Prudence Revisited

The use of expected-utility theory for decision-making by the trustees of a retirement fund

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Outline

- Introduction
- Argument of the utility function
- Functional form of the utility function
- Weighted average relative risk aversion
- Summary and conclusion
Introduction

Normative validity of EU theory:

- for trustees: Arrow (1951), Harsanyi (1975), Sen (1973)

Trusteeship requires prudence
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Argument of the utility function

Individual /DC member:

- current and future consumption (& bequests) (Von Neumann & Morgenstern, 1947; Savage, 1954)
- wealth at a time horizon (Tobin, 1958; Samuelson, 1969; Owen & Rabinovitch, 1983)
- wealth at retirement (Nielsen, unpublished);
- net replacement ratio at retirement (Thomson & Levitan, 2009)
- one-period-ahead funding ratio relative to reasonable expectations
Argument of the utility function

DB fund:

- one-period-ahead surplus (Cardinale et al, 2006)
- single-period surplus (Sherris, 1993)
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Functional form of the utility function

- HARA class has become standard

\[
r(x) = -\frac{u''(x)}{u'(x)} = \frac{1}{a + bx}
\]

\[
u(x) = \begin{cases} 
\frac{\gamma}{1-\gamma} \left( \frac{\alpha x}{\gamma} + \beta \right)^{1-\gamma} & \text{for } b \neq 0, \gamma \neq 1; \\
\ln(x + \beta) & \text{for } b \neq 0, \gamma = 1; \\
-\exp(\alpha x) & \text{for } b = 0.
\end{cases}
\]
Functional form of the utility function

Problem with the HARA class:

- It does not allow decreasing relative risk aversion
- Evidence of decreasing relative risk aversion (Projector & Weiss, 1966; Friend, 1973; Bossons, 1973; Cohn et al, 1975)
- Prudence suggests non-increasing relative risk aversion
Functional form of the utility function

- Booth (1995):

\[ u(z) = \begin{cases} 
\ln(z) & \text{for } z < 1; \\
\ln(z) + h & \text{for } z \geq 1 
\end{cases} \]
Functional form of the utility function

Khorasanee & Smith (1997):

\[ u(z) = \begin{cases} 
\ln(z) & \text{for } z \geq 1; \\
\rho^* \ln(z) & \text{for } z < 1; \rho^* \geq 1.
\end{cases} \]
Functional form of the utility function

Advantages of non-increasing relative risk aversion in the one-period-ahead funding ratio:

- consistent with empirical results
- not speculative, so consistency with prudence
- equivalent to non-increasing absolute risk aversion in the one-period-ahead force of return
- period may be defined as the interval between decisions
Functional form of the utility function

Elicitation of trustees’ utility functions:
- consensus
- compromise
- Harsanyi’s (1975) individualism postulate:

\[ u(x) = \sum_{m=1}^{M} c_m u_m(x) \]
Functional form of the utility function

Prudence (Kimball, 1990):

- coefficient of absolute prudence:
  \[ p(x) = -\frac{u''(x)}{u''(x)} \]

- coefficient of relative prudence:
  \[ \pi(x) = -x \frac{u'''(x)}{u''(x)} \]

Problems with Kimball’s prudence:

- Positive prudence does not necessarily imply positive risk aversion
- Positive prudence does not necessarily even imply non-satiation
In order to satisfy the requirements of prudence, the trustees’ utility function \( u(\bullet) \) should conform to the following criteria:

1. It should map the open interval \((0, +\infty)\) into the open interval \((-\infty, +\infty)\) or into a subset of that interval.

2. It should be at least twice differentiable and, if it is not thrice differentiable, then \( u''(\bullet) \) should have a finite number of finite jump-discontinuities.

3. The trustees should be unsatiated, so that, for all \( z \), \( u'(z) > 0 \).

4. For all \( z \), \( \gamma(z) \geq \gamma^* > 1 \).

5. For any \( z^* \) and for all \( z < z^* \), \( \gamma(z) \geq \gamma(z^*) \).
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Weighted average relative risk aversion: the basic WARRA class

\[ u(z) = \frac{u_0(z) + cu_\infty(z)}{1 + c}; \]

where:

\[ u_0(z) = \frac{z^{1-\gamma_0} - 1}{1 - \gamma_0}; \]

\[ u_\infty(z) = \frac{z^{1-\gamma_\infty} - 1}{1 - \gamma_\infty}; \]

\[ c > 0; \text{ and} \]

\[ \gamma_0 \geq \gamma_\infty > 1. \]
Weighted average relative risk aversion: the basic WARRA class

Relative risk aversion:

\[ \gamma(z) = \frac{\gamma_0 + c\gamma_\infty z^\lambda}{1 + cz^\lambda} \]

where:

\[ \lambda = \gamma_0 - \gamma_\infty \]

whence:

\[ \lim_{z \to 0} (\gamma(z)) = \gamma_0 \text{ and} \]

\[ \lim_{z \to \infty} (\gamma(z)) = \gamma_\infty \]

and:

\[ \frac{d}{dz}(\gamma(z)) \leq 0 \]
WARRA: parameterisation

Cumulative distribution

Average relative risk aversion
WARRA: parameterisation
WARRA: generalisation

\[ \gamma(z) = \frac{\gamma_0 + c\gamma_\infty z^\lambda}{1 + cz^\lambda} \]

\[ u(z) = \int_0^z \exp \left\{ -\int_1^y \frac{1}{x} \frac{\gamma_0 + c\gamma_\infty x^\lambda}{1 + cx^\lambda} \, dx \right\} \, dy \]
WARRA: generalisation

![Graph showing relative risk aversion and funding ratio with different values of lambda (1, 3, 10, inf)].

- **Relative risk aversion**
- **Funding ratio**
- **lambda = 1**
- **lambda = 3**
- **lambda = 10**
- **lambda = inf**
WARRA: extension to group utility aggregation

\[ u(z) = \frac{1}{M} \sum_{m=1}^{M} u_m(z) \]

where:

\[ u_m(z) = \frac{z^{1-\gamma_m} - 1}{1 - \gamma_m} \]

and hence:

\[ \gamma(z) = \frac{\sum_{m=1}^{M} \gamma_m z^{-\gamma_m}}{\sum_{m=1}^{M} \gamma_m z^{-\gamma_m}} ; \]

\[ \lim_{z \to 0} \gamma(z) = \gamma_1 ; \]

\[ \lim_{z \to \infty} \gamma(z) = \gamma_M ; \]

\[ \frac{d}{dz} \gamma(z) \leq 0 ; \text{ and} \]

\[ \lim_{z \to 1} \gamma(z) = \frac{1}{M} \sum_{m=1}^{M} \gamma_m . \]
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Summary and conclusion

Basic WARRA class:

\[ u(z) = \frac{u_0(z) + cu_\infty(z)}{1 + c}; \]

where:

\[ u_0(z) = \frac{z^{1-\gamma_0} - 1}{1 - \gamma_0}; \]

\[ u_\infty(z) = \frac{z^{1-\gamma_\infty} - 1}{1 - \gamma_\infty}; \]

\( c > 0; \) and \( \gamma_0 \geq \gamma_\infty > 1. \)

Generalisation of the basic class to faster transition:

\[ u(z) = \int_0^z \exp \left\{ -\int_1^y \frac{\gamma_0 + c\gamma_\infty x^2}{x(1 + cx^2)} \, dx \right\} \, dy \]

Extension of the basic class to group utility aggregation:

\[ u(z) = \frac{1}{M} \sum_{m=1}^M u_m(z) \]

where:

\[ u_m(z) = \frac{z^{1-\gamma_m} - 1}{1 - \gamma_m}. \]
Summary and conclusion

Operationalising prudent utility-based retirement-fund risk management:

- qualitative information to assist trustees in implementation of the process
- data to inform trustees about appropriate levels of RRA and transition rates
- risk-adjustment of returns earned
- application to allocation of assets, pricing of liabilities and reinsurance of risks
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Argument of the utility function

Cobb–Douglas utility function:

\[ u(L_t, A_t) = \left\{ u\left( \frac{L_t}{S_t \bar{a}_x} \right) \right\}^\alpha \left\{ u\left( \frac{A_t}{L_t} \right) \right\}^{1-\alpha} \]