A STOCHASTIC MODEL FOR THE SUSTAINABLE INVESTMENT POLICY IN A DEFINED BENEFIT PENSION FUND

By

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INTRODUCTION

- We study the optimal asset allocation problem in a defined benefit pension fund that accumulates resources for pension benefit payments and is allowed to invest its capital in a security market described by $n$ risky assets.
- Pension fund operates in continuous time (subject to financial and actuarial risks).
- Assume that the fund asset manager is concerned with taking into account the sustainability of the fund.
- Fund manager wishes to maximize his utility depending on an empiric indicator (ratio).
INTRODUCTION

- This is a finite time horizon control problem solved by using the dynamic programming approach.

- We obtain the Hamilton-Jacobi-Bellman equation and then we derive closed form expressions for the optimal investment policy.

- We perform a numerical application by using real Italian pension fund data.
We deal with a specific type of retirement program present in Italy: **Italian Professional Order Pension Fund (IPOPF)**;

- Private closed schemes; DB; mixed funded/PAYGO financial mechanism.
- Until 1995, IPOPFs were administered directly by the State that would step in in case of insolvency.
- After 1995, private pension plans have managed IPOPFs, without being sponsored by the State;
- The Italian Regulator is concerned with monitoring the financial self-sufficiency of the funds and in particular the financial sustainability established by the wealth management and by the portfolio investment strategy.
LITERATURE

- Over the last years, a vast literature regarding optimal asset allocation in pension fund has been developed. However, the analysis generally takes into account the fully funded system.

- Indeed, at the best of our knowledge, very few papers have dealt with mixed financing mechanisms in pension fund (e.g. see Menoncin, 2005)

- Our model generalizes a previous survey carried out by Hainaut and Devolder that deals with optimal investment strategy in a funded pension plan. They address the case of a fund manager concerned on a solvency the ratio.

- Indeed, in our model the Asset manager cares about a sustainability ratio in order to take into account the Italian legislation indicator (ratio of the fund value to a multiple of the current expenditure for pensions)
We consider a continuous time, complete and frictionless financial market consisting of \( n \) different risky assets:

\[
F_{i,t} \quad i = 1, \ldots, n \quad t \in [0,T]
\]

They are driven by geometric Brownian motions

\[
dF_{i,t} = m_i \cdot F_{i,t} \cdot dt + \sum_{j=1}^{n} \sigma_{ij} \cdot F_{i,t} \cdot dW_{j,t}^F
\]

with \( F_{i,0}, m_i, \sigma_{ij} \in \mathbb{R}^+ \)

Besides, \( \left( W_{j,t}^F \right)_{t \geq 0} \) is a \( (n) \)-dimensional vector Brownian motion defined on a probability space \( \left( \Omega^F, \mathcal{F}^F, P^F \right) \).
PROBLEM FORMULATION

- Assume that the stream of contributions \( \Gamma_t \) and of benefit \( B_t \) are stochastic and modeled by Ito processes:

\[
d\Gamma = \mu_\Gamma(t) \cdot \Gamma_t \, dt + \sigma_\Gamma(t) \cdot \Gamma_t \, dW_t^L \\
 dB = \mu_B(t) \cdot B_t \, dt + \sigma_B(t) \cdot B_t \, dW_t^L
\]

- Besides \( \mu_\Gamma(t), \mu_B(t), \sigma_\Gamma(t), \sigma_B(t) \) are deterministic functions.

- \( (W_t^L)_{t \geq 0} \) and \( (W_t^F)_{t \geq 0} \) are independent Brownian motions.

- The consumption process is given by the net payment stream \( B_t - \Gamma_t \)
PROBLEM FORMULATION

- Given the n-dimensional price process \( \{F_t\}_{t \geq 0} \), a Markovian relative portfolio strategy is any n-dimensional process \( \{\Pi_t\}_{t \geq 0} \) of the form \( \Pi_t = \Pi(t, F_t) \) with \( \Pi: \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n \).
- Add a balance constraint (pension fund borrowing is prohibited).
- The control space \( \mathcal{A} \subset \mathbb{R}^n \) is the set of investment polices s. t.
  \[
  \mathcal{A} = \left\{ \Pi = (\pi_i) : \pi_t^n = 1 - \sum_{i=1}^{n-1} \pi_t^i \right\}
  \]
  With \( \Pi_t \in \mathcal{A}, t \in [0,T] \).
- The fund wealth \( A \) satisfies
  \[
  dA_t = \left( m^T \Pi_t A_t + \Gamma_t - B_t \right) dt + A_t \Pi_t^T \sigma dW_t^F
  \]
  (risky assets have mean \( m_i \) and covariances \( \sigma_{ij} \)).
PROBLEM FORMULATION

- Assume the fund manager operates in a mixed funding-PAYG pension fund → goal of assuring the sustainability of the fund.
- Introduces the **sustainability ratio**: market value of the fund wealth divided by a reserve settled as a multiple of the current expenditure for pensions. Good indicator of sustainability for Italian regulators.
- **Sustainability reserve** of the fund:
  \[ R_t = \alpha \cdot B_t \quad \alpha \geq 0 \]
  \[ \int_{t_0}^{t} dR_t = \mu_B(t) R_t \, dt + \sigma_B(t) R_t \, dW_t \]
- **Sustainability ratio**:
  \[ S_t = \frac{A_t}{R_t} \]
**Proposition.** The process $S_t$ satisfies the SDE:

$$
    dS_t = \left( \left( m^T \Pi_t - \mu_B(t) + \sigma_B(t)^2 \right) S_t + \frac{1}{\alpha} (\psi_t - 1) \right) dt + \\
    \left( \Pi_t^T \sigma dW_t^F - \sigma_B(t) dW_t^L \right) S_t
$$

with $\psi_t = \Gamma_t / B_t$. Apply again Ito’s lemma:

$$
    d\psi_t = \psi_t \left( \mu_{\Gamma}(t) - \mu_B(t) + \sigma_{\Gamma}(t)^2 - \sigma_{\Gamma}(t) \cdot \sigma_B(t) \right) dt + \\
    \left( \sigma_{\Gamma}(t) - \sigma_B(t) \right) \psi_t dW_t^L
$$

- Managed pension fund represented by the stochastic differential equations with initial values $S_0, \psi_0$ and an investment policy $\Pi$ given by $dS_t$ and $d\psi_t \to X = (S, \Psi)$ is the state process.

\[1\]
\[0\]
The fund manager seeks to maximize continuously the utility arising from the sustainability ratio, via an adapted investment policy (the control variable of our model).

The **value function** $\mathcal{V} : R^+ \times R^2 \times \Pi \rightarrow R$ is

$$\mathcal{V}(t, X, \Pi) = \mathbb{E} \left[ \int_0^T U(t, X_{\Pi_t}, \Pi_t) dt + K(X_T^\Pi) \right]$$

where $\Pi$ is the set of admissible controls, $U$ is the utility function and $K$ is the bequest function.

The optimal value function $V : R^+ \times R^2 \rightarrow R$ is

$$V(t, X_t) = \sup_{\Pi \in \Pi} \mathcal{V}(t, X, \Pi)$$

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HJB EQUATION AND MAIN RESULTS

- An **optimal control** law $\Pi^{opt}$ is an admissible control such that $\mathcal{V}(\Pi^{opt}) = V$

- **Proposition.** If we assume that there exists an optimal control and the optimal value function is regular, $V(t, S_t, \Psi_t) \in C^{1,2,2}$
  - $V$ satisfies the HJB equation:

$$
\begin{cases}
V_t + \text{Sup}_{\Pi \in \Pi} \{ U(t, X_t, \Pi) + \mathcal{L}^{\Pi} V \} = 0 & \forall (t, X) \in (0, T) \times \mathbb{R}^2 \\
V(T, X) = U(X_T) & \forall X \in \mathbb{R}^2
\end{cases}
$$

- The supremum in the HJB equation above is achieved by the optimal control law $\Pi^{opt}(t, X)$.  

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THE OPTIMAL INVESTMENT STRATEGY

- We set $\Pi_t = M_1 \cdot \pi_t + M_2$ where $\pi_t$ is the vector of the (n-1) first elements of $\Pi_t$ and

$$M_1 = \begin{pmatrix}
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1 \\
-1 & \cdots & -1
\end{pmatrix}$$

$$M_2 = \begin{pmatrix}
0 \\
\vdots \\
0 \\
1
\end{pmatrix}$$

$$N_1 = M_1^T \cdot \Sigma \cdot M_1$$

$$N_2 = M_1^T \cdot \Sigma \cdot M_2$$

- Let differentiate the Hamiltonian with respect to $\pi_t$:

$$M_1^T \cdot m \cdot S \cdot V_S + S^2 \cdot V_{SS} \cdot M_1^T \cdot \Sigma \cdot (M_1 \cdot \pi_t + M_2) = 0$$

- We deduce:

$$\pi_t^{opt} = N_1^{-1} \cdot \left(- \frac{M_1^T \cdot m \cdot V_S}{S_t \cdot V_{SS}} - N_2 \right)$$
Thus, the complete **optimal strategy of investment** is

\[
\Pi_t^{opt} = -\frac{V^s}{S_t \cdot V_{ss}} \cdot P_1 \cdot m - P_2
\]

where \( P_1 = M_1 \cdot N_1^{-1} \cdot M_1^T \); \( P_2 = M_1 \cdot N_2^{-1} \cdot N_2 + M_2 \)

- Assume that the fund manager wishes to maximize continuously the quadratic utility arising from the sustainability ratio:

\[
U(S_t) = k \cdot S_t - (1-k) \cdot (S_t - TS(t))^2
\]

where \( k \in [0,1] \) is a parameter expressing the fund manager's preference for a large sustainability ratio and \( TS(t) \) is a deterministic function expressing a path for an ideal target sustainability ratio.
THE OPTIMAL INVESTMENT STRATEGY

- We postulate that the solution $V$ shares the same quadratic form

$$V(t, S_t, \Psi_t) = a(t, \Psi) \cdot S_t^2 + b(t, \Psi) \cdot S_t + c(t, \Psi)$$

$$V_S = 2a(t, \Psi) \cdot S_t + b(t, \Psi)$$

$$V_{SS} = 2a(t, \Psi)$$

$$V_t = \partial_t a(t, \Psi) \cdot S_t^2 + \partial_t b(t, \Psi) \cdot S_t + \partial_t c(t, \Psi)$$

- Straightforward calculation lead to the three following partial differential equations for $a(t, \Psi), b(t, \Psi)$ and $c(t, \Psi)$
\[ \partial_t a(t, \Psi) = -2a(t, \Psi) \left( -\mu_B(t) - m^T P_2 + P_2^T \Sigma P_1 m - m^T P_1 m \right) - \]
\[ -a(t, \Psi) \left( P_2^T \Sigma P_2 + \sigma_B(t)^2 + m^T P_1^T \Sigma P_1 m \right) + (1 - k) \]
\[ \partial_t b(t, \Psi) = -b(t, \Psi) \left( -\mu_B(t) - m^T P_2 + \sigma_B(t)^2 + P_2^T \Sigma P_1 m \right) + \]
\[ + \frac{1}{2} \left( m^T P_1 \Sigma m \right) - 2m^T P_1 m - 2a(t, \Psi) \cdot \left( (\Psi - 1) / \alpha \right) - k - 2(1 - k) TS(t) \]
\[ \partial_t c(t, \Psi) = -b(t, \Psi) \cdot \left( (\Psi - 1) / \alpha \right) + (1 - k) TS(t)^2 + \]
\[ + \frac{b(t, \Psi)^2}{2a(t, \Psi)} \cdot \left( m^T P_1 m - \frac{1}{2} m^T P_1 \Sigma P_1 m \right) \]

together with a set of terminal and boundary conditions.
NUMERICAL APPLICATION

- Consider data coming from a real Italian Professional Order pension fund (year 2009, in Euros).

<table>
<thead>
<tr>
<th></th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pension fund’s wealth (A)</td>
<td>1,600,000.00</td>
</tr>
<tr>
<td>Contributions (Γ)</td>
<td>423,898.092</td>
</tr>
<tr>
<td>Benefits (B)</td>
<td>365,481.348</td>
</tr>
<tr>
<td>Sustainability ratio (S) (α = 5)</td>
<td>0.88</td>
</tr>
<tr>
<td>Ψ</td>
<td>1.16</td>
</tr>
</tbody>
</table>

- this fund is in a position of actuarial surplus because the level of contributions exceeds the level of benefits.
- Current benefits multiplier is set equal to 5, according to the ratio monitored by the Italian Ministry.
- We used polynomial estimations for functions representing average benefits amount, average contributions amount and the target sustainability ratio path.
NUMERICAL APPLICATION

- Parameters of the polynomial regressions:

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\mu_B(t)$</th>
<th>$\mu_T(t)$</th>
<th>$TS(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_0$</td>
<td>1</td>
<td>1.0799</td>
<td>0.938425</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>$-1.65 \cdot 10^{-2}$</td>
<td>$6.7 \cdot 10^{-2}$</td>
<td>$-0.117125$</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>$4.3 \cdot 10^{-3}$</td>
<td>$-4.1 \cdot 10^{-3}$</td>
<td>$0.073305$</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>$-4.10 \cdot 10^{-5}$</td>
<td>$2 \cdot 10^{-4}$</td>
<td>$-0.015161$</td>
</tr>
<tr>
<td>$\beta_4$</td>
<td>$-2 \cdot 10^{-6}$</td>
<td>$-2 \cdot 10^{-6}$</td>
<td>$0.001471$</td>
</tr>
<tr>
<td>$\beta_5$</td>
<td>$3 \cdot 10^{-8}$</td>
<td>-</td>
<td>$-6.94 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>$\beta_6$</td>
<td>-</td>
<td>-</td>
<td>$1.27 \cdot 10^{-6}$</td>
</tr>
</tbody>
</table>
NUMERICAL APPLICATION

- We assume that the asset manager aims at investing in 15 financial assets whose expected returns and volatilities are:

<table>
<thead>
<tr>
<th>Monetary</th>
<th>3.5%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bond Gov EMU</td>
<td>4.55%</td>
</tr>
<tr>
<td>Bond Gov World ex EMU</td>
<td>4.55%</td>
</tr>
<tr>
<td>Bond Corp. EU</td>
<td>5.16%</td>
</tr>
<tr>
<td>Bond Corp. USA</td>
<td>5.34%</td>
</tr>
<tr>
<td>Bond Corp. High Yield</td>
<td>6.30%</td>
</tr>
<tr>
<td>Bond Emerg. Markets</td>
<td>6.83%</td>
</tr>
<tr>
<td>Inflation</td>
<td>4.64%</td>
</tr>
<tr>
<td>Equity EU</td>
<td>8.40%</td>
</tr>
<tr>
<td>Equity USA</td>
<td>8.93%</td>
</tr>
<tr>
<td>Equity Pacific</td>
<td>7.96%</td>
</tr>
<tr>
<td>Equity Emerging markets</td>
<td>15.15%</td>
</tr>
<tr>
<td>Hedge Funds</td>
<td>6.65%</td>
</tr>
<tr>
<td>Commodities</td>
<td>9.27%</td>
</tr>
<tr>
<td>Private Equity</td>
<td>12.16%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Monetary</th>
<th>1.0%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bond Gov EMU</td>
<td>5.0%</td>
</tr>
<tr>
<td>Bond Gov World ex EMU</td>
<td>4.9%</td>
</tr>
<tr>
<td>Bond Corp. EU</td>
<td>5.5%</td>
</tr>
<tr>
<td>Bond Corp. USA</td>
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<tr>
<td>Bond Corp. High Yield</td>
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<td>Bond Emerg. Markets</td>
<td>8.3%</td>
</tr>
<tr>
<td>Inflation</td>
<td>2.5%</td>
</tr>
<tr>
<td>Equity EU</td>
<td>19.7%</td>
</tr>
<tr>
<td>Equity USA</td>
<td>19.4%</td>
</tr>
<tr>
<td>Equity Pacific</td>
<td>20.8%</td>
</tr>
<tr>
<td>Equity Emerging markets</td>
<td>23.0%</td>
</tr>
<tr>
<td>Hedge Funds</td>
<td>8.2%</td>
</tr>
<tr>
<td>Commodities</td>
<td>19.5%</td>
</tr>
<tr>
<td>Private Equity</td>
<td>24.2%</td>
</tr>
</tbody>
</table>
NUMERICAL APPLICATION

- We assume that the asset manager sets his investment horizon in fifteen years ($T=15$); this is a long term time span according to pension funds' characteristics.

- In order to correctly describe the asset manager's risk aversion, we set the risk aversion coefficient $k$ equal to two different values, namely to 0.1 and to 0.6.

- We exhibit next the graphical solutions.
Optimal proportions of the fund, invested in the 15 asset classes detected in the previous subsection, for the coefficient $k=0.1$
Value related to the asset class Equity EU, $k=0.1$
NUMERICAL APPLICATION

- Value related to the complete asset allocation, $k=0.1$, $S=0.8$
• Value related to the complete asset allocation, $k=0.1$, $S=1.2$
Optimal proportions of the fund, invested in the 15 asset classes detected in the previous subsection, for the coefficient $k=0.6$
Numerical Application

- Value related to the asset class Equity EU, $k=0.6$
**NUMERICAL APPLICATION**

- Value related to the complete asset allocation, $k=0.6, S=0.8$
Value related to the complete asset allocation, $k=0.6$, $S=0.8$
FINAL REMARKS

- We have investigated the optimal dynamic portfolio allocation problem for a pension fund which operates in a mixed funded/PAYG financing regime.
- Particular set of Italian private closed pension scheme.
- We derived the stochastic dynamic for an empiric indicator which is the ratio of the fund wealth to a multiple of the current expenditure for pensions (the state variable of the problem).
- On the one hand, it is assumed that the asset manager aims at maximizing this empiric ratio, on the other hand we suppose he has an incentive to stay close to a certain target sustainability ratio path.
- Closed-form expressions for the investment strategy have been obtained.
MAIN REFERENCES

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