Abstract
This document deals with three types of interest rate financial derivatives traded in OTC international financial markets: bond options, caps - floors interest rates, and swap options. Then, some financial characteristics of these instruments are shown, mathematical details are explained to calculate their prices from martingales in the standard market model, and the valuation of computational examples are illustrated using the DerivaGem software. Therefore it shows, that having a financial, mathematical and computational perspective of these products and their corresponding valuation, provides reliable information for either the hedger or the issuer.

Keywords: Martingales, Stochastic Processes, Derivatives, Interest Rates.

Resumen
Este documento hace alusión a tres tipos de derivados financieros de tasas de interés que se negocian internacionalmente en el mercado OTC: opciones sobre bonos, caps - floors de tasas y opciones sobre swaps. Para ello, se realiza una descripción financiera de cada producto, se explican detalles matemáticos para calcular sus precios a partir de martingales en el modelo estándar de mercado, y se desarrollan algunos ejemplos computacionales de valoración usando el software DerivaGem. Se muestra así, que tener una perspectiva financiera, matemática y computacional de los productos y sus respectivas valoraciones, proporciona información confiable, o bien para un coberturista, o bien para un emisor.

Palabras Clave: Martingalas, Procesos Estocásticos, Derivados, Tasas de Interés.

*Professor University of Antioquia, School of Economic Sciences, Department of Statistics and Mathematics. Medellín - Colombia. E-mail cgrajal1@gmail.com
1 Introduction

In the Black - Scholes - Merton model [1], for european options pricing, it is established a set of assumptions, among them the one that states that free interest rate remains constant for the maturity of the financial instrument. Furthermore, such option pricing starts from the principle known as traditional risk neutral valuation. However, relaxing the assumption of constant rate is more consistent with the observed reality, on which it is understood that the dynamic rate obeys a stochastic nature. In addition, such action leads to new portfolio hedging strategies and new formulations for pricing interest rate derivatives.

So, in this new context, it is necessary to emphasize the importance of a result inherited from the theory of measure, called the equivalent martingale measure, from which it is possible to price a financial derivative, \( f \), through the extension of the principle of traditional risk neutral pricing and the establishment of a unit of measure, which is given by the price of a financial asset, \( g \), that is called numeraire. On the other hand, a stochastic process, \( X \), is a martingale if it has a zero drift, that is, \( dX = \sigma dw \), for a constant \( \sigma \) and \( w \) given by a Wiener process or a Brownian motion. So the principle mentioned states that the ratio \( f/g \) is a martingale in a world defined by the market price of risk of the asset underlying the derivative \( f \).

Due to this principle it is possible to evaluate three types of interest rate derivatives, traded in the international OTC market: bond options, caps - floors interest rates, and swap options. A long position in an bond option gives the right to buy the bond at a given date and at a given price. A cap (floor) contract provides hedging in the case that the floating rate, in a floating rate note, is greater (less) than a particular level of interest rate. On the other side, a swap option gives the holder the right to enter into a swap on a given date, to exchange a fixed rate for a floating rate, both rates applied over a certain capital. The valuation of such derivatives is framed in a generalized Black 76 model [2], from which the name of standard market model comes [3].

Section 2 establishes the principle of equivalent martingale measure in order to value the financial instruments mentioned. Section 3 illustrates the valuation of a cap derivative and it some of its sensitivity price measures are identified in a graphical way by using DerivaGem free software, all of it in order to provide valuable information for the issuer or the hedger of the instrument. Finally, the conclusions and agenda for future work are provided.

2 Interest rate derivatives. Martingales and Numeraires

First assume that a derivative, \( f \), depends on a single underlying asset, \( \theta \), which pays no dividends, and that their dynamics are given by
\[ d\theta = m\theta dt + s\theta dw \]
\[ df = \mu f dt + \sigma f dw \]

where \( m, s, \mu, \sigma \) are functions of \( \theta \) and \( t \); \( m \) and \( \mu \) represent, respectively, the growth rate of \( \theta \) and \( f \), while \( s \) and \( \sigma \) are the volatilities of the same variables. Consequently, it can be proved that any derivative \( f \) satisfies the relation \((\mu - r) = \lambda \sigma\). The constant \( \lambda \), defined in that way, is called the market price of risk of \( \theta \), and this is used to define the derivative expected excess return regarding the risk free rate, per unit of risk \( \sigma \). A traditional risk neutral measure (or a traditional risk neutral world) is defined as one where \( \lambda \) is zero, and therefore, \( \mu = r \). Alternatively, a measure (a world) can be chosen where \( \lambda \) is nonzero, and this implies a redefinition of the growth rate of \( f \) but not its volatility. The choice of the value of \( \lambda \) is also known as a change of measure. A particular value of \( \lambda \) leads to the real world, while if \( \lambda \) is equal to the volatility of the numeraire, the target world is called forward risk neutral measure (or forward risk neutral world) in regards to the numeraire, and this is how someone can travel through different worlds. Thus, the martingale change of measure states that, if \( \lambda \) is the volatility of the numeraire, then the ratio \( f/g \) is a martingale for all \( f \), i.e.,

\[ f_0 = g_0 E_g \left[ \frac{f_T}{g_T} \right] \quad (1) \]

where the subscripts 0 and \( T \) denote, respectively, the time \( t \) in which the related asset is valued, and \( E_g \) is the expected operator in the world defined by the volatility of the numeraire.

### 2.1 Traditional risk neutral measure

Usually, the numeraire \( g \) is taken as a monetary account in which a monetary unit, \( g_0 = 1 \), is capitalized at \( g_T = 1 \exp(rt) \) during time \( dT \), where \( r \), in a simpler way is deterministic. However, assuming a stochastic rate \( r_T \), the numeraire takes the form \( dg = r g dt \), and so, \( g_T = 1 \exp \left( \int_0^T r dt \right) = \exp(\tau T) \), where \( \tau \) is the average of the realized rate between 0 and \( T \). Finally, according to (1), like the volatility of the numeraire is zero, it means \( \lambda = 0 \), and for any derivative \( f \), \( f/g \) is a martingale in a traditional risk neutral world, i.e.,

\[ f_0 = \hat{E} [\exp (-\tau T) f_T] , \quad (2) \]

Where, for this equation, the symbolism \( E_g \) has been changed for \( \hat{E} \) for convenience of identification. It is in this context, where is valued a future price, \( F_{fut} \), of a underlying asset under a futures contract; there are also options valued on either stocks, indices, currencies or futures. The above contracts are often traded on a stock exchange.
2.2 Forward risk neutral measure

Alternatively to the above choice of numeraire, it is possible to opt for numeraire $g$ as the price at time $t$ of a zero coupon bond, $P(t,T)$, which pays 1 at maturity $T$. In consequence with Eq. (1), we have

$$f_0 = P(0,T)E_T[f_T],$$

(3)

where the symbolism $E_g$ has been changed for $E_T$ for convenience of identification. In this setting, for instance, either the forward price, $F_{fw}$, of the underlying asset in a forward contract or the price of an bond option are valued. It follows that if $f$ is the price of an option bond, then $f/P(t,T)$ is a martingale in a forward risk neutral world with respect to the numeraire $P(t,T)$.

However, a special case occurs when the derivative $f$ is a forward interest rate. If at time $t$, such rate is $R(t,T,T^*)$, applicable and composed at a later period of time $[T,T^*]$, and if $P(t,T^*)/P(t,T)$ is the forward price of a zero coupon bond in the same period, then it can be obtained that the forward rate is a martingale in a forward risk neutral world with respect to the zero coupon bond price $g_t = P(t,T^*)$, i.e.,

$$R(0,T,T^*) = E_{T^*}[f_T/g_T],$$

(4)

where $f_t = [P(t,T) - P(t,T^*)] / (T^* - T)$, and the symbolism $E_g$ has been modified for $E_{T^*}$ for convenience of identification. This case turns out to be suitable for valuing cap and floor interest rate contracts, which, respectively, are appropriate for hedging, at a time $t < T$, against an important rise or drop of the spot rate $R(T,T,T^*)$ compared to a fixed rate level $R_k$.

Lastly, it is considered that the derivative $f$ is an interest rate swap option. First, time $t$ is assumed to be the current time, the swap contract starts at $T_0 > t$, and the cash exchange between the companies that agreed the swap, take place at the dates $T_1, T_2, \ldots, T_N$. Furthermore, it is recognized that, for the position that receives the fixed rate, the value of the swap, $V_{sw}$, is given by $V_{sw} = B_{Tx} - B_{fl}$, where $B_{Tx}$ and $B_{fl}$ are the associated bond prices into the swap, in fixed and floating rates, respectively. In this way, if $s(t)$ is the forward swap rate, with $t \leq T_0$, then the fixed bond price at $t$ is $B_{t,fx} = P(t,T_N) + \sum_{i=0}^{N} s(t) (T_{i+1} - T_i) P(t,T_{i+1}) = 1P(t,T_N) + s(t)A(t)$, with $A(t) = \sum_{i=0}^{N} (T_{i+1} - T_i) P(t,T_{i+1})$. Meanwhile, if LIBOR is the floating rate bond reference, the value of the floating bond at $t$ is $B_{t,fl} = 1P(t,T_0)$, as each payment date into the swap, its value is the par. It is inferred that the forward swap rate is $s(t) = [P(t,T_0) - P(t,T_N)]/A(t)$. So $s(t)$ and $f/A(t)$ are both martingales in a forward risk neutral world regarding numeraire $A(t)$, and in particular it is deduced that

$$f_0 = A(0)E_A[f_T/A(T)],$$

(5)

where $E_g$ symbolism has been changed for $E_A$. for ease of use. The expected values appeared in equations (3), (4) and (5) are computed under the assumptions of the so called Standard Model, which takes for granted, that the expected
value of a market variable, $V_T$, at time $T$, under appropriated measure, follows
a lognormal distribution, with mean equal to its forward value, and being $\sigma \sqrt{T}$
the deviation of $\ln V_T$. As last point, it is assumed that the derivative payoffs are
discounted to present value with risk free rate. Moreover, derivative contracts
illustrated in this section are widely traded on international over the counter
markets.

3 Case: Cap rate pricing

This section develops an example to price a cap interest rate contract. These
results are a general form of the financial famous model Black 76 [2]. In this
case, the fixed rate $R_K$ is known as cap rate, and the realized rate, both defines
the benchmark comparison to $R_K$ and assumes a stochastic nature, which is
usually associated with the LIBOR rate observed at specified periods of time.
The period length is baptized tenor. Thus, for the cap rate design, the basic
elements to consider are $R_K$, the tenor, the benchmark rate $R_k$ (LIBOR) and the
contract maturity $T$. Now, if the inception of the contract is at $t_1$ and its payoffs
dates are at $t_2, t_3, \ldots, t_n, t_{n+1} = t_1 + T$, then a cap rate long position makes, at
$t_k+1$, the payoff $c_{k+1,k+1} = L \tau (R_k - R_K)^+$, $k = 1, \ldots, n$, where $\tau = t_{k+1} - t_k$,
$L$ is the principal on which interest is earned, and $R_k = R(t_k, t_k, t_{k+1})$.
Consequently, the amount $c_{t,k+1}$, seen at time $t < t_k$, can be understood as a
european call option on LIBOR, where the rate is observed at maturity $t_k$, and
where the strike price is $R_K$. In summary, the cap rate contract is a portfolio
of $n$ european call options with the features described. Now, to compute the
cap price $c_{t,k+1}$, at time $t$, it is convenient to use the numeraire $g_t = P(t, t_{k+1})$,
and set up the ratio $c/g$ as a martingale in a forward risk neutral with respect
to $g$. Therefore, according to [1], $c_{0,k+1} = P(0, t_{k+1})E_{k+1} [c_{k+1,k+1}]$. The next
sequences of inequalities are put in order to achieve the value of $c_{0,k+1}$,

\[
c_{0,k+1} = L \tau P(0, t_{k+1})E_{k+1} [(R_k - R_K)^+] = L \tau P(0, t_{k+1}) [E_{k+1} (R_k) N(d1) - R_K N(d2)] \\
= L \tau P(0, t_{k+1}) [E_{k+1} (R(t_k, t_k, t_{k+1})) N(d1) - R_K N(d2)] \\
= L \tau P(0, t_{k+1}) [E_{k+1} (R(t_0, t_k, t_{k+1})) N(d1) - R_K N(d2)] \\
= L \tau P(0, t_{k+1}) [F_k N(d1) - R_K N(d2)]
\]

where $F_k = R(t_0, t_k, t_{k+1})$, $d1 = [\ln (F_k/R_K) + \sigma_k^2 t_k/2] / (\sigma_k \sqrt{t_k})$, $d2 = d1 - \sigma_k \sqrt{t_k}$, and $\sigma_k \sqrt{t_k}$ is the volatility of the random variable $\ln (R_k)$ where $R_k$ is
assumed lognormal. In practice, $F_k$ is calculated from the zero curve LIBOR / SWAP.

In relation to previews discussion, the DerivaGem software [5] is a financial
derivatives free calculator that runs in Excel, and downloadable from the web page
www.rotman.utoronto.ca/ hull/software/. The preceeding lines help to
understand the different inputs and outputs of the software in order to price
a cap rate. The current valuation method is identified in the software under denomination *Pricing Model Black European*.

To numerically illustrate the valuation, according to the current notation, consider that we have the ensuing data: \( L = 100, \ t_1 = 0, \ T = 5 \) years, \( R_K = 5\% \), \( \sigma_k = 20\% \), and quarterly tenor. In addition, the zero curve assumed is \((0, 3.0\%), (1, 3.6\%), (2, 4.0\%), (3, 4.6\%), (4, 4.9\%), (5, 5.20\%)\). Thereby, the premium value obtained is roughly 3.7850, enough for hedging each of the caplets into the cap, against the likelihood, LIBOR rate rise above the 5% level on each payment date. Other outcomes generated by the software are shown in the four charts in figure 3, which exhibits from left to right and top to bottom, respectively, the price sensitivity respect the variation to the cap rate \( R_K \), to time, to the zero coupon curve when it is disturbed by a slight parallel movement, and to the volatility \( \sigma_k \). Each finding reflects a nonlinear relationship between the variables considered and allows to see that, although the cap rate contract can become in a hedging instrument for the hedger, it has a high delta and vega risk for the issuer, and those risk must to be hedge. In particular, could be measured in more detail the delta risk of the contract against a widespread shift (not parallel) in the zero coupon bond. In this sense a short rate model could be chosen and simulated under a suitable forward risk neutral measure, in order to generate a new term structure of interest rate.

**Figure 1**: Price and sensitivity of a cap rate derivative
Conclusion

Through a logic argument, some mathematical expressions have been used to value three interest rate derivatives, namely, bond options, caps - floors rates, and swap options, under suitable forward risk neutral measures and starting from the equivalent martingale measure principle. In particular, it has shown an argument that begins from a forward risk neutral measure, with respect to a zero coupon bond, and provides an analytical and straightforward formula for cap rate pricing. The DerivaGem software was employed to figure out the price of a given cap interest rate derivative, and to explore some measures of price sensitivity. The results suggest a high delta and vega risk, all of which can provide valuable information for either the hedger or the derivative issuer. Furthermore, a possible managerial and educational methodology to measure the delta risk more accurately, and of course, to cope with it, might be at a next working agenda.

References


