
Michael Adam, Raimond Maurer and Matthias Möller

Abstract
In the present paper we examine a concept which we claim to be more suitable than traditional ones for measuring chance and risk of a stock portfolio when options are included. After the basic excess chance measures and shortfall risk measures have been derived systematically, the connections between these measures are indicated. We subsequently use these measures to evaluate chance and risk of a generalized collar strategy. Closed formulas are given for the general case as well as for a logarithmic normally distributed stock price.

Résumé

Keywords
Generalized collar strategies, shortfall risk, excess chance.
1 Definition of the problem

The use of options in order to control shares or stock portfolios has in the meantime become a standard strategy in modern investment. Through the inclusion of options, investors succeed in influencing the original chance-risk-profile of a pure stock investment in a very flexible manner. Essential for a proper application of option strategies is that investors are informed of chances and risks of the resulting financial positions. The usually applied payment diagrams of uncertain final net worth positions merely represent a first indication for the professional investment practice, as they lead only to qualitative statements about chance and risk. Concerning portfolio-theory there is a strong need to represent chance and risk as measurable quantities. Since the publication of contributions by Bookstaber/Clarke (1983a, b, 1984, 1985) it has become clear that for risk evaluation of combined stock and option positions the traditional risk measures like variance and standard deviation will lead to unsatisfactory results. The reason for these problems is that the variance and standard deviation are both based on a symmetrical conception. Possible deviations below as well as above the expected value are understood as risk, while for an investor the latter corresponds to chance rather than to risk. This could be tolerated in case of a symmetrical return distribution, option strategies however lead to complex, asymmetrical distributions of probability.

Hence there is a need to cope better with the asymmetry of option strategies, using alternative risk measures. In literature most attention has been paid to a general class of shortfall risk measures with the well-known special cases shortfall probability, shortfall expectation and shortfall variance, as it was developed e.g. in Albrecht (1994a, b). For example these measures were used for evaluation of option strategies in the framework of historical and stochastic simulations in the contributions by Lewis (1990), Ferguson (1993), Marmer/Ng (1993), Zimmermann (1994) as well as by Albrecht/Maurer/Stephan (1995a). Starting with the paper by Albrecht/Maurer/Stephan (1995b) option strategies were evaluated on the basis of asymmetrical
chance measures related to asymmetrical risk measures, in the context of a historical simulation.

Principally historical simulations, stochastic simulations as well as a calculation of explicit analytical expressions can be used to retain the desirable risk and chance measures. Although historical analyses usually depend strongly on the used data base, this does not reduce the value of this method, as the investor is getting informed of empirical chances and risks of the option strategies examined. Stochastic simulations should certainly be used only in the case closed analytical expressions cannot be produced. In Albrecht/Maurer/Timpel (1995) closed analytical expressions were developed for the expectation, variance, shortfall probability, shortfall expectation and shortfall variance for normally as well as logarithmic normally distributed prices of the „underlying“ asset in context of a 1:1-collar. The objective of this work, on one hand, is to extend results derived in Albrecht/Maurer/Timpel (1995) for arbitrary hedge-ratios in the interval \([0, 1)\). On the other hand, the evaluation of option strategies on the basis of excess-chance-measures that was started in Albrecht/Maurer/Stephan (1995b) will be integrated in the analytical evaluation.

2 Excess-Chance and Shortfall-Risk of a Stock Portfolio Combined with Options

We assume that an investor is holding a stock with the actual value \(s_0\) and an uncertain value at time \(T\) denoted by the random variable \(S_T\). This stock is hedged by buying \(0 \leq \alpha_1 < 1\) European type put options with the exercise price \(x\), date of maturity \(T\) and option price \(p(x)\) at time 0. The long put position is financed (entirely, in part or even more than requested) by selling \(0 \leq \alpha_2 < 1\) European type call options with exercise price \(y > x\), option price \(c(y)\) at time 0 and with identical expiration date \(T\) as the put. For simplicity we assume no payments are to
be paid neither from the stock nor from the options and the price difference of the
options is to be invested or borrowed until \( T \) at the riskless rate of interest \( r \). Hence
the (positive or negative) cost of the option strategy equals \( \Delta = [\alpha_2 p(x) -
\alpha_2 c(y)] \exp(rT) \). We would like to remark that a lot of well-known option strategies
result as a special case of this general strategy: for \( \alpha_1 \to 1 \) and \( \alpha_2 \to 1 \) we get the
"collar strategy" or for \( p(x) - c(y) = 0 \) we obtain the "zero-cost-collar". Setting \( \alpha_2 = 0 \) we have the "1:1:1-Put-Hedge", setting \( \alpha_1 = 0 \) we have the "1:1:1-Covered-
Short-Call". The unprotected stock position results if we set \( \alpha_1 = \alpha_2 = 0 \) resp. \( x \to -\infty \) and \( y \to \infty \). All the considerations in this article are also valid for a portfolio
of stocks, if there exists a corresponding stock index which is the underlying for the
options in question.

The aim of this article is to evaluate chance and risk of the value at time \( T \) of the
combined position, i.e. of

\[
V_T = S_T + \alpha_1 \max(x - S_T, 0) - \alpha_2 \max(S_T - y, 0) - \Delta ,
\]

in the context of a general shortfall-risk and excess-chance concept developed by
Albrecht (1994 a, b).

Let us denote the minimally desired value of the combined position by \( m = m(T) \),
e.g. the value achievable by investing in a riskless zero bond or a value determined
by a minimum rate of interest, then we can split up \( V_T \) into (Albrecht 1994a, pp. 2-
3):

\[
V_T = m + V_T^*(m) - V_T^!(m).
\]
The excess-position $V_r^+(m) = \max(V_r - m, 0)$ characterize the chance of exceeding the minimal value $m$, the shortfall-position $V_r^-(m) = \max(m - V_r, 0)$ denotes the risk of failing this value.

Introducing a general loss function $L$ as in Albrecht (1994a) we obtain a one-dimensional measure for the risk of the shortfall-position by:

$$SR_m(V_r) := E[L(V_r^-(m))] . \quad (2.3)$$

If we chose $L(x) = x^n$, $n \in \mathbb{N}$, we get for $n = 0, 1, 2$ the well known risk measures shortfall-probability, shortfall-expectation and shortfall-variance (in the case $n = 0$ we define $\max(m - V_r, 0)^0 := I_{c=, m}(V)$, i.e:

$$SR_m(V_r) := E[V_r^-(m)^n]$$
$$= E[\max(m - V_r, 0)^n] = \text{LPM}_m^n(V) . \quad (2.4)$$

Instead of using the expectation $E(V_r)$ as a measure of chance, which we claim not to be suitable in our context, because of the inclusion of realisations $v_r < m$ considered as being negative in the calculation, we construct measures of chance in an analogous way to the above construction. Hence we introduce a general gain function $G$ and consider the measures:

$$EC_m(V_r) := E[G(V_r^+(m))] . \quad (2.5)$$

Choosing again $G$ to be the $0^{th}$, $1^{st}$ and $2^{nd}$ power we obtain the excess-probability, the excess-expectation and the excess-variance as measures of chance corresponding to the measures of shortfall-risk defined in (2.4).
In Albrecht/Maurer/Stephan (1995b) some formulas connecting the excess moments with the shortfall moments have been shown. So we get for the excess-probability UPM⁰ₘ and the shortfall-probability LPM⁰ₘ the following relationship

\[ \text{UPM}^{0}_m = 1 - \text{LPM}^{0}_m - \text{Prob}(V_{\tau} = m), \]  

(2.6)

the latter being important if \( m \) has a positive probability to be realized by \( V_{\tau} \). For the 1st upper and lower partial moments we get

\[ \text{UPM}^{1}_m = \text{LPM}^{1}_m + E(V_{\tau}) - m. \]  

(2.7)

Finally, we have for the 2nd partial moments

\[ \text{UPM}^{2}_m = \text{Var}(V_{\tau}) + (E(V_{\tau}) - m)^2 - \text{LPM}^{2}_m, \]  

(2.8)

where \( \text{Var}(V_{\tau}) \) is the variance of \( V_{\tau} \). So in what follows we will only calculate - besides the traditional measures \( E(V_{\tau}) \) and \( \text{Var}(V_{\tau}) \) - the shortfall-risk-measures LPM⁰ₘ, LPM¹ₘ and LPM²ₘ, the excess measures can then be derived by the formulas (2.6) to (2.8).

3 General results

The final wealth position according to the class of option strategies in (2.1) defined can also be written in dependence of the option parameters and the uncertain value of the underlying at the end of period \( (V = V_{\tau}, S = S_{\tau}) \):
\[ V = \begin{cases} (1 - \alpha_1)S + \alpha_1 x - \Delta & S \leq x \\ S - \Delta & x < S < y \\ (1 - \alpha_2)S + \alpha_2 y - \Delta & S \geq y. \end{cases} \] (3.1)

Assuming that \( S \) possesses a density function \( f \), then, because of the strong monotony of \( V \), \( V \) possesses the following density function \( g(v) \)

\[
g(v) = \begin{cases} g_1(v) = f \left( \frac{v - \alpha_1 x + \Delta}{1 - \alpha_1} \right) \frac{1}{1 - \alpha_1} & \alpha_1 x - \Delta \leq v \leq x - \Delta \\ g_2(v) = f(v + \Delta) & x - \Delta \leq v \leq y - \Delta \\ g_3(v) = f \left( \frac{v - \alpha_2 y + \Delta}{1 - \alpha_2} \right) \frac{1}{1 - \alpha_2} & v \geq y - \Delta \\ 0 & \text{otherwise}. \end{cases} \] (3.2)

For the expected final wealth position consequently we obtain

\[
E(V) = \int_{-\infty}^{\infty} v g(v) \, dv \\
= \int_{-\Delta}^{x - \Delta} v g_1(v) \, dv + \int_{x - \Delta}^{y - \Delta} v g_2(v) \, dv + \int_{y - \Delta}^{\infty} v g_3(v) \, dv.
\] (3.3)
For the variance we similarly obtain

\[ \text{Var}(V) = E(V^2) - [E(V)]^2 \]

\[ = \int_{\alpha x - \Delta}^{x - \Delta} v^2 g_1(v) \, dv + \int_{x - \Delta}^{y - \Delta} v^2 g_2(v) \, dv + \int_{y - \Delta}^{\infty} v^2 g_3(v) \, dv \]

\[ - \left[ \int_{\alpha x - \Delta}^{x - \Delta} v g_1(v) \, dv + \int_{x - \Delta}^{y - \Delta} v g_2(v) \, dv + \int_{y - \Delta}^{\infty} v g_3(v) \, dv \right]^2. \]

(3.4)

It is obvious, that the measures of shortfall-risk are depending on the magnitude of the target value \( m \). Therefore we have to examine four different cases:

Case 1: \( m < \alpha x - \Delta \)

In this case, the target value \( m \) is found below the possible minimum of the final wealth position. Therefore the shortfall-position is given by \( V = 0 \), and consequently we have \( \text{LPM}_m^0 = \text{LPM}_m^1 = \text{LPM}_m^2 = 0 \). According to (2.6) - (2.8) we obtain for the excess-chance-measures \( \text{UPM}_m^0 = 1 \), \( \text{UPM}_m^1 = E(V) - m \) and \( \text{UPM}_m^2 = \text{Var}(V) - [E(V) - m]^2 \).

Case 2: \( \alpha x - \Delta \leq m < x - \Delta \)

We set \( a = 1/(1 - \alpha_i) \) and \( b = (\Delta - \alpha_i x)/(1 - \alpha_i) \). Then we get for the shortfall-probability

\[ P(V \leq m) = P((1 - \alpha_i)S + \alpha_i x - \Delta \leq m) = P(S \leq am + b) = F(am + b). \]

(3.5)

For the shortfall-expectation we obtain
\[ E[\max(m - V, 0)] = \int_{x - \Delta}^{m} (m - v) g_1(v) dv \]  
(3.6)

and finally for the shortfall-variance

\[ E[\max(m - V, 0)^2] = \int_{x - \Delta}^{m} (m - v)^2 g_1(v) dv \]  
(3.7)

Case 3: \( x - \Delta \leq m < y - \Delta \)

In this case the shortfall-probability is equal to

\[ P(V \leq m) = P(S - \Delta \leq m) = F(m + \Delta). \]  
(3.8)

For the shortfall-expectation we obtain

\[ E[\max(m - V, 0)] = \int_{x - \Delta}^{x - \Delta} (m - v) g_1(v) dv + \int_{x - \Delta}^{m} (m - v) g_1(v) dv \]  
(3.9)

and finally we have for the shortfall-variance

\[ E[\max(m - V, 0)^2] = \int_{x - \Delta}^{x - \Delta} (m - v)^2 g_1(v) dv + \int_{x - \Delta}^{m} (m - v)^2 g_1(v) dv. \]  
(3.10)

Case 4: \( m \geq y - \Delta \)

In the last case we set \( c = 1/(1 - \alpha_2) \) and \( d = (\Delta - \alpha_2 y)/(1 - \alpha_2) \). Obviously we get for the shortfall-probability

\[ P(V \leq m) = P((1 - \alpha_2) S + \alpha_2 y - \Delta \leq m) = P(S \leq cm + d) = F(cm + d). \]  
(3.11)
The shortfall-expectation is given by

$$E[\max(m-V,0)] = \int_{-\Delta}^{x-\Delta} (m-v)g_1(v)dv + \int_{x-\Delta}^{y-\Delta} (m-v)g_2(v)dv + \int_{y-\Delta}^{m} (m-v)g_3(v)dv$$

(3.12)

and for the shortfall-variance we obtain

$$E[\max(m-V,0)^2] = \int_{-\Delta}^{x-\Delta} (m-v)^2g_1(v)dv + \int_{x-\Delta}^{y-\Delta} (m-v)^2g_2(v)dv + \int_{y-\Delta}^{m} (m-v)^2g_3(v)dv.$$  (3.13)

4 Results in case of a logarithmic normally distributed stock price

In what follows we focus on the analytical calculation for the general class of investment strategies given by (3.1). In order to gain closed form analytical results we have to make an reasonable assumption with respect to the distribution of $S_t$. In case of the standard assumption, that $\{S_t; 0 \leq t \leq T\}$ follows a geometrical Brownian motion process with constant drift $u$ and constant volatility $s$ we have, cf. HULL (1993, p. 210): $\ln S \sim N(\mu, \sigma^2)$ with $\mu = \ln s_t + (u - s^2/2)T$ and $\sigma^2 = s^2T$. As a consequence for the distribution of $S_t$ we consider the case of a logarithmic normal distribution with parameters $\mu$ and $\sigma^2$. It should be pointed out that in this case the well known Black/Scholes-formula is a suitable possibility to model the option prices at the begin of the period.

Let $\Phi$ denote the distribution function of the standard normal distribution and define $x_{LN} := (\ln x - \mu)/\sigma$. Then we obtain for the expected value given by (3.3):

$$E(V) = e^{-\mu \cdot \Delta} [1 - \alpha_1 \Phi(x_{LN} - \sigma) - \alpha_2 (1 - \Phi(y_{LN} - \sigma))]$$

$$+ \alpha_1 x \Phi(x_{LN}) + \alpha_2 y (1 - \Phi(y_{LN})) - \Delta.$$  (4.1)
For the variance we get:

\[ \text{Var}(V) = e^{2(\alpha_1 - \alpha_2)} [(\alpha_1^2 - 2\alpha_1) \Phi (x_{LN} - 2\sigma) + (1 - \alpha_1)^2 + (2\alpha_2 - \alpha_1^2) \Phi (y_{LN} - 2\sigma)] \]

\[ -2e^{\alpha_2 y - \alpha_2 y^2} \left[ (1 - \alpha_2)(\Delta - \alpha_2 y) + \alpha_1 (\alpha_1 x - x - \Delta) \Phi (x_{LN} - \sigma) \right] + \alpha_2 (y + \Delta - \alpha_2 y) \Phi (y_{LN} - \sigma) + (\alpha_1^2 x^2 - 2\Delta \alpha_1 x) \Phi (x_{LN}) \]

\[ + (2\Delta \alpha_2 y - \alpha_2 y^2) [1 - \Phi (y_{LN})] + \Delta^2 - \left[ e^{\alpha_2 y - \alpha_2 y^2} (1 - \alpha_1 \Phi (x_{LN} - \sigma)) \right] - \alpha_1 (1 - \Phi (y_{LN} - \sigma)) + \alpha_2 x \Phi (x_{LN}) + \alpha_2 y (1 - \Phi(y_{LN})) - \Delta^2. \quad (4.2) \]

Obviously, in the first case \( 0 < m < \alpha_1 x - \Delta \) all the shortfall-risk-measures are identically zero. More interesting are the cases two to four.

2. Case \( \alpha_1 x - \Delta < m < x - \Delta \)

Let \( a = 1/(1 - \alpha_1) \), \( b = (\Delta - \alpha_1 x)/(1 - \alpha_1) \) and define \( A_{LN} := (am + b)_{LN} = [\ln (am + b) - \mu]/\sigma \). Then we obtain for the shortfall-probability

\[ \text{LPM}^0_m = \Phi (A_{LN}) , \quad (4.3) \]

and for the shortfall-expectation

\[ \text{LPM}^1_m = (m + \Delta - \alpha_1 x) \Phi (A_{LN}) - (1 - \alpha_1) e^{\alpha_2 y - \alpha_2 y^2} \Phi (A_{LN} - \sigma) . \quad (4.4) \]
Finally we have for the shortfall-variance

\[
LPM_m^2 = \left\{ m^2 + 2m \frac{b}{a} + \frac{b^2}{a^2} \right\} \Phi (A_{LN}) \\
- \left\{ \frac{2m}{a} + \frac{2b}{a^2} \right\} e^{\frac{a}{\tau}} \Phi (A_{LN} - \sigma) \\
+ \frac{1}{a^2} e^{2(\mu + \sigma)} \Phi (A_{LN} - 2\sigma).
\]

(4.5)

3. Case \( x - \Delta < m < y - \Delta \)

Let \( M_{LN} := (m + \Delta)_{LN} = [\ln (m + \Delta) - \mu] / \sigma \) and define \( a \) and \( b \) like in the second case, then we obtain for the shortfall-probability

\[
LPM_m^0 = \Phi (M_{LN}),
\]

(4.6)

and for the shortfall-expectation we have

\[
LPM_m^1 = (m + \Delta) \Phi (M_{LN}) - \alpha_1 x \Phi (x_{LN}) \\
- e^{\frac{a}{\tau}} [\Phi (M_{LN} - \sigma) - \alpha_1 \Phi (x_{LN} - \sigma)].
\]

(4.7)
Finally for the shortfall-variance we obtain

\[ \text{LPM}_m^2 = (m + \Delta)^2 \Phi(M_{LN}) + [\alpha_1^2 x^2 - 2\alpha_1 x(m + \Delta)] \Phi(x_{LN}) \]
- \[ 2[\alpha_1^2 x - \alpha_1(x + m + \Delta)] e^{\frac{x}{\sigma}} \Phi(x_{LN} - \sigma) \]
- \[ 2(m + \Delta)e^{\frac{x}{\sigma}} \Phi(M_{LN} - \sigma) + (\alpha_1^2 - 2\alpha_1) e^{2(\mu - \sigma)^2} \Phi(x_{LN} - 2\sigma) \]
+ \[ e^{2(\mu - \sigma)^2} \Phi(M_{LN} - 2\sigma) \].

(4.8)

4. Case \( m > y - \Delta \)

In the final case we define \( c = 1/(1 - \alpha_2) \), \( d = (\Delta - \alpha_2 y)/(1 - \alpha_2) \) and \( C_{LN} := (cm + d)_{LN} = [\ln (cm + d) - \mu] / \sigma \). Hence, it follows for the shortfall-probability

\[ \text{LPM}_m^0 = \Phi(C_{LN}) \],

(4.9)

for the shortfall-expectation

\[ \text{LPM}_m^1 = (m + \Delta - \alpha_2 y) \Phi(C_{LN}) + \alpha_2 y \Phi(y_{LN}) \]
- \[ \alpha_1 x \Phi(x_{LN}) + e^{\frac{x}{\sigma}} [\alpha_1 \Phi(x_{LN} - \sigma) \]
- \[ \alpha_2 \Phi(y_{LN} - \sigma) - (1 - \alpha_2) \Phi(C_{LN} - \sigma) \].

(4.10)
and finally for the shortfall-variance

\[
LPM_1^2 = [\alpha_1 x - 2\alpha_1 (m + \Delta)] \Phi(x_{LN}) + [2\alpha_2 y(m + \Delta) - \alpha_2^2 y^2] \Phi(y_{LN}) \\
+ \left\{ m^2 + 2 \frac{md}{c} + \frac{d^2}{c^2} \right\} \Phi(C_{LN}) + e^\mu \cdot e^\sigma \frac{d^2}{c^2} [2[\alpha_1 (x + m + \Delta) - \alpha_1^2 x] \Phi(x_{LN} - \sigma)] \\
- 2[\alpha_2 (y + m + \Delta) - \alpha_2^2 y] \Phi(y_{LN} - \sigma) - \left( \frac{2m}{c} + \frac{2d}{c^2} \right) \Phi(C_{LN} - \sigma) \\
+ e^{2(L_s + \sigma)}[(\alpha_1^2 - 2\alpha_1) \Phi(x_{LN} - 2\sigma) - (\alpha_2^2 - 2\alpha_2) \Phi(y_{LN} - 2\sigma) + (1 - \alpha_2)^2 \Phi(C_{LN} - 2\sigma)] .
\]

(4.11)

It should be noted that for \( \alpha_1 \rightarrow 1 \) and \( \alpha_2 \rightarrow 1 \) our results (4.1), (4.2), (4.6) - (4.11) are including the analytical solutions for the expected value, the variance, and the shortfall-risk-measures obtained for the 1:1-Collar strategy in the work of Albrecht/Maurer/Timpel (1994) assuming a logarithmic normally-distributed wealth position of the underlying at the end of the period, as special cases.

5 Conclusion

In this contribution, we have carried on the discussion, starting with the work of Albrecht/Maurer/Timpel (1995), of closed form analytical solutions of shortfall-risk-measures in the context of option strategies. In addition to the former paper, we have derived solutions for a wider class of possible hedge-ratios and in addition for measures of excess-chance. First, we demonstrated how to obtain general results for measures of risk and chance respectively for a given set of parameters of the various option strategies, without assuming a special distribution of the underlying. Second, we developed analytical solutions in the event of a logarithmic normally distributed stock value, which is undoubtedly the most important case in modern capital market theory and in investment practice. Our results may be applied to a lot of problems in the field of risk-measurement and risk-management of optioned
portfolios. Further research should be done to put the excess-chance- and shortfall-risk-measures on a solid decision-theoretical basis. A possibility of doing this is to focus on risk-value-models introduced by Sarin/Weber (1993).
References


