OPTIMIZATION OF TIME STRUCTURE OF THE INVESTMENT PROJECT

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ABSTRACT

Problems of optimization of numerical characteristics of the investment project with the fixed payments and controlled moments of payments are considered. A Net Present Value is used as optimized characteristic. The effective algorithms are developed for the most important case when the discount-factor is constant, corresponding examples are given.

Keywords: investment project, Net Present Value, time structure.

The investment project $C [1,2]$ is a finite family of pairs $\{(c_i, t_i)\}$ where $t_0 < t_1 < \ldots < t_n$ are the moments of time and $c_i$ are the payments to the investor at the corresponding time moments (values $c_i$ are nonzero, there are positive and negative among $c_i$, negative $c_i$ correspond to the payments made by the investor). Vectors $C = (c_0, c_1, \ldots, c_n)$ and $R = (t_0, t_2, \ldots, t_n)$ we name a vector of payments and a time structure of the project accordingly. Financing of the project is carried out by the investor in the real financial environment. It is supposed that the investor has an opportunity of an alternative investment of means (in bank) and compares project’s financial effect with the alternative investment effect. Analytically this circumstance is reflected in the using of continuous discount function $f(t)$ defined for $t \geq 0$. It means that the sum $c/f(t)$ when $f(t)>0$ enclosed in bank at the initial time moment is equal to $c$ at the moment $t$. Function $f(t)$ is non-negative, not growing, $f(0) = 1$. We admit zero values of function of discounting - it corresponds to a default situation. Stability of financial conditions when the bank interest does not vary in time means that $f(t) = v^t$ where $v<1$ is a discount - factor.

Sometimes investor has the right to adjust time of payments in the certain frameworks.
The financial effect of the investment project \( C \) at function of discounting \( f(t) \) can be estimated as 
\[
NPV(C, R, f(t)) = \sum_{i=0}^{n} c_i f(t_i) \quad [1,2].
\]

Let's consider the following problems.

Problem 1. There are given numbers \( \tau_1, \tau_2, \ldots, \tau_n \) - minimally allowable time intervals between consecutive payments and the time horizon \( T \) – the time upper bound of the ending of project with a vector of payments \( C \). It is necessary to find time structure \( R = (t_0, t_1, \ldots, t_n) \) such that \( t_i - t_{i-1} \geq \tau_i \quad (i=1, \ldots, n) \), \( t_n \leq T \) and \( NPV \) of the project \( \{(c_i, t_i)\} \) is maximal.

Problem 2. Pairs of numbers \( \tau_1', \tau_1''; \tau_2', \tau_2''; \ldots; \tau_n', \tau_n'' \) are given - minimal and maximal allowable time intervals between consecutive payments i.e. inequalities \( \tau_i'' \geq t_i - t_{i-1} \geq \tau_i' \) should be carried out and the first payment should be made at the zero moment.

The set of allowable time structures, i.e. the structures satisfying the restrictions, obviously is compact subset of \( R^{n+1} \). This subset for a problem 2 is always nonempty, for a problem 1 it is so, if
\[
\sum_{i=1}^{n} \tau_i \geq T.\quad NPV \text{ as continuous function on the set of allowable time structures achieves a maximum in both problems. The optimum time structure can be nonunique, it is true for problems which are considered further too.}
\]

Some properties of solutions of these problems are described here and simple algorithms are constructed. We shall preliminary notice the following.

i. \( NPV \) of a project does not decrease at translation of positive payment \( c_i \) to some earlier time, as \( c_i f(p) \geq c_i f(q) \) at \( p < q \).

ii. Similarly, \( NPV \) of a project does not decrease at translation of negative payment to some later time.

PROPOSITION 1. Among optimal time structures for problems 1 and 2 exists such that for each payment (except for the first and last) even one of two adjacent time intervals is extreme (i.e. it is equal to \( \tau_i \) for a problem 1 and \( \tau_i' \) or \( \tau_i'' \) for a problem 2).

PROOF. If for the payment \( c_i \) this condition is wrong it can be displaced on an earlier and on later time moments. If \( \alpha \) is the time interval before payment \( c_i \) and \( \beta \) before payment \( c_{i+1} \) \( \alpha \geq \tau_i, \beta \geq \tau_i+1 \) for a problem 1 and \( \tau_i'' > \alpha > \tau_i', \tau_i'' > \beta > \tau_i' \) for a problem 2) then the moment of the payment \( c_i \) can be chosen as
- \( t_{i-1} + \tau_i \) at \( c_i > 0 \) for a problem 1,
- \( t_{i+1} - \tau_{i+1} \) at \( c_i < 0 \) for a problem 1,
- \( t_{i-1} + \tau'_i \) at \( c_i > 0, \tau''_i + \tau'_i \leq \alpha + \beta \) for a problem 2,
- \( t_{i-1} - \tau'_i \) at \( c_i > 0, \tau''_i + \tau'_i \geq \alpha + \beta \) for a problem 2,
- \( t_{i-1} + \tau'_i \) at \( c_i < 0, \tau''_i + \tau'_i \geq \alpha + \beta \) for a problem 2,
- \( t_{i-1} - \tau'_i \) at \( c_i < 0, \tau''_i + \tau'_i \leq \alpha + \beta \) for a problem 2.

It is obvious, that the received time structures are allowable and \( NPV \) of the received projects is not less than of initial one by properties i and ii.

REMARK 1. It is not necessary for the optimal time structure for a problem 1 that payment \( c_0 \) is realized at the moment 0 or payment \( c_n \) at the moment \( T \).

EXAMPLE 1. Let function \( f(t) \) be piecewise linear and is equal to 1 at \( t=0; 1/2 \) at \( t=6; 1/4 \) at \( t=10; 0 \) at \( t \geq 13 \). Let’s consider a vector of payments \((-1,1), \tau_1=1, T=12\). It is easy to check up that the optimum time structure is \((a, a+1)\) for \( 6 \leq a \leq 9 \).

REMARK 2. All intervals are not necessary maximal or minimal in optimal time structure for a problem 2.

EXAMPLE 2. Let function \( f(t) \) be piecewise linear is equal to 1 at \( t=0; 1/2 \) at \( t=3; 0 \) at \( t \geq 8 \), a vector of payments \((-1,1, -1,1)\) and admissible intervals between payments are 1-2, 1-10, 1-2. It is easy to check up that the optimal time structure is \((0,1, a, a+1)\) for \( 3 \leq a \leq 7 \), i.e. the interval between 2 and 3 payments does not coincide with the extreme values 1 or 10.

REMARK 3. One can see that interval between arbitrary positive payment and previous payment is always minimal in optimum time structure for a problem 1.

The optimal time structure can be described more precisely in the major special case of the function of discounting \( f(t) = \nu^t \). In particular the situations reflected in examples 1 and 2 are impossible. We add one more to constructions i, ii.

iii. Let’s consider some payments with the fixed time intervals between them \((c_1,t_1), (c_2,t_2), \ldots \) \((t_2-t_1=a_1, \ldots \)). The sum of discounted values of these payments is equal to \( c_1\nu^{t_1} + c_2\nu^{t_2} + c_3\nu^{t_3} + \ldots = \nu^{t_1}(c_1 + c_2\nu^{a_1} + c_3\nu^{a_1+a_2} + ...) \), i.e. for the counting of \( NPV \) such sequence of payments can be considered as one payment equals to \( c_1 + c_2\nu^{a_1} + c_3\nu^{a_1+a_2} + \ldots \) at the moment \( t_1 \) (merge of payments).

A construction such as iii is possible only for exponential functions of discounting. Further the major is that the sign on expression \( c_0f(t)+c_1f(t+a_1)+\ldots \) is kept at any values of parameters \( c_0, \ldots; a_1, \ldots \). We’ll consider the similar sums containing only two summands, i.e. we’ll suppose that the sign of expression \( c_0f(t)+c_1f(t+a) \) is kept at change \( t \) for any fixed values \( c_0, c_1, a \). This condition is equivalent to the following: “if \( c_0f(t)+c_1f(t+a)=0 \) for some \( t \) then it is true for arbitrary \( t' \)” as a
consequence of a continuity of function \( f(t) \). First of all, function \( f(t) \) should be positive: by monotonicity if \( f(p) = 0 \) at the some \( p \) then a discount function is equal to 0 on the set \([p, \infty)\). A function \( f(t) \) is not constant as \( f(0) = 1 \). If \( f(s) \neq f(s+a) \) then for \( c_0=1, \ c_1=-1 \) we have

\[
c_0 f(p) + c_1 f(p+a) = 0, \quad c_0 f(s) + c_1 f(s+a) \neq 0.
\]

Equivalence of equalities \( c_0 f(p) + c_1 f(p+a) = 0 \) and \( c_0 f(q) + c_1 f(q+a) = 0 \) gives equality

\[
\frac{f(p+a)}{f(p)} = \frac{f(q+a)}{f(q)}
\]

for any \( p, q, a \). But then we have \( f(p+a) = f(p)g(a) \). If \( p=0 \) then \( f(a) = g(a) \) i.e. a discount function satisfies to the standard functional equation which only solutions are exponents.

It is possible to reduce our problems to linear optimization problems in the case of constant discount-factor. We shall use designations \( u_i = v^i \) for this purpose.

Problem 1.

To maximize the linear function \( \sum_{i=0}^n c_i u_i \) if \( u_{i+1} \leq v^i, \ u_i, u_n \geq v^T \).

Problem 2

To maximize the linear function \( \sum_{i=0}^n c_i u_i \) if \( v^T, v \leq u_{i+1} \leq v^T, \ u_i, u_0 = 1 \).

Simple algorithms which are described further are based on substantive reasons and do not use these representations.

Let's consider some tolerable time structure for the problem 1. We'll fix the moments of negative payments. \( NPV \) of the project will not decrease at moving of positive payments to the earliest possible moments according to property i. For example, the part of the initial project \{(-50, 6), (30, 8), (60, 12)\} with minimally allowable intervals between the payments to be equal to 1 and 3, will be replaced on \{(-50, 6), (30, 7), (60, 10)\} at such transformation.

Result of such operation is the project with fixed time intervals between negative and the subsequent (possible, several) positive payments. It is possible to apply using property iii the operation of payments merge to these payments. Thus we replace the initial investment project by the project with in general smaller number of payments. The same procedure can be applied to new project. Continuing this process we shall finally receive the project where after arbitrary negative payments positive are not going. Such project has one of the following structures:

1. The first payments are positive, all other are negative;
2. All payments are positive;
3. All payments are negative.
In the first case NPV of the project does not increase according to properties i and ii at realization of all positive payments at the earliest tolerable moments of time and negative at the latest. In the second case payments must be necessary realized at the earliest tolerable and in the third at the latest tolerable time moments.

Thus for the initial project we have the following.

PROPOSITION 2. The optimal time structure for a problem 1 is realized even for one of the projects with the following time structure. At some value \( k = 0, 1, \ldots, n+1 \) such that for \( n+1 > k > 0 \) inequalities \( c_{k-1} > 0, \ c_k < 0 \) are fair, payments for \( i < k \) are realized at the earliest and for \( i \geq k \) at the latest tolerable time moments.

Thus the simple algorithm for the solving of a problem 1 is obtained: it is necessary to compare NPV of the projects with the time structures corresponding to various such values of \( k \).

EXAMPLE 3. Let’s consider a problem 1 for a vector of payments \((-50,100,60,-200,300)\) with the minimal time intervals between the consecutive payments be equal to \(2,1,3,2\) and \(T=12\). By the proposition 2, the optimal investment project is one of the next three, given in the table:

<table>
<thead>
<tr>
<th></th>
<th>50</th>
<th>100</th>
<th>60</th>
<th>200</th>
<th>300</th>
<th>(NPV, v=0.5)</th>
<th>(NPV, v=0.8)</th>
</tr>
</thead>
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<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>8</td>
<td>-19.4</td>
<td>42.8</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>10</td>
<td>12</td>
<td>-17.6</td>
<td>43.86</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>7</td>
<td>10</td>
<td>12</td>
<td>-1.2</td>
<td>17.5</td>
<td></td>
</tr>
</tbody>
</table>

\(NPV\) of projects are given in the last columns for two values of discount - factor. The third project is optimal for \( v=0.5 \), and the second one for \( v=0.8 \). Naturally the optimal time structure depends on a choice of discount-factor (norm of discount). The choice of the norm of discount is an independent economical problem.

Let's note some properties of optimal time structures for a problem 1.

1. The index of transition \( k \) does not change with a change of time horizon \( T \). It is a sequence of construction. Thus, it is easy to count the optimal time structure of the project at changing \( T \) on initial.

2. The index \( k \) can vary at change of interval boundaries between payments.

EXAMPLE 4. Let the vector of payments be \((-30,40,-60,70)\) with the minimal time intervals be equal to \(2,1,1\); \(T=10\); discount-factor \( v=0.9 \). It is easy to check up that the project \{(-30,0),(40,2),(-60,3),(70,4)\} is optimal. We shall replace the last of the minimal intervals on 3.
Then the optimum project will become \{(-30,0),(40,2),(-60,7),(70,10)\}. Thus, changes of time boundaries between negative payments does not influence on an index \(k\).

The solution of a problem 2 at constant discount-factor is described on the base of the given constructions.

**PROPOSITION 3.** In the optimal time structure for a problem 2 an interval between payments \(c_{k-1}\) and \(c_k\) is minimally allowable if \(\sum_{i=k}^{n} c_i v^i > 0\), it is maximally allowable if \(\sum_{i=k}^{n} c_i v^i < 0\). If this sum is equal to zero then this interval can be anyone - it does not influence on \(NPV\) of the project.

**PROOF.** We’ll carry out an induction on number of payments. The statement is obvious for two payments. Let the statement be true for a vector of payments \(C=(c_1,\ldots,c_n)\). We’ll prove the statement for a vector of payments \(C=(c_0,c_1,\ldots,c_n)\). Let it be \(R\) - the time structure described in the proposition, \(R_1\) - any time structure and \(R_1'\) - its part corresponding to the payments \((c_1,\ldots,c_n)\). We have \(NPV(C,R,v)=c_0+\sum_{i=1}^{n} v^i NPV(C_1,R_1',v)\) by the property iii. There is \(NPV(C_1,R_1',v)\geq NPV(C_1,R_1',v)\) under the assumption of induction. The necessary conclusion is true in view of a sign on value \(NPV(C_1,R_1',v)\) and monotoncity of positive function \(v^i\).

The following simple algorithm for solution of the problem 2 follows from the proposition 3. Let's choose some conditional time moment \(t_n\) for the last payment. Then by an induction we take \(t_{k-1}:=t_k-\tau_k'\) if \(\sum_{i=k}^{n} c_i v^i > 0\) and \(t_{k-1}:=t_k-\tau_k''\) if \(\sum_{i=k}^{n} c_i v^i \leq 0\). At last, we find final time structure: \(t_k:=t_k-t_0\).

**EXAMPLE 5.** Let consider a problem 2 for the vector of payments \((-50,55,-100,130,-70,90)\) with bounds of time intervals between consecutive payments 1–3, 2–4, 2–4, 1–3, 3–5 accordingly and \(v=0.7\). We take temporarily \(t_5=10\). As \(c_5=90>0\) then \(t_4=10-3=7\). As \(-70v^7+90v^{10}=-3.22<0\) then \(t_3=7-3=4\). As \(130v^4-70v^7+90v^{10}=27.9>0\) then \(t_2=4-2=2\). Further, \(-100v^2+130v^4-70v^7+90v^{10}=3.98>0\), hence, \(t_1=2-2=0\). At last, \(55v^0-100v^2+130v^4-70v^7+90v^{10}=58.98>0\) and \(t_0=0-1=-1\). Thus, in optimum time structure \(t_i = t_i+1\).

Finally we receive optimum time structure: \((0,1,3,5,8,11)\).

Similarly one can construct an optimum time structure in the following problem.
Problem 2a. There are given pairs of numbers \( \tau_1', \tau_1''; \tau_2', \tau_2''; \ldots; \tau_n', \tau_n'' \) as in a problem 2 and the moment of the last payment. It is necessary to find the time structure when \( NPV \) of the project is maximal.

The time structure should be constructed on increasing, i.e. it is necessary to calculate the sums \( \sum_{i=0}^{k} c_i v^i \). in contrast to the described algorithm.

Let's consider one more variant of a problem 2.

Problem 2b. The moment \( T \) of the last payment is fixed as addition to all conditions of a problem 2. It is necessary to find the time structure when \( NPV \) of the project is maximal.

The following statement is true.

PROPOSITION 4. Among optimal time structures for a problem 2b there exists such that no more than one time interval between payments is not extreme.

PROOF. Let it be two nonextreme time intervals: \( t_{k+1} - t_k \) and \( t_{s+1} - t_s \) where \( s > k \). We shall consider the sum \( S = \sum_{i=0}^{s} c_i v^i \). If \( S > 0 \) then it is possible to reduce all time moments \( t_{k+1}, \ldots, t_s \) on some value and thus \( NPV \) of the project grows. If \( S < 0 \) then it is possible to increase these moments with the same effect at last time, if \( S = 0 \) then it is possible to reduce number of nonextreme time intervals at preservation \( NPV \) by shifting of the specified payments.

It is obvious that the problem 2b has the solution only when \( \sum_{i=1}^{n} \tau_i' \leq T \leq \sum_{i=1}^{n} \tau_i'' \). It follows from the proposition 4 that it is necessary to search the optimal time structure among only finite set of time structures.

The algorithm of the solution of a problem 2b is based on the following statement.

PROPOSITION 5. The time structure for a problem 2b is nonoptimal if and only if there are numbers of payments \( s > k \) such that \( \sum_{i=0}^{s} c_i v^i > 0 \) and the interval \( t_{k+1} - t_k \) is not minimal and \( t_{s+1} - t_s \) is not maximal or \( \sum_{i=0}^{s} c_i v^i < 0 \) and the interval \( t_{k+1} - t_k \) is not maximal and \( t_{s+1} - t_s \) is not minimal.

It is necessary to prove only sufficiency as follows from the proof of the proposition 4. We consider any nonoptimal time structure \( (0 = t_0, \ldots, t_n = T) \) and compare it with the optimal one \( (0 = T_0, \ldots, T_n = T) \). We find such numbers of payments \( s > k \) that \( t_{k+1} - t_k > T_{k+1} - T_k \), \( t_{s+1} - t_s < T_{s+1} - T_s \) (for definiteness we'll suppose that it is so) or on the contrary and a difference \( s - k \) is minimal among
such pairs of indexes. The proposition is true if \( \sum_{i=k+1}^{s} c_i v^i > 0 \). The assumption \( \sum_{i=k+1}^{s} c_i v^i < 0 \) is in contradiction with an optimality of time structure \((T_0, \ldots, T_n)\). It is possible to change time structure of nonoptimal project when \( \sum_{i=k+1}^{s} c_i v^i = 0 \) with preservation of \( NPV \) and reducing of the number of indexes \( k \) with the property \( t_{k+1} - t_k \neq T_{k+1} - T_k \). Continuing process of transformation of time structure in the last case, we’ll come to the condition \( \sum_{i=k+1}^{s} c_i v^i \neq 0 \) on some step because the project is nonoptimal. The proposition is proved.

The algorithm of constructing of optimal time structure is the following. The initial time structure can be constructed in particular as follows.

Let \( k = \max \left\{ s : \sum_{i=0}^{s} \tau_i'' + \sum_{i=s+1}^{n} \tau_i' \leq T \right\} \). We assume that initial time structure is

\[
t_j = \begin{cases} 
\sum_{i=1}^{j} \tau_i'' & \text{when } j \leq s \\
T - \sum_{i=s+2}^{n} \tau_i' - \sum_{i=1}^{s} \tau_i'' & \text{when } j = s + 1 \\
T - \sum_{i=j+1}^{n} \tau_i' & \text{when } j \geq s + 2
\end{cases}
\]

Then the following iterative procedure is carried out. If the interval between initial and the next payments is not maximal then there exists an index \( k \) such, that the interval between \( k \)-th and the next payments is not minimal. If \( \sum_{i=1}^{k} c_i t_i' < 0 \) then all these payments will shift forward for extremely possible time and process is initiated anew, the following indexes \( k \) otherwise are looked through.

If the interval between 0 and the first payments is not minimal similar procedure with corresponding changes is made. Then we pass to the second interval. If there was any change of time structure process must start over again.

The described algorithm converges. The algorithm has complexity \( n^2 \). We shall demonstrate an example.
EXAMPLE 6. Let it be the vector of payments $(-30, -40, 60, 120, -30, -50, 70, 90)$, time intervals between all payments varies from 2 up to 6, $T=23$, $v=0.9$. Process of the solving of a problem we shall result as the table.

<table>
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<th>−40</th>
<th>60</th>
<th>120</th>
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The initial time structure constructed by the described method is given in the second line. The moments of time which should be changed are allocated as italic in each line except the last one.

It is interesting to analyze the problems of constructing of the optimal time structure when other characteristics of investment project such as Profitability Index are used.

The research supported by the Russian Foundation for Basic Researches.

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