This expected value depends on future survival rates $s\tilde{p}_{x,T}$, which are random variables viewed from time 0 because we are considering the mortality table to be random whereas in practice it is an estimate of future rates.

From (4), we can obtain

$$UL_t = \frac{1}{1 - k} \left( PBO_t - \sum_{i=1}^{n} A_{i,t} \right) \quad t = 1, 2, \ldots, T. \quad (8)$$

Equations (1) and (2) imply that the total fund available for investment at time $t$, $\sum_{i=1}^{n} A_{i,t}$ in (8), can be expressed as

$$\sum_{i=1}^{n} A_{i,t} = \sum_{i=1}^{n} A_{i,t-1}(1 + r_{i,t}) + C + k \left[ PBO_t - \sum_{i=1}^{n} A_{i,t-1}(1 + r_{i,t}) - C \right]$$

$$= (1 - k) \sum_{i=1}^{n} A_{i,t-1}(1 + r_{i,t}) + (1 - k)C + k \cdot PBO_t. \quad (9)$$

Assume the plan sponsor invests a proportion $w_i$ of the initial accumulated fund $PA_0$ and the latter contributions $C + k \cdot UL_t$ in asset $i$, $i = 1, 2, \ldots, n$. The value of asset $i$ at time $t$ in the
portfolio (9) can be solved recursively:

\[
A_{i,t} = (1 - k)A_{i,t-1}(1 + r_{i,t}) + (1 - k)w_iC + kw_i \cdot PBO_t + (1 - k)w_iC + k w_i \cdot PBO_t \\
= (1 - k) [(1 - k)A_{i,t-2}(1 + r_{i,t-1}) + (1 - k)w_iC + k w_i \cdot PBO_{t-1}] (1 + r_{i,t}) \\
+ (1 - k)w_iC + k w_i \cdot PBO_t \\
= (1 - k)^2 A_{i,t-2}(1 + r_{i,t-1})(1 + r_{i,t}) + w_iC [(1 - k)^2(1 + r_{i,t}) + (1 - k)] \\
+ w_i k [(1 - k) \cdot PBO_{t-1}(1 + r_{i,t}) + PBO_t] \\
= \ldots .
\]

Finally, we get

\[
A_{i,t} = (1 - k)^t A_{i,0}(1 + r_{i,1})(1 + r_{i,2}) \cdots (1 + r_{i,t-1})(1 + r_{i,t}) \\
+ w_iC[(1 - k)^t(1 + r_{i,2})(1 + r_{i,3}) \cdots (1 + r_{i,t}) \\
+ (1 - k)^{t-1}(1 + r_{i,3})(1 + r_{i,4}) \cdots (1 + r_{i,t}) + \ldots \\
+ (1 - k)^3(1 + r_{i,t-1})(1 + r_{i,t}) + (1 - k)^2(1 + r_{i,t}) + (1 - k)] \\
+ w_i k[(1 - k)^{t-1} \cdot PBO_1(1 + r_{i,2})(1 + r_{i,3}) \cdots (1 + r_{i,t}) \\
+ (1 - k)^{t-2} \cdot PBO_2(1 + r_{i,3})(1 + r_{i,4}) \cdots (1 + r_{i,t}) + \ldots \\
+ (1 - k)^2 \cdot PBO_{t-2}(1 + r_{i,t-1})(1 + r_{i,t}) \\
+ (1 - k) \cdot PBO_{t-1}(1 + r_{i,t}) + PBO_t],
\]

where \( A_{i,0} = w_i M \) for \( i = 1, 2, \ldots, n \).

Based on (11), it is straightforward to show that at time \( T \) the size of unfunded liability equals

\[
UL_T = \frac{1}{1 - k} \left( PBO_T - \sum_{i=1}^{n} A_{i,T} \right) \\
= \sum_{i=1}^{n} w_i(H_i - G_i),
\]
where

\[
H_t = Ba(x(T)) - (1 - k)^T - 1 PA_0 (1 + r_{i,1}) (1 + r_{i,2}) (1 + r_{i,3}) \cdots (1 + r_{i,T}) \\
- k(1 - k)^T - 2 \frac{Ba(x(T))}{(1 + \rho)^T - 1} (1 + r_{i,2}) (1 + r_{i,3}) \cdots (1 + r_{i,T}) \\
- k(1 - k)^T - 3 \frac{Ba(x(T))}{(1 + \rho)^T - 2} (1 + r_{i,3}) (1 + r_{i,4}) \cdots (1 + r_{i,T}) - \cdots \\
- k(1 - k) \frac{Ba(x(T))}{(1 + \rho)^T} (1 + r_{i,T-1}) (1 + r_{i,T}) - \frac{Ba(x(T))}{(1 + \rho)} (1 + r_{i,T})
\]

(13)

and

\[
G_t = C[(1 - k)^T - 1 (1 + r_{i,2}) (1 + r_{i,3}) \cdots (1 + r_{i,T}) \\
+ (1 - k)^T - 2 (1 + r_{i,3}) (1 + r_{i,4}) \cdots (1 + r_{i,T}) + \cdots \\
+ (1 - k)(1 + r_{i,T}) + 1].
\]

(14)

The plan makes periodic contributions to meet future benefit obligations. Those contributions constitute the plan’s total pension cost. Following Maurer et al. (2009), we define total pension cost \( TPC \) as the present value of all normal contributions \( C \), supplementary contributions \( SC_t \) and withdrawals \( W_t \):

\[
TPC = \sum_{t=1}^{T} \frac{C + SC_t (1 + \psi_1) - W_t (1 - \psi_2)}{(1 + \rho)^t},
\]

(15)

where

\[
SC_t = \max\{k \cdot UL_t, 0\}
\]

and

\[
W_t = \max\{-k \cdot UL_t, 0\}.
\]

The constants \( \psi_1 \) and \( \psi_2 \) are penalty factors on supplementary contributions \( SC_t \) and withdrawals \( W_t \) respectively. The penalty factor \( \psi_1 \) accounts for the opportunity cost the plan sponsor incurs due to unexpected mandatory supplementary contributions \( SC_t \) that could have been invested in positive net present value projects. And \( \psi_2 \) takes into account the tax benefits the plan forgoes when it reduces the plan’s normal contribution.

2.2. Objective function and optimization problem. Following (Haberman et al., 2000) and others, we consider a pension plan that aims to minimizes solvency risk and contribution risk. As
in much of the DB pension literature, we model this risk with a quadratic function and measure the supplementary contributions \(k \cdot UL_t\) using the spread method of the fund amortization (3). In addition, the budget constraint requires that the plan control its total pension cost.

Taking the above into consideration, we propose a \(T\)-period model where time ranges from \(t = 0, 1, 2, \cdots, T\) and the decision is taken at time \(t = 0\). Our optimization problem is to solve for the asset weights \(w = [w_1, w_2, \ldots, w_n]\) and the normal contribution \(C\), so as to minimize the size of unfunded liability at \(T\) in a mean square sense subject to an expected total pension cost constraint.

\[
\begin{align*}
\text{Minimize} & \quad E \left[ (UL_T)^2 \right] \\
\text{subject to} & \quad E(UL_T) = 0 \\
& \quad E(TPC) \leq \zeta \\
& \quad w_i \geq 0, \quad i = 1, 2, \ldots, n \\
& \quad \sum_{i=1}^{n} w_i = 1 \\
& \quad C \geq 0.
\end{align*}
\]

(16)

In (16) the expected total pension cost \(E(TPC)\) is not higher than a preset target level \(\zeta\). The constraint \(w_i \geq 0\) implies short-selling is not allowed. In addition, we specify the expected unfunded liability is zero. Thus (16) is a mean-variance pension funding problem with a constraint on expected total pension costs.

2.3. Example. Here we present a numerical example to show how to achieve the optimal normal contribution and asset allocation by solving optimization problem (16). We consider a cohort, all joining the plan at age \(x_0 = 45\) at \(t = 0\), and retiring at \(T = 20\) when they reach age \(x = 65\). We estimate that the benefit payment rate is \(c\) and number of survivors at age \(x\) is \(n\), so that \(B = nc = 10\) million. The plan will pay benefits to survivors at times \(T + 1, T + 2, \ldots\) so for each of the \(n\) retirees, the present value of benefits is \(c \cdot a(x(T))\), and the aggregate present value is the sum of \(n\) independent identically distributed benefits \(nc \cdot a(x(T)) = 10 \cdot a(x(T))\) million.
The initial pension fund is $M = 5 million at $t = 0$. The pension funds are invested in three assets: S&P 500 index $A_{1,t}$, Merrill Lynch corporate bond index $A_{2,t}$ and 3-month T-bill $A_{3,t}$, with rates of returns $r_{1,t}$, $r_{2,t}$ and $r_{3,t}$ in period $t$ respectively. The plan sets the pension valuation rate at $\rho = 0.08$ and the life annuity factor discount rate at $r = 0.05$. It will amortize the unfunded liability over $m = 7$ years. The plan’s target expected total pension cost is specified as $\zeta = 24$ million. Moreover, following Maurer et al. (2009) we assume the penalty factors on supplementary contributions and withdrawals are both equal to $\psi_1 = \psi_2 = 0.2$.

To model pension assets and mortality dynamics, we consider a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with the filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ where $\mathcal{F}_t$ is the information available up to time $t$. The filtration $\mathbb{F}$ consists of two subfiltrations, which is denoted as $\mathbb{F} = \mathbb{F}^F \vee \mathbb{F}^M$ where $\mathbb{F}^F$ contains information about the financial market and $\mathbb{F}^M$ contains information about the mortality evolution. We assume the two subfiltrations $\mathbb{F}^F$ and $\mathbb{F}^M$ are independent. That is, the stochastic processes for assets and for mortality rates are independent.

2.3.1. **Financial market model.** We describe the process of the S&P 500 index at time $t$, $A_{1,t}$, as the combination of a Brownian motion and a compound Poisson process as follows:

$$\frac{dA_{1,t}}{A_{1,t}} = \begin{cases} \alpha_1 \, dt + \sigma_1 \, dW_{1t}, & \text{if the Poisson event does not occur at time } t; \\ (\alpha_1 - \lambda_1 k_1) \, dt + \sigma_1 \, dW_{1t} + (Y_1 - 1), & \text{if the Poisson event occurs at time } t. \end{cases}$$

(17)

The constant $\alpha_1$ is the drift of the S&P 500 index; $\sigma_1$ is its instantaneous volatility, conditional on no jumps. $W_{1t}$ is a standard Brownian motion with mean 0 and variance $t$. $\lambda_1$ is the mean number of arrivals per unit time of a Poisson process $N_{1t}$. The Poisson random measure $N_{1t}^1$ counts the number of jumps of a particular size $(Y_1 - 1)$ during a time interval of $(0, t)$. The Poisson process $N_{1t}^1$ and the standard Brownian motion $W_{1t}$ are independent. The parameter $k_1$ is $k_1 \equiv E(Y_1 - 1)$ where $E(Y_1 - 1)$ is the expected percentage change in the S&P 500 index if a Poisson event occurs.

The S&P 500 index, $A_{1,t}$, will be continuous most of the time with finite jumps occurring at discrete points of time. The part “$\sigma_1 \, dW_{1t}$” describes the instantaneous part of an unanticipated “normal” S&P index change, and “$Y_1 - 1$” captures the “abnormal” shocks. If $\lambda_1 = 0$, then $Y_1 - 1 = 0$ and it is the same as the standard stochastic model without jumps. If $\alpha_1$, $\sigma_1$, $\lambda_1$ and $k_1$
are constants, we can solve the differential equation (17) as

\[
\frac{A_{1,t}}{A_{1,0}} = \exp \left[ \left( \alpha_1 - \frac{1}{2} \sigma_1^2 - \lambda_1 k_1 \right) t + \sigma_1 W_{1t} \right] Y(N^1_t). \tag{18}
\]

The cumulative jump size \(Y(N^1_t) = 1\) if \(N^1_t = 0\) and \(Y(N^1_t) = \prod_{j=1}^{N^1_t} Y_{1j}\) for \(N^1_t \geq 1\) where the jump sizes, \(Y_{1j}\) for \(j = 1, 2, \ldots\), are independent and identically distributed as lognormal random variables so that \(\log Y_{1j}\) is a standard normal random variable with mean parameter \(m_1\) and volatility parameter \(s_1\).

From (18), we can derive the S&P 500 index value \(A_{1,t+\Delta}\), given \(\mathcal{F}_t\) resulting in

\[
A_{1,t+\Delta} | \mathcal{F}_t = A_{1,t} \exp \left[ \left( \alpha_1 - \frac{1}{2} \sigma_1^2 - \lambda_1 k_1 \right) \Delta + \sigma_1 \Delta W_{1t} \right] \prod_{j>N^1_t} Y_{1j}, \tag{19}
\]

where \(\mathcal{F}_t\) is the information set up to time \(t\).

Similarly, the Merrill Lynch corporate bond index \(A_{2,t+\Delta}\) given \(\mathcal{F}_t\) is similarly defined as follows,

\[
A_{2,t+\Delta} | \mathcal{F}_t = A_{2,t} \exp \left[ \left( \alpha_2 - \frac{1}{2} \sigma_2^2 - \lambda_2 k_2 \right) \Delta + \sigma_2 \Delta W_{2t} \right] \prod_{j>N^2_t} Y_{2j}, \tag{20}
\]

where \(\alpha_2\) is the drift of the Merrill Lynch corporate bond index; \(\sigma_2\) is the instantaneous volatility, conditional on no jumps. \(W_{2t}\) is a standard Brownian motion with mean 0 and variance \(t\). The jump sizes are independent lognormal random variables with parameters \(m_2\) and \(s_2\). The jumps \(Y_{1j}\) and \(Y_{2i}\) are independent for all \(i\) and \(j\). The covariance of \(W_{1t}\) and \(W_{2t}\) is

\[
\text{Cov}(W_{1t}, W_{2t}) = \rho_{12} \sigma_1 \sigma_2 t, \tag{21}
\]

where \(\rho_{12}\) is the parameter measuring the dependence of the stock and bond indices.

In addition, we assume the 3-month T-bill evolves according to a geometric Brownian motion as follows:

\[
\frac{dA_{3,t}}{A_{3,t}} = \alpha_3 \, dt + \sigma_3 \, dW_{3t}, \tag{22}
\]
Table 1. Maximum likelihood parameter estimates of three pension assets

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<th>Estimate</th>
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</tr>
</tbody>
</table>

where $\alpha_3$, $\sigma_3$ and $W_{3t}$ are the drift, volatility and standard Brownian motion of the 3-month T-bill.

The differential equation (22) can be solved as

$$A_{3,t+\Delta} | \mathcal{F}_t = A_{3,t} \exp \left[ (\alpha_3 - \frac{1}{2} \sigma_3^2) \Delta + \sigma_3 \Delta W_{3t} \right].$$

(23)

We further assume the 3-month T-bill is uncorrelated with the S&P 500 index and the Merrill Lynch corporate bond index,

$$\text{Cov}(W_{1t}, W_{3t}) = 0, \text{ and } \text{Cov}(W_{2t}, W_{3t}) = 0.$$

We estimate models (19), (20) and (23) based on monthly data from March 1988 to December 2010. For the S&P 500 index and the Merrill Lynch corporate bond index, we rely on the time series data provided by the DataStream. The monthly 3-month T-bill rates are obtained from FRED at Federal Reserve Bank of St. Louis.\(^3\) We use the monthly data to increase the number of observations for model calibration. Since we are interested in annual returns, we convert our monthly maximum likelihood estimates to annual estimates. Those annual estimates are presented in Table 1.

Table 1 indicates that the S&P 500 index has a higher expected log return ($\alpha_1 = 0.1081$) and volatility ($\sigma_1 = 0.1069$) than the Merrill Lynch corporate bond index ($\alpha_2 = 0.0794, \sigma_2 = 0.0481$) and the 3-month T-bill ($\alpha_3 = 0.0523, \sigma_3 = 0.0094$). The expected number of jumps per year for the stock index is $\lambda_1 = 0.2946$. For the corporate bond index the expected number of jumps per year is $\lambda_2 = 0.0080$. Given a jump event occurs, the expected log returns of the jump size is $m_1 = -0.0272$ for the stock index and $m_2 = -0.0744$ for the bond index. In addition, the

\(^3\)http://research.stlouisfed.org.
the stock index and the corporate bond index are positively correlated with correlation parameter \( \rho_{12} = 0.3380 \).

2.3.2. **Stochastic mortality model.** We apply the Lee and Carter (1992) model to describe pension mortality rates. This model incorporates both the age-specific mortality variation and the general time trend of mortality evolution for all ages. In this form of the Lee-Carter model the logarithm of the one-year death rate \( q_{x,t} \) at age \( x \) \((x = 0, 1, 2, \ldots)\) in year \( t \) \((t = 1, 2, \ldots, K)\) is

\[
\ln q_{x,t} = a_x + b_x \gamma_t + \epsilon_{x,t}, \tag{24}
\]

where \( a_x \) and \( b_x \) are the age-specific parameters.

The transitory shock \( \epsilon_{x,t} \) is a normally distributed variable with zero mean. The time-series common risk factor \( \gamma_t \) affects the mortality rates of all ages in year \( t \). Lee and Carter (1992) assume \( \gamma_t \) follows a random walk with drift \( g \),

\[
\gamma_t = \gamma_{t-1} + g + e_t, \tag{25}
\]

where the error term \( e_t \) is normally distributed with a zero mean and a variance \( \sigma_{\gamma} \).

The solution of (24) is not unique so two constraints are imposed:

\[
\sum_x b_x = 1, \quad \text{and} \quad \sum_t \gamma_t = 0. \tag{26}
\]

These two constraints imply that the intercept \( a_x \) is simply the empirical average of age \( x \) over time:

\[
a_x = \frac{\sum_{t=1}^{K} \ln q_{x,t}}{K}, \tag{27}
\]

where \( K \) is the length of the time series of mortality data. We follow the Lee and Carter (1992) proposal incorporating singular value decomposition to estimate \( b_x \) for each age \( x \) and \( \gamma_t \) for each year \( t \).

We assume the pension plan has the same mortality experience as that of the US male population and it estimates model (24) following the Lee and Carter (1992) procedure. The tables for years
1901 to 1999 are from the Human Life Table Database and the tables for 2000 to 2007 are from the Human Mortality Database, published by the University of California, Berkeley and Max Planck Institute for Demographic Research. The estimated $\gamma_t$ are shown in Figure 1. Based on the ages $x = 0, 1, 2, \ldots$ and the time series of $\gamma_t$ where $t = 1901, 1902, \ldots, 2006, 2007$, we obtain $g = -0.20$ and $\sigma_\gamma = 0.63$ for (25).

Next, we forecast the mortality rates $q_{x,2008+j}$ for $j = 0, 1, \ldots, T, T+1, \ldots$ and $x = 65, 66, \ldots$. We first simulate the error term $e_{2008+j}$ and then add the constant $g = -0.20$ to each simulated $e_{2008+j}$ to get a $\gamma_{2008+j}$ where $j = 0, 1, 2, \ldots, T, T+1, \ldots$. Given the estimated $a_x$’s and $b_x$’s and the simulated $\gamma_{2008+j}$’s, we use the model (24) to calculate simulated future mortality rates

$$
\tilde{q}_{x,2008+j} = e^{a_x+b_x\gamma_{2008+j}+\epsilon_{x,2008+j}}, \quad j = 0, 1, 2, \ldots, T, T+1, \ldots
$$

To simplify notation, now we use $\tilde{q}_{x,j}$ in place of $\tilde{q}_{x,2008+j}$.

The one-year forecasted survival probability is

$$
\tilde{p}_{x+j,j} = 1 - \tilde{q}_{x+j,j}.
$$

---

4Available at www.mortality.org or www.humanmortality.de (data downloaded on February 25, 2011).

5To conserve space, the parameter estimates of $a_x$ and $b_x$ are not reported but available upon request.
After generating a path of survival rates $\tilde{p}_{x,T}, \tilde{p}_{x+1,T+1}, \tilde{p}_{x+2,T+2}, \ldots$ with the Lee-Carter model for $x = 65$ at time $T = 20$, we calculate the value of the life annuity $a(x(T))$ using (7):

$$a(x(T)) = \sum_{s=1}^{\infty} v^s s \tilde{p}_{x,T} = \sum_{s=1}^{\infty} v^s s \tilde{p}_{65,20}$$

(28)

where

$$s \tilde{p}_{65,20} = \tilde{p}_{65,20} \cdot \tilde{p}_{66,21} \cdots \tilde{p}_{65+s-1,T+s-1}.$$

With the value of $a(x(T))$ for that path, the value of the pension liability at time $T$ is simply

$$PBO_T = Ba(x(T)).$$

2.3.3. **Optimization results.** We set year 2007 as the base year $t = 0$ and ran a Monte Carlo simulation with 10,000 iterations. Although in our numerical example we only consider three assets, the model can handle a high number of assets.

Next we solved the optimization problem (16) to obtain the optimal asset allocation and normal contribution. The row with $g = -0.20$ in Table 2 shows the results. On average, about $w_1 = 11.13\%$ of the strategic portfolio funds is allocated to the S&P 500 index, $w_2 = 44.49\%$ is allocated to the the Merrill Lynch corporate bond index, and the remaining funds are invested in the 3-month T-bill with $w_3 = 44.38\%$. In addition, given $E(UL_T) = 0$, the optimal normal contribution equals $C = $0.67 million per year, which ensures the lowest funding variation $E[(UL_T)^2]$ and the expected $TPC$ equal to $\zeta = $24 million. In this base case, the two tail-risk measures, the 95%-level conditional value at risk of $UL_T$ (CVaR$_{95\%}(UL_T)$) and the 95%-level conditional value at risk of $TPC$ (CVaR$_{95\%}(TPC)$) equal 10.69 and 28.43 respectively.

2.4. **Longevity risks.** Mortality has improved overtime. In the previous example, which we call the base case, the common risk factor $\gamma_t$ shows that on average the mortality improvement rate across all ages is 20% per year from 1901 to 2007 (i.e. $g = -0.20$). However, the tendency of pension participants to live longer than what the historical data suggest has been increasingly attracting the attention of plan sponsors, regulators, actuaries, academia and others. This is so called longevity risk, the risk that the mortality of pensioners improves at a higher rate than expected.
Accordingly, an interesting question for the pension plan is: if the mortality improves more than the level in the base case with $g = -0.20$, how will it change the optimal asset allocation and normal contribution?

The impact of longevity risk on the optimal strategy can be studied easily with our model. In particular, we consider mortality dynamics under which the pension liability is an increasing function of survival probabilities, as the higher survival probabilities will yield the longer lifetime and the longer duration of the annuity payments. In a sensitivity analysis of the model with respect to the mortality improvement parameter $g$, we vary the parameter from $-0.25$ to $-0.40$. A more negative $g$ means a greater level of mortality improvement. Table 2 shows the effect of changing $g$ on the optimization results given all other parameters constant. As expected, a more negative value of $g$ leads to a riskier portfolio. That is, if future life expectancies of the pension participants increases, which means the plan is more likely to be underfunded, the plan should take more financial risk to accumulate sufficient funds to cover the claim (Josa-Fombellida and Rincón-Zapatero, 2004). For example, when $g = -0.20$, the plan should invest 55.62% ($= 11.13% + 44.49%$) in risky assets. However, when $g = -0.40$, this proportion increases to 72.58% ($= 14.63% + 57.95%$). Moreover, the higher level of mortality improvement results in the higher variation of plan unfunded liability $E[(UL_T)^2]$ and the higher downside risk in terms of $\text{CVaR}_{95%}(UL_T)$ and $\text{CVaR}_{95%}(TPC)$. When the mortality improvement parameter $g$ decreases from the base case $-0.20$ to $-0.40$, the variation $E[(UL_T)^2]$ increases by 78.5% from 32.46 to 57.95, and $\text{CVaR}_{95%}(UL_T)$ rises by 33.2% from 10.69 to 14.24. Table 2 summarizes the sensitivity of optimal solutions to longevity risk.
It is difficult for a plan to predict future mortality rates. The resulting uncertainty can immediately affect the stability of the pension plan through, at least, wrong asset allocations and normal contribution levels when the plan underestimates future mortality improvements. Therefore, if the pension plan can successfully hedge its longevity risk, it will suffer much less from financial distress due to unexpected mortality improvement. Now we turn our attention to an analysis of optimal hedging decisions. In the next two sections, we discuss two pension longevity risk hedging strategies: the ground-up hedging strategy and the excess-risk hedging strategy, which are discussed in detail in the next two sections.

3. MANAGING PENSION LONGEVITY RISKS WITH THE GROUND-UP HEDGING STRATEGY

In this section, we study the optimal hedging decision for a pension plan when it implements the ground-up hedging strategy. This strategy covers the proportion \( h \), \( 0 \leq h \leq 1 \), of the pension liability represented by the shaded area \( h\text{Ba}(x(T)) \) in Figure 2, which is transferred to the hedge provider. The hedge provider will pay a proportion \( h \) of benefits due to retirees at time \( T + 1, T + 2, \ldots \).

Let \( E[Ba(x(T))] = \bar{Ba}(x(T)) \). At time \( t = 0 \), the pension plan starts with an accumulated fund \( M \) and cedes a portion \( h \) of the plan liability \( Ba(x(T)) \) to a third party (e.g. longevity-security
investors), who accepts the obligation for a price equal to

$$HP_1 = \frac{h(1 + \delta_1)Ba(x(T))}{(1 + \rho)^T},$$

where $\delta_1$ is the hedging cost per dollar hedged. The hedging cost $\delta_1$ covers risk premium, issuance cost and administrative expenses of the longevity risk taker, and the time and resources spent by the plan to transfer its longevity risk. As a result, the pension plan retains a liability of $(1 - h)Ba(x(T))$ and its liability at the end of period $t$ becomes

$$PBO'_t = (1 - h)PBO_t$$
$$= \frac{(1 - h)Ba(x(T))}{(1 + \rho)^{T-t}} \quad t = 1, 2, \ldots, T.$$

Meanwhile, the plan invests in the capital market to generate investment income. Given that the plan hedges a fraction $h$ of its pension liability by paying $HP_1$, the total amount of pension assets available for investment at $t = 0$ is

$$PA'_0 = M' = M - HP_1,$$

which is lower than that in the no-hedging case where $PA_0 = M$. In the accumulation phase, the annual normal contribution $C$ and supplemental contribution $SC_t$ are also invested. The investment activity generates an amount, $A'_{i,t}$, at $t$ which is calculated following (11) by replacing $A_{i,0}$ with $A'_{i,0} = w_iM'$ and $PBO_i$ with $PBO'_i$.

The optimal investment and contribution policy is subject to a total pension cost constraint. Compared with the no-hedging case, the ground-up hedging strategy has an additional cost component — the hedging cost. For each dollar hedged, the plan pays $\delta_1$ dollars to set up the contract. After hedging $hBa(x(T))$ of the plan liability, the plan pays total pension costs of $TPC'$ at time 0 calculated as follows:

$$TPC' = \frac{h\delta_1Ba(x(T))}{(1 + \rho)^T} + \sum_{t=1}^{T} \frac{C + SC_t(1 + \psi_1) - W_t(1 - \psi_2)}{(1 + \rho)^t},$$

(29)

where $h\delta_1Ba(x(T))/(1 + \rho)^T$ is the present value of total hedging cost.
3.1. **How to transfer longevity risk with the ground-up hedging strategy?** With the ground-up hedging strategy, the new size of unfunded liability at time $T$ equals

$$UL'_T = \frac{1}{1-k} \left( PBO'_T - \sum_{i=1}^{n} A'_{i,T} \right).$$

(30)

where

$$PBO'_T = (1-h)Ba(x(T))$$

(31)

and $A'_{i,T}$ is obtained according to (11) by replacing $A_{i,0}$ with $A'_{i,0} = w_i M'$ and $PBO_t$ with $PBO'_t$ when $t = T$.

With this change in mind, the optimization problem of the pension plan with respect to the asset weights $w = [w_1, w_2, \ldots, w_n]$, normal contribution $C$ and hedge ratio $h$ can be expressed as:

Minimize $E \left[ (UL'_T)^2 \right]$

subject to $E(UL'_T) = 0$

$$E(TPC') \leq \zeta$$

$$\frac{h(1 + \delta_1)B\bar{a}(x(T))}{(1 + \rho)^T} \leq M$$

$$0 \leq h \leq 1$$

$w_i \geq 0$, \hspace{1cm} $i = 1, 2, \ldots, n$

$$\sum_{i=1}^{n} w_i = 1$$

$C \geq 0$.

The constraint

$$\frac{h(1 + \delta_1)B\bar{a}(x(T))}{(1 + \rho)^T} \leq M$$

ensures the premium does not exceed the pension fund at $t = 0$. That is, the plan does not borrow money to pay for the cost of risk transferred.

3.2. **Example.** Here we continue the example in Section 2.3, but now assume that the plan implements a ground-up hedging strategy. The results are shown in Table 3. When hedging is costless
(δ₁ = 0), subject to the total pension cost constraint ζ = $24 million, the plan hedges 10.41% of the pension liability and achieve the lowest pension variation $E[(UL^T)^2] = 23.72$, which is lower than that in the no-hedging case $E[(UL^T)^2] = 32.46$. This can be explained by two effects. First, the plan retains a lower longevity risk by ceding some of it to a third party. Second, the optimal asset portfolio is relatively less risky. The risky assets accounts for 52.86% (= 10.59% + 42.27%) of invested funds when the plan hedges without cost, compared to 55.62% (= 11.13% + 44.49%) in the no-hedging situation. Both effects attribute to the lower funding variation.

When the plan hedges, it incurs an explicit cost and an implicit cost. The explicit cost is the hedging cost δ₁ for each dollar hedged. The implicit cost arises from the reduced fund available for investment since the fund available for investment decreases from M to M’. The plan sacrifices the higher expected return in the risky asset class in exchange for the low longevity risk. However, longevity risk hedging is attractive only if its benefits exceed costs. Table 3 shows a negative relation between the hedge ratio h and the hedge cost rate δ₁. That is, the plan chooses to hedge less as hedging becomes more expensive. As δ₁ increases from 0 to 0.188, the hedge ratio h decreases from 10.41% to 1.51%. When δ₁ goes up to 0.190 and above, no longevity risk will
be ceded. Moreover, as the hedge ratio $h$ goes down, the plan’s funding variation $\mathbb{E}[(U_L^t)^2]$ increases due to, at least, higher longevity risk.

4. Managing Pension Longevity Risks with the Excess-risk Hedging Strategy

Although the ground-up hedging strategy in Section 3 reduces longevity risk, it has two disadvantages. First, it provides a protection that the plan may not need. The plan can predict its future payments to some extent and it is only uncertain about the amount exceeding its expectation. The cost of hedging is usually proportional to the amount of coverage. As such, the ground-up strategy is expensive. In addition, this strategy is capital intensive. It requires a large up-front premium, which the plan may not afford. The first longevity bond, the EIB bond issued in November 2004, is an example of the ground-up hedging strategy. It is a 25-year bond that provides a protection equal to the amount of a fixed annuity, £50 million, multiplied by the percentage of the reference population still alive at each anniversary. The coverage of this bond is illustrated as the solid filled bars in Figure 3.

The EIB bond offered a hedge to pension plans but the EIB did not sell. The ground-up protection structure, at least partially, explains its failure (Lin and Cox, 2008). A more attractive structure is the one that only covers the annuity payment that exceeds a certain strike level. The modified EIB bond we suggest is shown in Figure 4. The unfilled area of each bar in Figure 4 represents
the risk retained by the plan. The plan only cedes the longevity risk above this level, which is represented as the solid filled bars. The risk covered by our proposed EIB bond in the present value term at $T$ is

$$\sum_{s=1}^{25} v^s \max [B_s \tilde{p}_{x,T} - B_s \hat{p}_{x,T}, 0],$$

where $B_s \hat{p}_{x,T}$ is the strike level for year $s$, $s = 1, 2, \ldots, 25$ when $T = 0$. This illustrates the second strategy we examine in this section: the excess-risk hedging strategy.

With the excess-risk strategy, the plan needs to determine a threshold level for each of the years $T+1, T+2, \ldots$ above which to transfer the longevity risk. Let $s \tilde{p}_{x,T} = E[s \tilde{p}_{x,T}]$. Suppose at time 0, the plan transfers the risk that exceeds the plan’s expectation after the plan participants reach the retirement age $x$ at time $T$,

$$\sum_{s=1}^{\infty} v^s \max [B_s \tilde{p}_{x,T} - B_s \tilde{p}_{x,T}, 0], \quad s = 1, 2, \ldots$$

which can be viewed as a set of European call options written at 0 and exercised at $T+1, T+2, \ldots$. The excess-risk strategy transfers a proportion $h$ of the risk exceeding the strike level, in this example, $\sum_{s=1}^{\infty} v^s \max [B_s \tilde{p}_{x,T} - B_s \tilde{p}_{x,T}, 0]$. Of course, the plan can set a different threshold given its risk tolerance, for example, one or two standard deviation above the mean $s \tilde{p}_{x,T}, s = 1, 2, \ldots$.

In Section 4.3, we investigate how a change in the strike level will change our conclusion. Because
the excess-risk strategy only cedes the high-end risk, in general, it requires a lower cash outflow for the protection than the ground-up strategy.

If the plan hedges a proportion $h$ of (34), it needs to pay a price equal to

$$HP_2 = \frac{h(1 + \delta_2)E [\sum_{s=1}^{\infty} v^s \max [B_s\tilde{p}_{x,T} - B_s\bar{p}_{x,T}, 0]]}{(1 + \rho)^T},$$

where $\delta_2$ is the hedging cost per dollar hedged in the excess-risk hedging strategy and $E [\sum_{s=1}^{\infty} v^s \max [B_s\tilde{p}_{x,T} - B_s\bar{p}_{x,T}, 0]]/(1 + \rho)^T$ is the present value of expected payments from the longevity risk taker at $T$. As such, the plan’s liability at the end of period $t$ becomes

$$PBO''_t = (1 - h)Ba(x(T)) + h \sum_{s=1}^{\infty} v^s \min [B_s\tilde{p}_{x,T}, B_s\bar{p}_{x,T}] / (1 + \rho)^T$$

$$+ \sum_{t=1}^{T} C + SC_t(1 + \psi_1) - W_t(1 - \psi_2) / (1 + \rho)^t,$$

(35)

If the pension benefit due at time $T + s, s = 1, 2, \ldots$ is lower than the strike level $B_s\tilde{p}_{x,T}$, the plan will not exercise the option and its liability is $B_s\tilde{p}_{x,T}$. If the pension liability at $T + s$ is higher than the strike level $B_s\tilde{p}_{x,T}$, the plan will exercise the option and it is only responsible for $(1 - h)B_s\tilde{p}_{x,T} + hB_s\bar{p}_{x,T}$.

After the plan buys the longevity hedge, the fund $M''$ it can invest in the capital market at $t = 0$ is

$$M'' = M - HP_2.$$

A model for finding optimal asset allocation, the normal contribution and the hedge ratio takes into account total pension costs. The excess-risk strategy has a total pension cost $TPC''$ as follows:

$$TPC'' = \frac{h_2E [\sum_{s=1}^{\infty} v^s \max [B_s\tilde{p}_{x,T} - B_s\bar{p}_{x,T}, 0]]}{(1 + \rho)^T} + \sum_{t=1}^{T} C + SC_t(1 + \psi_1) - W_t(1 - \psi_2) / (1 + \rho)^t,$$

(36)

where $h_2E [\sum_{s=1}^{\infty} v^s \max [B_s\tilde{p}_{x,T} - B_s\bar{p}_{x,T}, 0]]/(1 + \rho)^T$ is the hedging cost. Recall that $TPC'$ in (29) is the total pension cost of the ground-up hedging strategy. Both $TPC'$ and $TPC''$ have a hedging cost component. However, the hedging cost components of these two strategies are not equal due to different hedging structures and possibly different unit hedging costs ($\delta_1$ vs. $\delta_2$). This
can lead to very different optimal hedging solutions for these two strategies, which we will discuss later.

4.1. How to manage longevity risks with the excess-risk hedging strategy? When the plan applies the excess-risk hedging strategy, the size of its unfunded liability at time $T$ is

$$UL''_T = \frac{1}{1-k} \left( PBO''_T - \sum_{i=1}^{n} A''_{i,T} \right),$$

(37)

where the pension liability at time $T$ is:

$$PBO''_T = (1-h)Ba(x(T)) + h \sum_{s=1}^{\infty} v^s \min \left[ B_s \tilde{p}_{x,T}, B_s \bar{p}_{x,T} \right]$$

(38)

and the accumulated fund invested in asset $i$ at time $T$, $A''_{i,T}$, is calculated from (11) by replacing $A_{i,0}$ with $A''_{i,0} = w_i M''$ and $PBO_t$ with $PBO'_t$ when $t = T$.

The optimal excess-risk hedging strategy is to solve the following optimization problem:

Minimize $w, C, h$

$$E \left[ (UL''_T)^2 \right]$$

subject to

$$E(UL''_T) = 0$$

$$E(TPC'') \leq \zeta$$

$$h(1 + \delta_2)E \left[ \sum_{s=1}^{\infty} v^s \max \left[ B_s \tilde{p}_{x,T} - B_s \bar{p}_{x,T}, 0 \right] \right] \leq M$$

$$0 \leq h \leq 1$$

$$w_i \geq 0, \quad i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} w_i = 1$$

$$C \geq 0.$$  

(39)

The constraint

$$h(1 + \delta_2)E \left[ \sum_{s=1}^{\infty} v^s \max \left[ B_s \tilde{p}_{x,T} - B_s \bar{p}_{x,T}, 0 \right] \right] \leq M$$

ensures the premium does not exceed the pension fund at $t = 0$. 
4.2. **Example.** Continue the example in Section 2.3. This time the plan adopts the excess-risk hedging strategy with the strike level \( \bar{p}_{x,T}, s = 1, 2, \ldots \) and \( T = 20 \). It solves problem (39) to obtain the optimal normal contribution \( C \), the asset allocations \( w_i \), and hedge ratio \( h \) that will minimize the funding variation \( \text{E}[(UL'_T)^2] \) subject to \( \text{E}(UL'_T) = 0 \) and \( \text{E}(TPC) = \zeta = $24 \) million.

Table 4 summarizes the results for various unit hedging costs \( \delta_2 \). An interesting pattern is that as long as \( \delta_2 \) is not greater than 0.22, the plan will cede the entire longevity risk that exceeds the strike level \( (h = 100\%) \). In those cases, the funding variations \( \text{E}[(UL'_T)^2] \) are all lower than the case without hedging longevity risk in Table 2. However, the hedging will not do anything good when \( \delta_2 \geq 0.26 \) so the plan chooses to retain all of the risks \( (h = 0\%) \).

The hedge ratios \( h \) in Table 4 are much higher than those in Table 3, which creates an apparent contrast between the excess-risk strategy and the ground-up strategy. This can be explained by the more attractive structure and the lower cost of the excess-risk strategy. The excess risk strategy only covers the payment above the strike level. However, the ground-up strategy covers the entire risk, including the payment the plan can predict reasonably. As such, the ground-up strategy requires a higher price, reflecting the higher future payment from the hedge provider. This means the plan may give up a large proportion of the initial fund \( M \) and forgo possibly higher returns in the capital market. In this case the plan decides to hedge less. Moreover, the hedging cost increases with the amount of coverage. The extra coverage provided by the ground-up strategy increases hedging cost. This further discourages the plan from hedging with this instrument. In contrast, the excess-risk strategy focuses on the protection the plan really needs. In our example, as long as the unit hedging cost is not too high \( (\delta_2 \leq 0.22) \), the plan will hedge all of the excess longevity risk.

One must use caution when comparing the funding variation column \( \text{E}[(UL'_T)^2] \) in Table 3 with \( \text{E}[(UL'_T)^2] \) in Table 4. One might find it striking that many values of \( \text{E}[(UL'_T)^2] \) in Table 4 are higher than \( \text{E}[(UL'_T)^2] \)'s in Table 3 and then might conclude that the excess-risk strategy has a higher longevity risk. This is not the case because it does not recognize that the mean squared unfunded liability reflects not only the longevity risk, but also the investment risk. Actually, the higher variation \( \text{E}[(UL'_T)^2] \) in the excess-risk strategy arises from the higher amount available
Table 4. Optimal Excess-risk Hedging Strategies with Different Assumptions on Hedging Cost Parameter δ. Given ζ = $24 Million, g = −0.20 and the strike level $s \bar{p}_{x,T}$, $s = 1, 2, \ldots$ and $T = 20$.

<table>
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<th>δ_2</th>
<th>w_1</th>
<th>w_2</th>
<th>w_3</th>
<th>C</th>
<th>h</th>
<th>E [(UL''_T)^2]</th>
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<td>11.30%</td>
<td>45.16%</td>
<td>43.54%</td>
<td>0.646</td>
<td>100%</td>
<td>32.249</td>
</tr>
<tr>
<td>0.21</td>
<td>11.31%</td>
<td>45.21%</td>
<td>43.47%</td>
<td>0.644</td>
<td>100%</td>
<td>32.316</td>
</tr>
<tr>
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<td>45.27%</td>
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<td>47.29%</td>
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<td>44.71%</td>
<td>44.12%</td>
<td>0.662</td>
<td>22.04%</td>
<td>32.505</td>
</tr>
<tr>
<td>0.26</td>
<td>11.11%</td>
<td>44.51%</td>
<td>43.38%</td>
<td>0.669</td>
<td>0%</td>
<td>32.512</td>
</tr>
</tbody>
</table>

for asset investments (i.e. $M'' > M'$). In our example, the expected annuity payment $B \bar{a}(x(T))$ equals $111.94$ million and the expected excess-risk payment is

$$E \left[ \sum_{s=1}^{\infty} v^s \max [B_s \bar{p}_{x,T} - B_s \bar{p}_{x,T}, 0] \right] = $1.58 million. $$

Assuming $\delta_1 = \delta_2 = 0.10$, the price of the ground-up hedging strategy paid to the risk taker at $t = 0$ is

$$HP_1 = \frac{h(1 + \delta_1)B \bar{a}(x(T))}{(1 + \rho)^T} = \frac{9.46% \times (1 + 0.10) \times 111.94}{1.08^{20}} = $2.50 million,$$

while the price of the excess-risk strategy is only

$$HP_2 = \frac{h(1 + \delta_2)E \left[ \sum_{s=1}^{\infty} v^s \max [B_s \bar{p}_{x,T} - B_s \bar{p}_{x,T}, 0] \right]}{(1 + \rho)^T} = \frac{100% \times (1 + 0.10) \times 1.58}{1.08^{20}} = $0.37 million.$$

This implies that the initial fund available for asset investment in the ground-up strategy is

$$PA'_0 = M' = 5 - 2.50 = $2.50 million.$$
TABLE 5. Optimal Excess-risk Hedging Strategies with Different Assumptions on Hedging Cost Parameter $\delta_2$. Given $\zeta = 24$ Million, $g = -0.20$ and the strike level 

$$B_s \tilde{p}_{x,T} + \sigma_{B_s \tilde{p}_{x,T}}, \ s = 1, 2, \ldots \text{ and } T = 20.$$ 

<table>
<thead>
<tr>
<th>$\delta_2$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>$C$</th>
<th>$h$</th>
<th>$E[(UL''_T)^2]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>11.07%</td>
<td>44.31%</td>
<td>44.62%</td>
<td>0.673</td>
<td>100%</td>
<td>31.961</td>
</tr>
<tr>
<td>0.04</td>
<td>11.08%</td>
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<td>100%</td>
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<td>100%</td>
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<td>0.16</td>
<td>11.11%</td>
<td>44.48%</td>
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<td>100%</td>
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<tr>
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<td>100%</td>
<td>32.223</td>
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<tr>
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<td>11.13%</td>
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<td>0.666</td>
<td>100%</td>
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<td>44.23%</td>
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<td>32.354</td>
</tr>
<tr>
<td>0.32</td>
<td>11.15%</td>
<td>44.63%</td>
<td>44.22%</td>
<td>0.664</td>
<td>90.09%</td>
<td>32.379</td>
</tr>
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<td>0.34</td>
<td>11.15%</td>
<td>44.64%</td>
<td>44.21%</td>
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<td>0.36</td>
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<td>44.64%</td>
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</tr>
<tr>
<td>0.38</td>
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<td>65.99%</td>
<td>32.440</td>
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<td>44.51%</td>
<td>44.38%</td>
<td>0.669</td>
<td>0%</td>
<td>32.512</td>
</tr>
</tbody>
</table>

which is much lower than that in the excess-risk strategy

$$P A''_0 = M'' = 5 - 0.37 = 4.63 \text{ million}.$$ 

Since the risky assets’ volatility, in general, is higher than that of longevity risk, we observe the higher funding variation but the lower longevity risk in the excess-risk strategy.

4.3. **Optimization with different strike levels.** We next turn our attention to the optimal solutions with different strike levels. In the earlier example we set the strike level at the expected benefit payment $s \tilde{p}_{x,T}, \ s = 1, 2, \ldots \text{ and } T = 20$. The plan may want to set a higher level such as the benefit payment one or two standard deviations above the expected payment, i.e. $B_s \tilde{p}_{x,T} + \sigma_{B_s \tilde{p}_{x,T}}$ or $B_s \tilde{p}_{x,T} + 2\sigma_{B_s \tilde{p}_{x,T}}$ where $\sigma_{B_s \tilde{p}_{x,T}}$ is the standard deviation of $B_s \tilde{p}_{x,T}$.

Table 5 shows the results with the strike level $B_s \tilde{p}_{x,T} + \sigma_{B_s \tilde{p}_{x,T}}$ and Table 6 is for the situation with the strike level $B_s \tilde{p}_{x,T} + 2\sigma_{B_s \tilde{p}_{x,T}}$. Both tables present a pattern similar to that in Table 4: the plan transfers the entire longevity risk ($h = 100\%$) as long as the unit hedging cost $\delta_2$ is not too high. An interesting finding is that as the strike level goes up, the maximum allowed $\delta_2$ to find a feasible optimization solution also goes up. The maximum allowed $\delta_2$ increases from
TABLE 6. Optimal Excess-risk Hedging Strategies with Different Assumptions on Hedging Cost Parameter $\delta_2$ Given $\zeta = $24 Million, $g = -0.20$ and the strike level $B_s\tilde{p}_{x,T} + 2\sigma_{B_s\tilde{p}_{x,T}}$, $s = 1, 2, \ldots$ and $T = 20$.

<table>
<thead>
<tr>
<th>$\delta_2$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>$C$</th>
<th>$h$</th>
<th>$E[\left(U'\delta\right)^2]_T$</th>
</tr>
</thead>
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<td>0</td>
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<td>44.55%</td>
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<td>100%</td>
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<tr>
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<td>100%</td>
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<tr>
<td>0.4</td>
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<td>44.41%</td>
<td>44.51%</td>
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<td>100%</td>
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</tr>
<tr>
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<td>100%</td>
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</tr>
<tr>
<td>0.8</td>
<td>11.09%</td>
<td>44.46%</td>
<td>44.45%</td>
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</tr>
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<td>44.47%</td>
<td>44.43%</td>
<td>0.670</td>
<td>100%</td>
<td>32.318</td>
</tr>
<tr>
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<td>44.41%</td>
<td>0.669</td>
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<td>32.512</td>
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</tbody>
</table>

0.25 to 0.38 when the strike level increases from $B_s\tilde{p}_{x,T}$ to $B_s\tilde{p}_{x,T} + \sigma_{B_s\tilde{p}_{x,T}}$ and further jumps to 2.4 when the strike level is $B_s\tilde{p}_{x,T} + 2\sigma_{B_s\tilde{p}_{x,T}}$, $s = 1, 2, \ldots$ and $T = 20$. This jump is not surprising and is consistent with what we observe in the catastrophe reinsurance market. Froot and O’Connell (1999) find that the industry price in the catastrophe reinsurance market is several times of ceded exposure. With the strike level set at $B_s\tilde{p}_{x,T} + 2\sigma_{B_s\tilde{p}_{x,T}}$, the high unit hedging cost $\delta_2$ reflects the risk aversion of pension plans and hedge providers towards catastrophe longevity risk since the high-end risks are more difficult to predict and if they occur, they will cause serious financial consequences. A risk averse plan is willing to pay a higher $\delta_2$ to hedge extreme longevity risk. Therefore, among these three cases, the case with the highest strike level $B_s\tilde{p}_{x,T} + 2\sigma_{B_s\tilde{p}_{x,T}}$, $s = 1, 2, \ldots$ and $T = 20$ has the highest maximum unit hedging cost $\delta_2$ that still keeps hedging attractive to the pension plan.

5. IMPLICATIONS FOR LONGEVITY SECURITIZATION

Packaging longevity risk through securitization was first advocated by Blake and Burrows (2001) to help pension plans transfer longevity risk. Since then, it has been widely discussed by both academics and practitioners (Lin and Cox, 2005, Blake et al., 2006, Cairns et al., 2006, Cox and Lin, ...
However, most of the possible capital solutions for longevity risk remain theoretical. Up to now, the market for longevity securities has not taken off, although there have been limited experiments with products assuming longevity risk (Standard & Poor’s, 2010). Several investment banks are actively exploring this space, but we have observed very few pension plans hedge longevity risk in the capital market. Our optimization results provide important insights for longevity securitization which may draw pension plans to this market and allow transaction activity to grow.

First, the market should design attractive longevity securities that can attract pension plans to those new instruments. As our results indicate, the pension plans are inclined to transferring more longevity risk with the excess-risk hedging strategy since it is less capital intensive and more cost effective. In July 2008, Canada Life traded a survivor swap with a group of insurance-linked securities and other investors. This is the first publicly-known longevity security successfully executed. Although this deal has no pension plan involvement, it does provide support for the pension plans to pursue the excess-risk strategy. As what we observe in the market, this strategy has gained momentum. Notably, six longevity swaps were completed in the U.K. in 2009 covering liabilities of approximately £4.1 billion (Brcic and Brisebois, 2010).

Second, longevity securities should not be too expensive. The EIB longevity bond offer in 2004 received much attention in the financial press because it was considered as the first longevity security. However, no pension funds and life insurance companies subscribed to the deal and it failed. Besides technical issues such as design problems we discussed earlier, Lin and Cox (2008) and others conclude that its price was too high. Our model indicates that the longevity hedge ratio is a decreasing function of hedging cost. If the hedging cost per each dollar hedged ($\delta_1$ or $\delta_2$) is too high, the plan will not transfer the longevity risk. In this sense, our model can be used to explain the failure of the EIB bond from the perspective of pension asset-liability optimization.

Our model can also be used to explain the reluctance of pension plans to use capital markets to hedge longevity risk. The current high transaction costs in the capital market discourage the plans from transferring longevity risk. This problem can be mitigated by standardizing longevity transactions. For example, we can promote consistent best market practices and publish tradable
longevity indices. Such efforts will provide greater transparency and confidence for this market, and increase transaction activity. Standardization can also alleviate the problems related to investor education and barriers to entry because it facilitates investor efforts to understand those complex longevity securities and promotes price convergence between buyers and sellers. The longevity market is expected to develop more standardization to allow a broader range of investor groups to enter the market. As pointed out by Manish Kapoor, “We’ll see more efforts at standardization between transactions, and as that occurs, increased liquidity—albeit to a limited extent—and the ability to novate or exit structures. It will be interesting to see how the pricing changes as capital market participants other than insurers and reinsurers enter [longevity securitization markets]” (Standard & Poor’s, 2010). As liquidity increases, the cost of longevity securities will go down and we expect more pension plans to enter into these transactions.

6. CONCLUSION

This paper proposes a model to identify the optimal contribution, capital allocation and longevity risk hedging strategies that minimize pension funding risk for a DB pension plan with a single cohort in the accumulation phase. Given a target expected total pension cost and zero expected unfunded liability, we minimize the plan’s funding variation upon the employees’ retirement. In this setup, the total pension cost includes a penalty cost caused by deviation of the overall contribution from the normal contribution.

Then we investigate how sensitive the pension funding status is to longevity risk. Populations all over the world have enjoyed increasing life expectancy in the last century. This trend is expected to continue in the future (KPMG LLP, 2008). It makes clear the need to study the impact of mortality improvement on the plan’s asset-liability management. With our optimization model, the numerical examples show that as pensioners live longer, given a fixed expected total pension cost, the pension plan will make less normal contribute and invest more in risky assets. All of these lead to a higher plan risk as expressed as funding variation.
To mitigate the longevity effect, as another contribution of this paper, we examine the plan’s optimal longevity risk management decision and compare two longevity risk hedging strategies—the ground-up hedging strategies and the excess-risk hedging strategy. Our optimization results show that the longevity hedge ratio is negatively related to the hedging cost. The plan tends to hedge more with the excess-risk strategy due to its lower total hedging cost and more attractive structure. Our results also explain the failure of the 2004 EIB longevity bond and the emergence of longevity swaps in recent years: The EIB bond has a design similar to the ground-up strategy and the longevity swaps have the same structure as the excess-risk strategy. As far as we know, this is the first paper that incorporates longevity risk management into a pension plan’s asset-liability optimization problem.

We believe this paper presents the starting point of a new way of thinking about pension longevity risk management in the optimization context. As such, it leaves some questions unanswered and in turn opens lines for further research. First, notice that we only consider pension longevity risk management. This is based on the assumption of no hedging on pension funds investment. We would likely obtain richer results from the model in which the plan makes hedging decisions on both asset and liability risks. Second, we study the plan’s optimization decision at one moment. It would be of interest to investigate a multi-period decision model for adjusting contribution and asset allocation in each period such as Bogentoft et al. (2001). Further questions involve to what extent our model manages the plan’s downside risk. In this paper, to some extent, we reduce the plan’s extreme risk by minimizing its funding variation. But to directly control the downside risk, we could include a condition value at risk constraint to our optimization problem. We leave these questions for future research.

REFERENCES


