Pricing S-forwards via the Risk Margin under Solvency II

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Goal of the paper

- We need to estimate the market price of longevity risk (premium that a life insurer or pension plan might be willing to pay to release such a risk)

- In an incomplete market, such as the longevity-linked securities market, it is not possible to estimate a unique market price of risk

- Goal of the paper: investigating the possibility to derive the market price of longevity risk through the risk margin implicit in the evaluation of the technical provisions as required by Solvency II

- Technical provisions calculated as the sum of their best estimate plus a risk margin (the market value of the uncertainty on insurance obligations)

- Some authors suggested to link the pricing to the amount that the insurer should hold to cover unexpected losses. E.g.:
  - Börger (2010): risk margin considered as the maximum price a life insurer would be willing to pay to transfer LR via securitization
Mortality data coming from Italian male population: period 1974-2007, age 60-90

Mortality model chosen between the ones compared in Cairns et al. 2008

<table>
<thead>
<tr>
<th>Model</th>
<th>MLL</th>
<th>BIC</th>
<th>Rank(MLL)</th>
<th>Rank(BIC)</th>
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<tbody>
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<td>RH</td>
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<td>-6,754</td>
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<td>Currie APC</td>
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<td>CBD-2</td>
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<td>CBD-3</td>
<td>-6,250</td>
<td>-6,709</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>
Mortality model: CBD-1

- Death probabilities described by
  \[
  \text{logit}(q_{x,t}) = \ln\left(\frac{q_{x,t}}{1 - q_{x,t}}\right) = k_t^{(1)} + k_t^{(2)}(x - \bar{x}) + \gamma_c^{(3)}
  \]

- Estimated parameters

- Forecast of parameters via a multivariate ARIMA model
  \[
  K_{s+1} = K_s + \phi(K_{s-2} - K_{s-1}) + \mu + CZ_{s+1}
  \]
  with \(K_s\) vector of parameters \(k_t^{(1)}\), \(k_t^{(2)}\) and \(\gamma_c^{(3)}\)
Results: parameters estimation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>ARIMA</th>
<th>$\sigma^2$</th>
<th>$\mu$</th>
<th>$\phi$</th>
</tr>
</thead>
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<tr>
<td>$k^{(1)}$</td>
<td>(0,1,0)</td>
<td>0.000759</td>
<td>-0.020394</td>
<td>0</td>
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<tr>
<td>$k^{(2)}$</td>
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<td>0.000001</td>
<td>0.001015</td>
<td>0</td>
</tr>
<tr>
<td>$\gamma^{(3)}$</td>
<td>(1,1,0)</td>
<td>0.000414</td>
<td>0.011075</td>
<td>-0.569975</td>
</tr>
</tbody>
</table>

$C = \begin{pmatrix}
0.027548 & 0 & 0 \\
0.000469 & 0.001077 & 0 \\
0 & 0 & 0.020358
\end{pmatrix}$

Table 2: Fitted parameters of the ARIMA models

Figure 3: Parameters estimation with corresponding 5% and 95% prediction intervals, years 1974-2067.
The dynamics under a risk-neutral measure $Q$ equivalent to the current real-world measure $P$ becomes

$$K_{s+1} = K_s + \phi(K_{s-1} - K_s) + \mu + C(Z^Q_{s+1} - \lambda)$$

with $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ market prices of longevity risk associated with $Z^{(1)}$, $Z^{(2)}$, $Z^{(3)}$ under $Q$.
Insurer with a portfolio of pure endowments paying a lump sum (1€) at maturity T to a cohort of \( l_x \) individuals all aged \( x \) at initial time

- \( \hat{l}_{x+T} \) : expected number of survivors to age \( x+T \) at time \( T \)
- \( l_{x+T} \) : realized number of survivors to age \( x+T \) at time \( T \)

Exposure to risk of systematic deviations between \( l_{x+T} \) and \( \hat{l}_{x+T} \)

\[ l_{x+T} - \hat{l}_{x+T} \] : losses experienced by the insurer at time \( T \)
S-forward: agreement between two counterparties to exchange at maturity T an amount equal to the realized survival rate of a given population cohort (floating rate payment), in return for a fixed survival rate agreed at the inception of the contract (fixed rate payment), is a zero-coupon fixed-for-floating survival swap

- $\pi \ (\pi \geq 0)$: fixed proportional S-forward premium, set in a way that the swap value is zero at inception $\rightarrow$ market value of fixed leg = market value of floating leg

Compared with other mortality securities – e.g. longevity bonds – S-forwards:

- Involve lower transactions cost
- More flexible and tailor-made to meet different needs
- Do not require the existence of a liquid market
The basic survivor swap value at time zero to the fixed-rate payer is:

\[
\text{Swap value} = V\left[ l_{x+T} \right] - V\left[ (1 + \pi)\hat{l}_{x+T} \right]
\]

- market price of the floating leg in t=0
- market price of the fixed leg in t=0

Assuming the independence between interest rate and mortality:

\[
V\left[ (1 + \pi)\hat{l}_{x+T} \right] = (1 + \pi)\hat{l}_{x+T} d(0,T)
\]

- expected present value of the fixed leg under the real-world probability measure

\[
V\left[ l_{x+T} \right] = E^*\left( l_{x+T} \right) d(0,T)
\]

- expected present value of the floating leg under a risk-adjusted probability measure

\[
d(0,T) : \text{risk-free discount factor}
\]

\[
\pi = \frac{E^*\left( l_{x+T} \right)}{\hat{l}_{x+T}} - 1 = \frac{rP_x^*}{T\hat{P}_x} - 1
\]

Survivor swap premium
Estimation of the risk-adjusted probabilities

- Question: how to estimate the risk-adjusted survival probabilities?

- We propose an approach using the risk margin implicit in the evaluation of the technical provision under Solvency 2
Under Solvency 2:

- market value of liabilities = best estimate + Risk Margin

The Risk Margin (RM) is used to provide a risk adjustment of the best estimate liabilities → sort of “market prices of liabilities”

The cost-of-capital (CoC) approach

- RM = the cost of providing an amount of capital equal to the Solvency Capital Requirements necessary to support the insurance obligations over their lifetime

- Cost of capital rate set to 6% (QIS 5 technical specifications):

\[ RM_t = 6\% \sum_{i=t}^{T-1} SCR_i \cdot d(t,i) \]
Solvency Capital Requirement (SCR) under Solvency II

- SCR defined as the amount of economic capital that an insurance company needs to hold to absorb unexpected losses over a 1-year at a 99.5% percentile level
- Parameters of the standard formula determined to reflect a 99.5% Value-at-Risk (VaR) on the unexpected loss on a 1 year time horizon

Solvency II Standard formula

- SCR for longevity risk = change in the net value of assets minus liabilities due to a longevity shock

\[
SCR_t = (\Delta \text{NAV}_t | \text{longevity shock}) = V'_t - \hat{V}_t = l_{x,0} \left( T p'_{x,0} - T \hat{p}_{x,0} \right) d(t, T)
\]

- Longevity shock = permanent 20% reduction of mortality rates at each age
- One level of possible shock \(\rightarrow\) deterministic setting
Risk margin and market price of longevity risk

- Assumption: the insurance company is only exposed to longevity risk
- An insurer completely hedged against longevity risk: no solvency capital for longevity risk to provide $\Rightarrow$ SCR=$0$ $\Rightarrow$ RM=$0$
- Assumption: the insurer is interested in securitizing its LR if the transaction price is lower or equal to the PV of the future CoC required in presence of LR
- Risk Margin $\Rightarrow$ maximum price the insurance company would be willing to pay for longevity risk securitization (see Börger (2010))
- Drawbacks:
  - The company’s expected cost of capital $\mu$ may be lower than the Risk Margin
  - A company might accept a higher market price of risk for strategic reasons (e.g. difficulties in raising capital and risk of increasing cost of capital in the future).
  - Different companies might accept different longevity risk prices
- The Risk Margin $\Rightarrow$ a starting point for the pricing of longevity derivatives
Estimation of the risk-adjusted probabilities from the risk margin

- Solvency capital requirements (SCR) in t=0 for a portfolio of pure endowment

\[
SCR_0 = V_0' - \hat{V}_0 = l'_{x+T,T} \cdot d(0,T) - \hat{l}'_{x+T,T} \cdot d(0,T) = l_{x,0} \left( T \ p'_{x,0} - T \hat{p}_{x,0} \right) \cdot d(0,T)
\]

Difference between technical provision with longevity shock and the best estimate

Difference between survival probabilities with longevity shock and the best estimate

- SCR in t=0 after a S-forward with maturity T (portfolio maturity)

\[
SCR^S_0 = \left( l_{x,0} - l^S_{x,0} \right) \cdot \left( T \ p'_{x,0} - T \hat{p}_{x,0} \right) \cdot d(0,T)
\]

If \( l_{x,0} = l^S_{x,0} \)
then: \( SCR^S_0 = 0 \)

- SCR requirements in t after a S-forward with maturity T (portfolio maturity)

\[
SCR^S_t = \left( l_{x,0} - l^S_{x,0} \right) \cdot \left( T \ p'_{x,0} - T \hat{p}_{x,0} \right) \cdot d(t,T)
\]

If \( l_{x,0} = l^S_{x,0} \)
then: \( SCR^S_t = 0 \)

Assumptions: no credit risk and no basis risk
Estimation of the risk-adjusted probabilities from the risk margin

- Risk margin in t=0

\[ RM_0 = 6\% \cdot \sum_{i=0}^{T-1} SCR_i \cdot d(0, i) = 6\% \cdot l_{x,0} \sum_{i=0}^{T-1} (T \cdot p'_{x,0} - h \hat{p}_{x,0}) \cdot d(0, T) = 6\% \cdot l_{x,0} \cdot T \cdot (T \cdot p'_{x,0} - h \hat{p}_{x,0}) \cdot d(0, T) \]

- Risk margin in t=0 after a S-forward with maturity T

\[ RM_S^0 = 6\% \cdot \sum_{i=0}^{T-1} SCR_S \cdot d(0, i) = 6\% \cdot T \cdot (l_{x,0} - l_{x,0}^S) \cdot (T \cdot p'_{x,0} - T \hat{p}_{x,0}) \cdot d(0, T) \]

- Measure of the amount of RM saved by the insurer entering in a S-forward:

\[ RM_0 - RM_S^0 = 6\% \cdot T \cdot l_{x,0}^S (T \cdot p'_{x,0} - T \hat{p}_{x,0}) \cdot d(0, T) \rightarrow \pi \cdot \hat{l}_{x+T, T}^S \cdot d(0, T) \]

\[ \pi^{\text{max}} = e^{\delta^{\text{max}} T} - 1 = \frac{6\% \cdot T \cdot (T \cdot p'_{x,0} - T \hat{p}_{x,0})}{T \hat{p}_{x,0}} \]

- RM saving = maximum premium to be paid by the insurer for hedging LR
Market price of longevity risk

- Assuming that \( \pi = \pi^{\text{max}} \):

- We find the “maximum” risk-neutral probabilities

\[
E_Q[T p_{x,0}] = T \hat{p}_{x,0} + 6\% \cdot T \cdot (T p'_{x,0} - T \hat{p}_{x,0})
\]

\[
E_Q[T p_{x,T}] = E_Q \left[ \prod_{s=0}^{T-1} p_{x+s,t+s} \right] = E_Q \left[ \prod_{s=0}^{T-1} \frac{1}{1 + \exp \left( k_{t+s}^{(1)} + k_{t+s}^{(2)} (x - \bar{x}) + \gamma_{t-x}^{(3)} \right)} \right]
\]

- And then the market price of longevity risk implicit in the risk-neutral probability measure:

\[
\lambda = \arg \min_{\lambda} \sum_{i=1}^{m} \left[ \delta_i(\lambda) - \delta_i^{\max} \right]^2
\]

where: \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \) and:

\( m = \) number of S-forwards the model is calibrated to

and:

\[
\delta_i^{\max} = \ln \left[ \frac{T_i \hat{p}_{x_i,0} + 6\% \cdot T_i \left( T_i p'_{x_i,0} - T_i \hat{p}_{x_i,0} \right)} {T_i \hat{p}_{x_i,0}} \right] \frac{1}{T_i}
\]

\[
\delta_i(\lambda) = \ln \left\{ \frac{E_{Q(\lambda)} \left[ T_i p_{x_i,0} \right]} {T_i \hat{p}_{x_i,0}} \right\} \frac{1}{T_i}
\]
Numerical application

- Portfolio of pure endowments
- Initial cohort of $l_x=10,000$ policyholders aged $x$ in year 2007 (t=0)
- Death counts for age 60-90 and years 1974-2007
- Risk-free interest rate term structure taken from CEIOPS: year 2007
- Assumptions: no credit risk and no basis risk
Results: S-forward maximum price

Males

<table>
<thead>
<tr>
<th>Year of birth:</th>
<th>1957</th>
<th>1952</th>
<th>1947</th>
<th>1942</th>
<th>1937</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>$T$</td>
<td>$\pi^{max}$</td>
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<tr>
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<td>0.325354</td>
<td>0.701456</td>
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</table>

Table 3: Values of the S-forward “maximum price” $\pi^{max}$ for different maturities, $T$
Results: S-forward maximum spread

Males

<table>
<thead>
<tr>
<th>Year of birth:</th>
<th>1957</th>
<th>1952</th>
<th>1947</th>
<th>1942</th>
<th>1937</th>
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</thead>
<tbody>
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<td>T</td>
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<td>0.017716</td>
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<td>-</td>
</tr>
</tbody>
</table>

Table 4: Values of the S-forward “maximum spread” $\delta_{\text{max}}$ for different maturities, $T$. 
Results: market price of longevity risk

The vector $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ calibrated to five S-forwards price with:

- same maturity: $T=10$
- different ages: 50, 55, 60, 65, 70 years
- evaluated in the year 2007

$\lambda = (0.924316, 0.066557, 0.024869)$
Further research

- Compare the results with other pricing methodology
- Effect of different approximations on $SCR$, evaluation in the standard formula
- Adoption of an internal model instead of the standard formula
- More complex insurance portfolio (life annuities) and more complex hedging instruments (Plain vanilla survivor swaps)
- …

Thank you for your attention


Human Mortality Database. University of California, Berkeley (USA) and Max Planck Institute for Demographic Research, Rostock (Germany), 2010.

