Arbitrage Valuation and Bounds for Sinking-Fund Bonds with Multiple Sinking-Fund Dates

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Abstract
This paper tackles the problem of pricing, under interest-rate risk, a default-free sinking-fund bond which allows its issuer to recurrently retire part of the issue before maturity by (a) a lottery call at par, or (b) on open market repurchase. By directly modelling zero-coupon bonds as diffusions driven by a single-dimensional Brownian motion, we supply a pricing formula for the sinking-fund bond based on a backward induction procedure which exploits, at each step, the martingale approach to the valuation of contingent-claims. With more than one sinking-fund date before maturity, however, this pricing formula is not in closed form, not even for relatively simple parametrizations of the process for zero-coupon bonds, so that a numerical approach is called for. Since the complexity of such an approach increases exponentially with the number of sinking-fund dates before maturity, we provide arbitrage-based lower and upper bounds for the sinking-fund bond price, whose computation is almost effortless when zero-coupon bonds’ prices depend on the spot rate only, with the spot rate modeled as in Cox, Ingersoll and Ross (1985). Numerical comparisons between the price of the sinking-fund bond obtained via Monte Carlo simulation and our lower and upper bounds are illustrated for different choices of parameters.

Résumé
L’objet de cet article est de proposer une méthode d’évaluation pour les emprunts amortissables qui offrent au débiteur l’option, à chaque échéance, d’acheter directement sur le marché les obligations à rembourser. Nous prenons en considération le cas d’un emprunt avec plus de deux échéances et supposons qu’il n’y a pas de risque de défaillance. Pour cet emprunt nous obtenons une formule d’évaluation qui est basée sur une procédure d’induction rétroactive. De plus, nous démontrons que telle valeur doit être contenue entre deux bornes qui résulteront extrêmement faciles à calculer. Nous terminons avec la présentation de quelques résultats numériques.
Keywords
Sinking-fund bonds, multiple sinking-fund dates, interest rate risk, martingale approach, CIR model, Monte Carlo simulation.

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1. Introduction

Fixed-income securities, in particular bonds issued by corporations, very often embed various types of options, some favoring the issuer, some others the holders. Among the options favoring the issuer, one embedded in the majority of industrial bonds, and in a good share of those issued by utilities, is the delivery option associated with a sinking-fund provision (see Thompson and Norgaard (1967), and Wilson and Fabozzi (1990)). A sinking-fund provision is a clause, specifically established in the indenture, which requires its issuer to retire portions (usually substantial) of the bond before maturity, and the delivery option associated with this provision consists in the fact that the issuer can choose between retiring these portions (a) by a lottery call, usually at par, or (b) by buying them back on the market. A fraction of the issue is retired every year, although certain sinking-fund provisions may be delayed until after some years have elapsed since the bond has been issued.

In this paper, we present a framework for valuing a sinking-fund bond whose indenture requires the issuer to retire, before the maturity $t_n$, the fractions $C_j$ of the outstanding principal at the times $t_j < t_n$. We assume that the only source of uncertainty relevant in the valuation problem is constituted by a stochastic term structure of interest rates, and we model directly the zero-coupon bond prices as diffusion processes all driven by the same one-dimensional brownian motion. We remark that this assumption is compatible with both the class of single factor models with affine yield, members of which are the frameworks pioneered by Vasicek (1977) and Cox, Ingersoll and Ross (1985, CIR henceforth), and the class of whole-yield models originated in Ho and Lee (1986), and formalized in continuous time by Heath, Jarrow and Morton (1992). To produce a valuation formula for the price of a sinking-fund bond, we assume that the bond is held competitively, and we rely on the martingale theory for the valuation of contingent-claims put forward by Harrison and Kreps (1979) and Harrison and Pliska (1981, 1983), and summarized in Duffie (1992). We show how to apply this approach at each sinking-fund date to obtain, by means of a backward induction procedure, a recursive valuation formula for the bond under scrutiny. We observe however that, unless a unique sinking-fund date is present before maturity, our recursive formula does not posses a closed form, not even for relatively simple parametrizations of the process governing the evolution of zero-coupon bonds. For a realistic number of sinking-fund dates, moreover, even a numerical approach seems to be not simple to implement, since it is likely to require the availability of a very considerable amount of CPU time.
To present a (partial) answer to this problem, we develop arbitrage-based lower and upper bounds for the sinking-fund bond price, and we show that, when zero-coupon bond prices are monotonic functions of a unique state variable, these bounds become portfolios of zero-coupon bonds and options on them only. We consider then a situation in which zero-coupon bonds are assumed to depend on the spot rate of interest only, with the spot rate following the CIR model. We recall that, under no arbitrage, such a situation is compatible with our assumption that zero-coupon bonds' prices are diffusions driven by a unique standard brownian motion. Since in the CIR framework both zero-coupon bonds and zero-coupon bonds options are in closed form, moreover, our lower and upper bounds for the sinking-fund bond price can be expressed in closed form as well, hence readily computable. Some numerical experiments show that these bounds deliver a good first-hand estimate of the sinking-fund bond price, even for realistic numbers of sinking-fund dates before maturity.

The literature on the valuation of sinking-fund bonds has so far restricted its attention to the case in which a single sinking-fund date is present before maturity. In particular, Ho and Singer (1984) have provided a valuation formula for the case in which interest-rate risk is absent, but the sinking-fund bond is subject to default. The symmetric case, in which only interest-rate risk matters, has been discussed by Ho (1985) and Bacinello, Ortu and Stucchi (1995). The first author models directly the dynamics of zero-coupon bonds, yet with a non-stochastic volatility for their instantaneous return, while Bacinello, Ortu and Stucchi model the term structure alternatively as in Vasicek (1977), and in CIR. As compared to these contributions, here we approach the valuation problem under the more realistic assumption that more than one sinking-fund date may be present before maturity. Our general recursive valuation procedure, moreover, requires neither the instantaneous return on zero-coupon bonds to depend on time only, nor that a single state-variable framework be employed.

The paper is structured as follows. In the next section we formalize the main features of a sinking-fund bond, and present our assumptions on the term structure. In Section 3 we briefly review the martingale methodology for pricing contingent claims, and we then apply it to our valuation problem, to obtain the recursive valuation formula for the sinking-fund bond price. Lower and upper bounds on this price are developed in Section 4, while Section 5 shows how the results specialize when the CIR model is employed, and presents some numerical results for this case. Concluding remarks are collected in Section 6.
2. The valuation framework: definitions and assumptions

We describe in this section the structure of a sinking-fund bond whose indenture requires the issuer to retire the principal in \( n \geq 2 \) instalments. To this end, we consider securities markets populated by rational and non-satiated agents, all sharing the same information. We assume markets to be perfectly competitive, frictionless and free of arbitrage opportunities, and the traded securities to be all perfectly divisible, and default-risk-free. In this framework, the unique source of uncertainty is assumed to be the stochastic behaviour of the term structure of interest-rates, behaviour which we model at the end of the present section.

To formalize the intertemporal cash-flow generated by the sinking-fund bond, issued at time \( t_0 \) and maturing at time \( t_n \), we introduce the following notation:

\[
\begin{align*}
\{ t_j \}_{j=1}^{n-1} & \quad \text{sinking-fund dates, at which the issuer has to retire a certain fraction of the issue and to make coupon payments, with } t_0 < t_1 < \ldots < t_{n-1} < t_n; \\
\{ C_j \}_{j=1}^{n-1} & \quad \text{fractions of the principal to be retired at } t_j. \text{ At any sinking-fund date } t_j, \text{ the issuer can choose between (a) repurchasing } C_j \text{ at its market price, or} \text{ (b) calling } C_j \text{ by lottery at par (the delivery option);} \\
\{ Q_j \}_{j=0}^{n-1} & \quad \text{principal outstanding at any time } t \in \{ t_j, t_{j+1} \}, \text{i.e. } Q_j = \sum_{k>j} C_k. \text{ At any sinking-fund date } t_j, \text{ in particular, } Q_j \text{ represents the principal still floating right after the fraction } C_j \text{ has been retired;} \\
\delta & \quad \text{continuously compounded coupon-rate paid on the sinking-fund bond;} \\
\{ I_j \}_{j=1}^{n-1} & \quad \text{coupon payment made at } t_j, \text{ with } I_j = Q_{j-1} [\delta(t_j - t_{j-1}) - 1]; \\
I_n + C_n & \quad \text{balloon payment at maturity, with } I_n + C_n = C_n e^{\delta(t_n - t_{n-1})}.
\end{align*}
\]

We assume that, at any time \( t \), the floating-supply of the sinking-fund bond is held competitively, i.e. there are no accumulators, or hoarders, that, by going long enough on the bond, are able to force its price away from the purely competitive one. If the entire issue was privately held by a monopolist, in fact, the delivery option would become valueless, while if the bond was publicly held but accumulators were present, the valuation problem would call for a strategic approach (see Dunn and Spatt (1984), and, for a signalling approach to the choice of the amortization schedule, Wu (1993)).
From now on, we let \( B_{SF}(t) \) denote the time \( t \) price of the sinking-fund bond, net of coupon payments when \( t = t_j \). \( B_{SF}(t) \) is then the market value of the stream of liabilities \( \{ L_j \}_{j > t} \) with \( L_j \) defined as

\[
L_j = \begin{cases} 
I_j + C_j \min \{1, \frac{B_{SF}(t_j)}{Q_j} \} & j = 1, \ldots, n-1 \\
I_n + C_n & j = n
\end{cases}
\]

Relation (2.1) translates formally the fact that at any sinking-fund date the issuer, between retiring \( C_j \) by lottery at par and repurchasing it on the market, will choose in such a way to minimize the outflow. To see this, let \( x_j \) be the unit price of the sinking-fund bond, net of interests and just prior to retiring the fraction \( C_j \), and observe that the optimal decision requires an outflow of \( C_j \min \{1, x_j\} \). When it is optimal to sink \( C_j \) by an open market operation, it must be \( x_j = \frac{B_{SF}(t_j) + C_j}{Q_j + C_j} \) or, equivalently, \( x_j = \frac{B_{SF}(t_j)}{Q_j} \), so that in this case the unit prices before and after the sinking-fund operation coincide.

When, on the contrary, \( C_j \) is retired by lottery at par, it must be \( x_j = \frac{B_{SF}(t_j) + C_j}{Q_j + C_j} \), a quantity exceeding 1 if and only if the same holds true for \( \frac{B_{SF}(t_j)}{Q_j} \). Hence \( \min \{1, x_j\} = \min \{1, \frac{B_{SF}(t_j)}{Q_j} \} \), which justifies (2.1). Simple manipulations show that (2.1) can be also rewritten as

\[
L_j = \begin{cases} 
I_j + C_j - \frac{C_j}{Q_j} \max \{ Q_j - B_{SF}(t_j), 0 \} & j = 1, \ldots, n-1 \\
I_n + C_n & j = n
\end{cases}
\]

so that the difference between \( (C_j + I_j) \) and \( L_j \) when \( j < n \) can be interpreted as \( \frac{C_j}{Q_j} \) times the payoff at maturity, \( t_j \), of a put option on \( B_{SF} \) with exercise price \( Q_j \), a fact that will become useful in developing arbitrage-based bounds on the sinking-fund bond price.

We introduce now our assumptions and notation concerning the term structure. We assume that at any time \( t \in [t_j, t_{j+1}] \), \( j = 1, \ldots, n-1 \), zero-coupon bonds with maturities \( t_j \) and \( t_{j+1} \) are traded, and that a zero-coupon with maturity \( t_n \) is available between \( t_{n-1} \) and \( t_n \). We model bond prices as diffusion processes all driven by the same \( \mathbb{R} \)-valued process \( W \), a standard brownian motion under a given probability measure \( P \). More precisely, we assume \( b_j(t) \), the price at time \( t \leq t_j \) of a zero-coupon bond with maturity \( t_j \), to be solution to the stochastic differential equation
(2.3) \[ \frac{db_j(t)}{bj(t)} = \mu_j(b_j(t),t)dt + \sigma_j(b_j(t),t)dW(t) \]

with \( b_j(t_j) = 1 \) for all \( j \), \( b_j(t) > 0 \) for all \( t < t_j \) since arbitrage opportunities are assumed away, and \( \mu_j, \sigma_j \) continuous functions. We remark that our assumption on the stochastic evolution of zero-coupon bonds is compatible with both the class of single factor models with affine yield, members of which are the frameworks pioneered by Vasicek (1977) and Cox, Ingersoll and Ross (1985), and the class of whole yield-curve models put forward by Ho and Lee (1986), and Heath, Jarrow and Morton (1992).

From now on we assume that, for any \( j = 1, 2, \ldots, n-1 \), the following conditions hold for all \( t \in [t_{j-1}, t_j) \):

C1) \( \sigma_{j+1}(b_{j+1}(t),t) - \sigma_j(b_j(t),t) \neq 0 \). Moreover, the random variable \( q(t) = \frac{\mu_{j+1}(b_{j+1}(t),t) - \mu_j(b_j(t),t)}{\sigma_{j+1}(b_{j+1}(t),t) - \sigma_j(b_j(t),t)} \) has finite variance;

C2) the random variable \( \exp \left\{ \frac{1}{2} \int_{t_{j-1}}^{t_j} q_j^2(t) \, dt \right\} \) has finite expectation;

C3) the random variable \( \xi_j = \exp \left\{ - \int_{t_{j-1}}^{t_j} q_j(t) \, dW(t) - \frac{1}{2} \int_{t_{j-1}}^{t_j} q_j^2(t) \, dt \right\} \) has finite variance.

In the following section we sketch briefly, and in a non-technical way, the martingale methodology to pricing by arbitrage a contingent claim delivering, at maturity, a random payoff with finite variance, and we apply then this methodology to the valuation of a sinking-fund bond with multiple sinking-fund dates.

3. The martingale approach to the valuation problem

The aim of this section is to produce a pricing formula for the sinking-fund bond, formula based on a backward induction procedure which exploits, at each sinking-fund date, the martingale approach to the valuation of contingent-claims (for a detailed exposition of this approach, see Duffie (1992)). To recall the basic results which make this approach effective under our assumptions, we concentrate the attention on a given time interval \( [t_{j-1}, t_j] \) for \( j \in n \), in which zero-coupon bonds with maturities \( t_j \) and \( t_{j+1} \) are traded. We term “normalized” the market in which the prices of all traded securities are
expressed in units of $b_j$, and let $b_j(t) = \frac{b_{j+1}(t)}{b_j(t)}$. Recall now that an arbitrage opportunity is a self-financing trading strategy which, when started at time $t_{j-1}$, does not require any outflow, while it produces at time $t_j$ a payoff (almost) surely non-negative, and strictly positive with positive probability. It can be shown that the normalized market is free of arbitrage opportunities if and only if the "original" one is so. Moreover, a sufficient condition for the normalized economy to be arbitrage-free is the existence of a probability measure $\mathbb{P}^j$ which, along with satisfying certain technical conditions, transforms the process $b^*_j$ into a martingale. Letting then $\mathcal{F} = \{\mathcal{F}_t, t \in [t_{j-1}, t_j]\}$ denote the standard filtration generated by the brownian motion $W$, define $\mathbb{P}^j$ by

\begin{equation}
\mathbb{P}^j(A) = E[1_A \xi_j] \quad \forall A \in \mathcal{F}_t
\end{equation}

where the expectation is taken with respect to the probability $P$ under which $W$ is a standard brownian motion, $1_A$ denotes the indicator function of the set $A$, and the random variable $\xi_j$ is defined in Section 2. It can be shown that, under conditions C1 to C3 introduced in the previous section, $\mathbb{P}^j$ transforms $b^*_j$ into a martingale, so that both the normalized and the original markets are free of arbitrage opportunities. Given any finite-variance payoff $Z_j$ available at time $t_j$, conditions C1 to C3 guarantee moreover that, in the normalized economy, there exists a self-financing trading strategy which produces at time $t_j$ the payoff $Z_j$. More precisely, there exists an $\mathbb{R}^2$-valued stochastic process $\theta^j = (\theta^j_0, \theta^j_1)$, where $\theta^j_0$ stands for the amount invested in the riskless security present in the normalized economy, while $\theta^j_1$ stands for the one invested in the normalized bond with maturity $t_{j+1}$, such that the process $\theta^j_0 + \theta^j_1 b^*_j$ is a $\mathbb{P}^j$-martingale, and $Z_j = \theta^j_0(t_j) + \theta^j_1(t_j)b^*_j(t_j)$. Absence of arbitrage opportunities implies then that, at any time $t \in [t_{j-1}, t_j]$, the price of $Z_j$ in the normalized market, $\pi^*_j(Z_j)$, must satisfy

\begin{equation}
\pi^*_j(Z_j) = \theta^j_0(t) + \theta^j_1(t)b^*_j(t) = E^{\mathbb{P}^j}\left[\theta^j_0(t_j) + \theta^j_1(t_j)b^*_j(t_j) \mid \mathcal{F}_t\right] = E^{\mathbb{P}^j}[Z_j \mid \mathcal{F}_t]
\end{equation}

where $E^{\mathbb{P}^j} \cdot \mid \mathcal{F}_t$ denotes expectation under the probability measure $\mathbb{P}^j$, and conditional on information up to time $t$. Since the time $t_j$ prices in the original and normalized markets coincide, $Z_j$ represents also a payoff in the original market, whose price at time $t$, $\pi_t(Z_j)$, is linked to the normalized one by the relation $\pi^*_j(Z_j) = \frac{\pi_t(Z_j)}{b_j(t)}$. Under no arbitrage, then, the time $t$ price of $Z_j$ is supplied by

\begin{equation}
\pi_t(Z_j) = b_j(t)\pi^*_j(Z_j) = b_j(t)E^{\mathbb{P}^j}[Z_j \mid \mathcal{F}_t] \quad \forall t \in [t_{j-1}, t_j]
\end{equation}
We exploit now relation (3.3) to obtain, by means of a recursive backward procedure, a formula for the price $B^S_F(t)$ of the sinking-fund bond at any time $t$ before maturity. To this end, we observe first that, when $t$ belongs to the time interval $[t_{n-1}, t_n]$, $B^S_F(t)$ is just the market value of the balloon payment $L_n = I_n + C_n$, so that

$$B^S_F(t) = b_n(t)(I_n + C_n) \quad t_{n-1} \leq t < t_n$$

Let now $t$ belong to the interval $[t_{j-1}, t_j)$, with $j=1, \ldots, n-1$, and observe that, by no arbitrage, $B^S_F(t)$ is the time $t$ value of the payoff $Z_j = L_j + B^S_F(t_j)$, i.e. of the liability $L_j$ plus the time $t_j$ value of the stream of liabilities $\{L_k\}_{k>j}$. Relation (3.3) yields then

$$B^S_F(t) = b_j(t) E^{b_j}[L_j + B^S_F(t_j) \mid \mathcal{F}_{t_j}] \quad t_{j-1} \leq t < t_j$$

By coupling (3.4) and (3.5), we obtain the following recursive relation between the sinking-fund bond prices at any sinking-fund date $\{t_1, \ldots, t_{n-1}\}$:

$$B^S_F(t_k) = \begin{cases} b_n(t_{n-1})(I_n + C_n) & \text{if } k = n-1 \\ b_{k+1}(t_k) E^{b_{k+1}}[L_{k+1} + B^S_F(t_{k+1}) \mid \mathcal{F}_{t_k}] & \text{if } k = n-2, \ldots, 1 \end{cases}$$

Relations (3.4) to (3.6) supply a procedure that can be employed to value the sinking-fund bond at any time $t$ before its maturity. In particular, if $t > t_{n-1}$, one computes $B^S_F(t)$ directly from (3.4). If $t$ is in any of the intervals $[t_{j-1}, t_j)$, with $j < n$, one first obtains $B^S_F(t_j)$ from (3.6), and then computes $B^S_F(t)$ by means of (3.5).

It is interesting to observe that, when only one sinking-fund date remains before maturity (i.e. $t_{n-2} \leq t < t_{n-1}$), exploiting (2.2), the linearity of the conditional expectation operator, and the fact that, on the interval $[t_{n-2}, t_{n-1}]$, the process $\frac{b_n(t)}{b_{n-1}(t)}$ is a $\mathbb{P}^{n-1}$-martingale, the procedure just described yields

$$B^S_F(t) = b_n(t_n)(I_n + C_n) + b_{n-1}(t)(I_{n-1} + C_{n-1}) - b_{n-1}(t) C_{n-1} E^{b_{n-1}}[\max\{Q_{n-1} - b_n(t_{n-1})(I_n + C_n), 0\} \mid \mathcal{F}_{t}]$$

or equivalently, since $Q_{n-1} = C_n$ and $(I_n + C_n) = C_n e^{\delta(t_n - t_{n-1})}$,

$$B^S_F(t) = b_n(t_n)(I_n + C_n) + b_{n-1}(t)(I_{n-1} + C_{n-1}) - C_{n-1} e^{\delta(t_n - t_{n-1})} b_{n-1}(t) E^{b_{n-1}}[\max\{e^{\delta(t_n - t_{n-1})} - b_n(t_{n-1}), 0\} \mid \mathcal{F}_{t}]$$
The last term in the RHS of (3.8) is the time t price of \( C_{n-1} e^{\delta(t_n-t_{n-1})} \) European put options on a zero-coupon bond with maturity \( t_n \), options expiring at \( t_{n-1} \) and with exercise price \( e^{-\delta(t_n-t_{n-1})} \). By suitably specializing the parameters characterizing the stochastic evolution of zero-coupon bonds, one can express the price of these bond-options in closed form, hence obtaining a closed formula for \( \text{BSF}(t) \). This is the case when, for example, the evolution of zero-coupon bonds is compatible with the single-factor models for the term structure proposed by Vasicek (1977) or Cox, Ingersoll and Ross (1985). For a detailed analysis of the comparative statics of (3.8) when these single-factor models are employed, we refer the interested reader to Bacinello, Ortu and Stucchi (1995).

When more than one sinking-fund date is left before maturity (i.e. \( t < t_{n-2} \)), instead, a closed formula for \( \text{BSF}(t) \) is unavailable even under relatively simple parametrizations of the dynamics of zero-coupon bonds. In Section 5 we revert to Monte Carlo simulation to compute \( \text{BSF}(t) \) under the assumption that the evolution of zero-coupon bonds is compatible with the CIR framework. We observe however that, since our valuation procedure requires the computation of nested conditional expectations, its complexity increases exponentially with the number of sinking-fund dates before maturity. It is therefore particularly useful to produce a sufficiently "small" interval in which \( \text{BSF}(t) \) must lie. We do this in the next section, where we show that the absence of arbitrage opportunities forces the price of the sinking-fund bond to lie between a lower and an upper bound whose computation is particularly simple, at least when compared to the numerical valuation of \( \text{BSF}(t) \).

4. Arbitrage bounds for the sinking-fund bond price

To provide arbitrage-based upper and lower bounds for the sinking-fund bond value, we follow Ho (1985) and Bacinello, Ortu and Stucchi (1995), and assume that, along with the sinking-fund bond, the following instruments are traded:

i) **Corresponding Serial Bond.** It is a bond with the same date of issuance \( t_0 \), coupon rate \( \delta \), and amortization schedule \( \{C_j, t_j\}_{j=1}^{n} \) of the sinking-fund bond, but requiring its issuer to sink \( C_j \) anyway by lottery at par. In what follows, we denote by \( B^S(t) \) the time \( t \) price of the corresponding serial bond, with the understanding that, when \( t = t_j \), \( B^S(t_j) \) is net of the instalment \( R_j = I_j + C_j \), with \( I_j \) defined in Section 2.
ii) **Corresponding Coupon Bond.** It is a coupon bond with unit face value, and same coupon rate $\delta$, coupon payment dates \( \{ t_j \}_{j=1}^{n-1} \), and maturity $t_n$ as the sinking-fund and serial ones, so that the coupons \( \{ \beta_j \}_{j=1}^{n-1} \) satisfy $\beta_j = \frac{t_j}{Q_{j-1}}$, while the balloon payment at maturity is given by $\beta_n = \frac{R_n}{Q_{n-1}}$. Henceforth, we denote by $B_{CP}(t)$ the time $t$ price of the corresponding coupon bond, with the understanding that $B_{CP}(t)$ is net of coupon payments when $t=t_j$.

We assume moreover that, at any time $t \in [t_{j-1}, t_j)$, $j=1, \ldots, n-1$, european-style call and put options on the sinking-fund bond, with maturities $\{ t_k \}_{k=j}^{n-1}$ corresponding to the sinking-fund dates following $t$, and exercise prices $Q_k$, are traded. We denote by $c_{SF}(t, t_k, Q_k)$, respectively $p_{SF}(t, t_k, Q_k)$, the time $t$ prices of such options. The next result, both interesting per se and instrumental in obtaining our arbitrage-based bounds, relates the sinking-fund bond to the assets just introduced.

**Proposition 4.1** Under no-arbitrage, for all $t \in [t_{j-1}, t_j)$, $j=1, \ldots, n$, the prices of the sinking-fund bond and the corresponding serial and coupon ones are related via

\[
B_{SF}(t) = \begin{cases} 
B^S (t) - \sum_{k=j}^{n-1} \alpha_k p_{SF}(t, t_k, Q_k) & j=1, \ldots, n-1 \\
B^S (t) & j=n 
\end{cases}
\]

\[
B_{SF}(t) = Q_{j-1} \left[ B_{CP}(t) - \sum_{k=j}^{n-1} \gamma_k c_{SF}(t, t_k, Q_k) \right] & j=1, \ldots, n-1 \\
Q_{n-1} B_{CP}(t) & j=n
\]

where, for $k=1, \ldots, n-1$, $\alpha_k = \frac{C_k}{Q_k}$ and $\gamma_k = \frac{\alpha_k}{Q_{k-1}} = \frac{C_k}{Q_k Q_{k-1}}$.

**Proof.** Since relation (4.1) is a straightforward consequence of (2.2) and the no arbitrage assumption, we only need to establish (4.2). To this end, observe first that the way in which the corresponding coupon bond is defined implies immediately that (4.2) holds for $j=n$. To prove that it holds for all $j<n$, we proceed by induction on the number of sinking-fund dates remaining before maturity. To do so, we notice first that, for $j<n$, simple algebraic manipulations on (2.1) allow us to rewrite the liability $L_j$ as

\[
L_j = t_j + \frac{C_j}{Q_j} B_{SF}(t_j) - \frac{C_j}{Q_j} \max \left\{ B_{SF}(t_j) - Q_j, 0 \right\}
\]
so that, by (3.5), for all \( t \in [t_{j-1}, t_j) \), we have

\[
B^{SF}(t) = b_j(t) E^B \left[ \tilde{j}_j + \frac{I_j + Q_j}{Q_j} B^{SF}(t_j) - \alpha_j \max \left\{ B^{SF}(t_j) - Q_j, 0 \right\} \bigg| \mathcal{F}_t \right]
\]

or equivalently, since \( \tilde{j}_j = \frac{I_j}{Q_j} \) and \( Q_{j-1} = C_j + Q_j \),

\[
(4.3) \quad B^{SF}(t) = Q_{j-1} \left\{ b_j(t) E^B \left[ \tilde{j}_j + \frac{B^{SF}(t_j)}{Q_j} \bigg| \mathcal{F}_t \right] - \gamma c^{SF}(t, t_j, Q_j) \right\}
\]

To verify that, for all \( t \in [t_{n-2}, t_{n-1}) \), \( B^{SF}(t) = Q_{n-2} [B^{CP}(t) - \gamma_{n-1} c^{SF}(t, t_{n-1}, Q_{n-1})] \), i.e. that (4.2) holds when only one sinking-fund date remains before maturity, let \( j = n-1 \) in (4.3), recall that \( \frac{B^{SF}(t_{n-1})}{Q_{n-1}} = B^{CP}(t_{n-1}) \), and observe that, exploiting (3.3), \( B^{CP}(t) = \pi_t (j_{n-1} + B^{CP}(t_{n-1})) = b_{n-1}(t) E^{B^{n-1}} \left[ j_{n-1} + B^{CP}(t_{n-1}) \bigg| \mathcal{F}_t \right] \). Suppose now that (4.2) holds when \( h-1 \) sinking-fund dates remain before maturity, i.e. for all \( t \in [t_{n-h}, t_{n-h+1}) \), with \( 2 \leq h \leq n-1 \). Hence, letting for ease of notation \( j = n-h \),

\[
B^{SF}(t_j) = Q_j \left[ B^{CP}(t_j) - \sum_{k=j+1}^{n-1} \gamma_k c^{SF}(t_j, t_k, Q_k) \right]
\]

so that, substituting \( B^{SF}(t_j) \) into (4.3), and exploiting the linearity of the conditional expectation operator, we have for all \( t \in [t_{j-1}, t_j) \),

\[
B^{SF}(t) = Q_{j-1} \left\{ b_j(t) E^B \left[ \tilde{j}_j + B^{CP}(t_j) \bigg| \mathcal{F}_t \right] - \sum_{k=j+1}^{n-1} \gamma_k b_j(t) E^B \left[ c^{SF}(t_j, t_k, Q_k) \bigg| \mathcal{F}_t \right] - \gamma c^{SF}(t, t_j, Q_j) \right\}
\]

Observing that \( b_j(t) E^B \left[ \tilde{j}_j + B^{CP}(t_j) \bigg| \mathcal{F}_t \right] = B^{CP}(t) \), and that \( b_j(t) E^B \left[ c^{SF}(t_j, t_k, Q_k) \bigg| \mathcal{F}_t \right] = c^{SF}(t, t_k, Q_k) \) since for \( k > j \) the process \( \frac{c^{SF}(t, t_k, Q_k)}{b_j(t)} \) is a \( \mathcal{B} \)-martingale on \( [t_{j-1}, t_j] \), the last relation becomes indeed

\[
B^{SF}(t) = Q_{j-1} \left[ B^{CP}(t) - \sum_{k=j}^{n-1} \gamma_k c^{SF}(t, t_k, Q_k) \right]
\]

which, recalling the position \( j = n-h \), shows that (4.2) holds when \( h \) sinking-fund dates remain before maturity, i.e. for all \( t \in [t_{n-h-1}, t_{n-h}) \).
It is interesting to observe that, when a single sinking-fund date is present, i.e. \( n=2 \), then only options on zero-coupon bonds intervene in (4.1) and (4.2). This fact allows us to claim that Proposition 4.1 extends, to the case of multiple sinking-fund dates, Proposition 1 in Ho (1985) and Proposition 2.1 in Bacinello, Ortu and Stucchi (1995).

We assume now that, along with european-style call and put options on the sinking-fund bond, also european-style call and put options on both the corresponding serial and coupon bonds are traded. Once again, we assume that, at any time \( t \in [t_{j-1}, t_j) \), \( j=1,...,n-1 \), options of such kinds are traded with maturities \( \{ t_k \}_{k=j}^{n-1} \) correspondent to the sinking-fund dates following \( t \), and exercise prices \( Q_k \) for the options on the corresponding serial, and 1 for the options on the corresponding coupon. We denote then by \( c^S(t, t_k, Q_k) \), \( p^S(t, t_k, Q_k) \) the time \( t \) values of european call, respectively put, options on the corresponding serial, and with \( c^C(t, t_k, 1) \), \( p^C(t, t_k, 1) \) the time \( t \) prices of european call, respectively put, options on the corresponding coupon bond. We have then

**Proposition 4.2** Under no-arbitrage, for all \( t \in [t_{j-1}, t_j) \), \( j=1,...,n-1 \), the sinking-fund bond price satisfies

\[
(4.4) \quad L_j(t) \leq B^SF(t) \leq U_j(t) \leq \min \{ B^S(t), Q_{j-1}B^C(t) \}
\]

where

\[
L_j(t) = Q_{j-1} \left[ B^C(t) - \sum_{k=j}^{n-1} \gamma_k \min \{ c^S(t, t_k, Q_k), Q_k c^C(t, t_k, 1) \} \right]
\]

\[
U_j(t) = B^S(t) - \sum_{k=j}^{n-1} \alpha_k \max \{ p^S(t, t_k, Q_k), Q_k p^C(t, t_k, 1) \}
\]

**Proof.** To establish the lower bound on \( B^SF(t) \), we observe first of all that, as an immediate consequence of Proposition 4.1, we have for all \( k=1,...,n-1 \)

\[
(4.5) \quad B^SF(t_k) \leq \min \{ B^S(t_k), Q_k B^C(t_k) \}
\]

so that

\[
\max \left\{ B^SF(t_k) - Q_k, 0 \right\} \leq \max \left\{ \min \left\{ B^S(t_k), Q_k B^C(t_k) \right\} - Q_k, 0 \right\}.
\]

or equivalently, since \( \max \{ \min \{ A, B \} - C, 0 \} = \min \{ \max \{ A-C, 0 \}, \max \{ B-C, 0 \} \} \) for all \( A, B, C \in \mathbb{R} \),

\[
\max \left\{ B^SF(t_k) - Q_k, 0 \right\} \leq \min \left\{ \max \{ B^S(t_k) - Q_k, 0 \}, Q_k \max \{ B^C(t_k) - 1, 0 \} \right\}
\]
Observing now that the LHS of the above inequality is the payoff at maturity of a European call option on the sinking-fund bond, with maturity $t_k$ and exercise price $Q_k$, while the RHS is the minimum between the payoffs at maturity, $u_k$, of a call with exercise price $Q_k$ on the serial, and of $Q_k$ calls with unit exercise price on the coupon bond, absence of arbitrage opportunities implies that for all $t < t_k$, $k=1,...,n-1$

$$c^s(t,t_k,Q_k) \leq \min \left\{ c^s(t,t_k,Q_k), Q_k c^c_p(t,t_k,1) \right\}$$

This last inequality, together with (4.2) in Proposition 4.1, yields

$$B^s(t) = Q_{j-1} \left[ B^c_p(t) - \sum_{k=j}^{n-1} \gamma_k c^s(t,t_k,Q_k) \right]$$

$$\geq Q_{j-1} \left[ B^c_p(t) - \sum_{k=j}^{n-1} \gamma_k \min \left\{ c^s(t,t_k,Q_k), Q_k c^c_p(t,t_k,1) \right\} \right]$$

which shows that, for all $t \in [t_{j-1},t_j)$, $j=1,...,n-1$, $L_j(t)$ is indeed a lower bound on $B^s(t)$. To verify that $U_j(t)$ is an upper bound on the sinking-fund bond price, exploit once again (4.5) to obtain, for all $k=1,...,n-1$

$$\max \left\{ Q_k - B^s(t_k), 0 \right\} \geq \max \left\{ Q_k - \min \left\{ B^s(t_k), Q_k B^c_p(t_k) \right\}, 0 \right\}$$

or equivalently, since $\max \left\{ C-\min\{A, B\}, 0 \right\} = \max \left\{ \max\{C-A, 0\}, \max\{C-B, 0\} \right\}$ for any real numbers $A, B, C$,

$$\max \left\{ Q_k - B^s(t_k), 0 \right\} \geq \max \left\{ \max\{Q_k - B^s(t_k), 0\}, Q_k \max\{1 - B^c_p(t_k), 0\} \right\}$$

a relation linking the payoffs at maturity of put options on the sinking-fund, corresponding serial, and corresponding coupon bonds. Absence of arbitrage opportunities implies then that, for all $t < t_k$, $k=1,...,n-1$

$$p^s(t,t_k,Q_k) \geq \max \left\{ p^s(t,t_k,Q_k), Q_k p^c_p(t,t_k,1) \right\}$$

which, together with (4.1) in Proposition 4.1, yields

$$B^s(t) = B^s(t) - \sum_{k=j}^{n-1} \alpha_k p^s(t,t_k,Q_k)$$

$$\leq B^s(t) - \sum_{k=j}^{n-1} \alpha_k \max \left\{ p^s(t,t_k,Q_k), Q_k p^c_p(t,t_k,1) \right\}$$
which shows that, for all \( t \in [t_{j-1}, t_j) \), \( j = 1, \ldots, n-1 \), \( U_j(t) \) is an upper bound for \( B^{SF}(t) \).

To conclude, we prove that \( U_j(t) \) is an upper bound at least as good as 
\[
\min \left\{ B^{SF}(t), Q_{j-1}B^{CP}(t) \right\}.
\]
Since \( U_j(t) \leq B^{SF}(t) \) is obvious, we only need to show, for all \( t \in [t_{j-1}, t_j) \), \( j = 1, \ldots, n-1 \), \( U_j(t) \leq Q_{j-1}B^{CP}(t) \). Since \( U_j(t) \leq B^{SF}(t) - \sum_{k=j}^{n-1} \alpha_k Q_k p^{CP}(t, t_k, 1) \), moreover, it is enough to establish that

\[
B^{SF}(t) - \sum_{k=j}^{n-1} \alpha_k Q_k p^{CP}(t, t_k, 1) \leq Q_{j-1}B^{CP}(t) \tag{4.6}
\]

To this end, denote by \( b_k(t) \) the time \( t \) price of a zero-coupon bond with maturity \( t_k > t \), obtained, under the assumptions on the market for zero-coupon bonds spelled out in Section 2, as the time \( t \) value of a suitable rollover strategy of the traded zero-coupon bonds. Observe that, under no arbitrage, for all \( t \in [t_{j-1}, t_j) \) we have \( B^{CP}(t) = \sum_{k=j}^{n} \beta_{bk} Q_{k} p^{CP}(t, t_k, 1) \), and apply then put-call parity to each one of the options intervening in the LHS of (4.6) to obtain

\[
p^{CP}(t, t_k, 1) = b_k(t) - \sum_{h=k+1}^{n} \beta_{bh} b_h(t) + c^{CP}(t, t_k, 1) \quad k = 1, \ldots, n-1
\]

Substituting this expression back into (4.6), observing that, under no-arbitrage, for all \( t \in [t_{j-1}, t_j) \) we have \( B^{SF}(t) = \sum_{k=j}^{n} (I_k + C_k) b_k(t) \), and recalling that \( \alpha_k Q_k = C_k \), \( I_k = Q_{k-1} \beta_{k} \) if \( k < n \), \( I_n = Q_{n-1}(\beta_{n} - 1) \), and \( Q_{k-1} = \sum_{h=k}^{n} C_h \), we obtain

\[
B^{SF}(t) - \sum_{k=j}^{n-1} \alpha_k Q_k p^{CP}(t, t_k, 1) = \sum_{k=j}^{n} Q_{k-1} \beta_{bk} b_k(t) + \sum_{k=j}^{n-1} C_k \sum_{h=k+1}^{n} \beta_{bh} b_h(t) - \sum_{k=j}^{n-1} C_k c^{CP}(t, t_k, 1)
\]

\[
= \sum_{k=j}^{n} C_k \sum_{h=k}^{n} \beta_{bh} b_h(t) + \sum_{k=j}^{n-1} C_k \sum_{h=k+1}^{n} \beta_{bh} b_h(t) - \sum_{k=j}^{n-1} C_k c^{CP}(t, t_k, 1)
\]

\[
= Q_{j-1}B^{CP}(t) - \sum_{k=j}^{n-1} C_k c^{CP}(t, t_k, 1)
\]

which shows that (4.6) holds for all \( t \in [t_{j-1}, t_j) \), \( j = 1, \ldots, n-1 \), and concludes the proof.

It is now particularly interesting to consider the case in which the assumptions on the zero-coupon bonds spelled out in Section 2 are compatible with the fact that zero-coupon bonds' prices are monotonic functions of a unique state variable \( x \). When this is case,
indeed, the lower and upper bounds for BS\(F(t)\) obtained in Proposition 4.2 can be expressed in terms of zero-coupon bonds and options on them only. To see this, we denote by \(c(t, t_k, t_h, K)\) the time \(t\) prices of european call, respectively put, options on a zero-coupon bond expiring at \(t_h\), options with maturity date \(t_k\) and exercise price \(K\), and state 

**Proposition 4.3** Suppose that, for all \(t < t_{n-2}\) and \(j\) such that \(t_j > t\), \(b_j(t) = b_j(x, t)\) is a monotonic function of the state variable \(x\). For all \(k = 1, ..., n - 1\), moreover, let \(x^Q_k\) be a solution to \(\sum_{h > k} R_h b_h(x, t_k) = Q_k\), and \(x^1_k\) a solution to \(\sum_{h > k} J_h b_h(x, t_k) = 1\). Letting then \(K^Q_k = b_h(x^Q_k, t_k)\) and \(K^1_k = b_h(x^1_k, t_k)\), for all \(t \in [t_{j-1}, t_j)\), \(j = 1, ..., n - 1\) we have 

\[
L_j(t) = \left[ \sum_{k = 1}^{n} J_k b_k(x, t) \right] \\
- \sum_{k = j}^{n-1} M_k \min \left\{ \sum_{h = k+1}^{n} R_h c(t, t_k, t_h, K^Q_{kh}), \sum_{h = k+1}^{n} J_h c(t, t_k, t_h, K^1_{kh}) \right\} \\
(4.7)
\]

\[
U_j(t) = \sum_{k = j}^{n} R_k b_k(x, t) \\
- \sum_{k = j}^{n-1} \alpha_k \max \left\{ \sum_{h = k+1}^{n} R_h p(t, t_k, t_h, K^Q_{kh}), \sum_{h = k+1}^{n} J_h p(t, t_k, t_h, K^1_{kh}) \right\} \\
(4.8)
\]

**Proof.** To prove (4.7), recall first that, for all \(t \in [t_{j-1}, t_j)\), \(j = 1, ..., n - 1\), the absence of arbitrage opportunities implies \(B_S(t) = x^Q_j R_k b_k(x, t)\) and \(B^{CP}_S(t) = x^Q_j J_k b_k(x, t)\). Under our hypotheses, and exploiting in particular the monotonicity with respect to \(x\) of the zero-coupon bonds, for all \(k = 1, ..., n - 1\) the payoff at maturity \(t_k\) of a call option on the corresponding serial bond, with exercise price \(Q_k\), becomes 

\[
\max \left\{ \sum_{h = k+1}^{n} R_h b_h(x, t_k) - Q_k, 0 \right\} = \max \left\{ \sum_{h = k+1}^{n} R_h \left( b_h(x, t_k) - b_h(x^Q_k, t_k) \right), 0 \right\} \\
= \sum_{h = k+1}^{n} R_h \max \left\{ b_h(x, t_k) - K^Q_{kh}, 0 \right\}
\]

so that, by no-arbitrage, 

\[
c^S(t, t_k, Q_k) = \sum_{h = k+1}^{n} R_h c(t, t_k, t_h, K^Q_{kh}) \text{ for } t < t_k.
\]

By the same token, the payoff at maturity \(t_k\) of a call option on the corresponding coupon bond, with unit exercise price, becomes
\[
\max \left\{ \sum_{h=k+1}^{n} J_h b_h(x,t_k) - 1, 0 \right\} = \max \left\{ \sum_{h=k+1}^{n} J_h [b_h(x,t_k) - b_h(x_{k+1},t_k)], 0 \right\} \\
= \sum_{h=k+1}^{n} J_h \max \left\{ b_h(x,t_k) - K_{kh}^1, 0 \right\}
\]

from which \( e^{CP(t,t_k,1)} = \sum_{h=k+1}^{n} J_h e(t,t_k,t_h,K_{kh}^1) \), which proves (4.7). Similar arguments, involving however the payoffs at maturity of the put options on the corresponding serial and coupon bonds, lead to (4.8).

The fact that, via relations (4.7) and (4.8), the bounds \( L_j(t) \) and \( U_j(t) \) depend only on zero-coupon bonds and options on them is particularly appealing when zero-coupon bond options prices are in closed form. A situation in which zero-coupon bonds depend on a single state variable in a way compatible with the assumptions spelled out in Section 2, the hypotheses in Proposition 4.3 are met, and a closed form is available for zero-coupon bond options, occurs when the state variable \( x \) is identified with the spot rate of interest, with evolution described by the CIR model. We analyse this situation in the next Section, where we also present numerical results for both the sinking-fund bond price \( B_{SF}(t) \) and the lower and upper bounds \( L_j(t) \), \( U_j(t) \). We argue in particular that, while the numerical computation of \( B_{SF}(t) \) is massively time-consuming even for bonds sunk in few dates, the computation of the bounds in (4.7) and (4.8) is practically effortless, and yet these bounds supply a good first-hand estimate of the sinking-fund bond price.

5. Valuation and numerical results in the CIR framework

In this section we assume that zero-coupon bond prices are functions of the spot rate of interest \( r \) prevailing on the market, i.e. \( b_j(t) = b_j(r,t) \), with the evolution of \( r \) described by the CIR model. In the first part of the section, we show how this framework can be made compatible with the assumptions on the dynamics of zero-coupon bond prices spelled out in Section 2. We recall then how the general valuation formula for contingent claims presented in Section 3 specializes in the CIR model, and we argue that in this case the upper and lower bounds for \( B_{SF}(t) \) can be expressed in closed form. Numerical valuations of the sinking-fund bond price, and of the lower and upper bounds, are presented in the second part of the section.
5.1 Valuation and bounds in the CIR framework

We assume that the spot rate $r$ is solution to the stochastic differential equation

$$dr(t) = a [\theta - r(t)] dt + \eta \sqrt{r(t)} dW(t)$$

(5.1.1)

where $a$, $\theta$ and $\eta$ are positive constants, while $W$ is the $\mathbb{R}$-valued standard brownian motion already introduced in Section 2. To show the compatibility with our general framework, we concentrate the attention on the time interval $[t_{j-1}, t_j]$, with $j \in \mathbb{N}$. It is well known (see, for instance, Cox, Ingersoll and Ross (1981), and Artzner and Delbaen (1989)) that, when the spot rate of interest is the unique state variable, arbitrage opportunities are ruled out of the market if and only if there exists a process $\lambda$, independent from the maturities of the traded bonds and known as the market price of interest rate risk, satisfying certain regularity conditions and such that

$$\lambda(r,t) = \frac{m_k(r,t) - r}{s_k(r,t)}$$

(5.1.2)

where

$$m_k(r,t) = [b_k(r,t)]^{-1} \left\{ \frac{\partial b_k(r,t)}{\partial r} a(\theta - r) + \frac{\partial b_k(r,t)}{\partial t} + \frac{1}{2} \eta^2 r \frac{\partial^2 b_k(r,t)}{\partial r^2} \right\}$$

and

$$s_k(r,t) = [b_k(r,t)]^{-1} \frac{\partial b_k(r,t)}{\partial r} \eta \sqrt{r}$$

are the drift and volatility of the instantaneous rate of return on zero-coupon bonds, obtained by Ito's Lemma. Assuming that $\lambda(r,t) = \frac{\rho \sqrt{r}}{\theta}$ for some constant $\rho \neq a$, and substituting the expressions for $m_k(r,t)$ and $s_k(r,t)$ into (5.1.2), one obtains a partial differential equation in $b_k(r,t)$, whose unique solution satisfying $b_k(r,t_k)=1$ is

$$b_k(r,t) = \exp \left\{ -r A(t_k-t) + B(t_k-t) \right\}$$

(5.1.3)

where

$$A(\tau) = \frac{2 (e^{\tau r} - 1)}{2 \gamma + (a+\rho+\gamma)(e^{\tau r} - 1)} ,$$

$$B(\tau) = \frac{2a\theta}{\eta^2} \left\{ \ln (2\gamma) + (a+\rho+\gamma) \frac{\tau}{2} - \ln \left[ 2\gamma + (a+\rho+\gamma)(e^{\tau r} - 1) \right] \right\} ,$$

$$\gamma = \sqrt{(a+\rho)^2 + 2\eta^2}.$$
Solving then (5.1.3) for \( r \), and substituting back into the expressions for \( m_k(r,t) \) and \( s_k(r,t) \), simple manipulations show that the price \( b_k \) is indeed solution to a stochastic differential equation as in (2.3) of Section 2, with

\[
\mu_k(b_k,t) = \frac{B(t_k-t) - \ln b_k}{A(t_k-t)} [1 - \rho A(t_k-t)]
\]

\[
\sigma_k(b_k,t) = -\eta \sqrt{A(t_k-t) [B(t_k-t) - \ln b_k]}.
\]

In order to obtain numerical estimates of the sinking-fund bond price, it is useful to observe that, in the CIR framework, the valuation formula \( \pi_t(Z_j) = b_j(t)E^P[Z_j|\mathcal{F}_t] \) supplied by (3.3) in Section 3 can be rewritten \( \forall t \in [t_{j-1},t_j] \) as

\[
(5.1.4) \quad \pi_t(Z_j) = E \left[ Z_j \exp \left\{ -\frac{\rho}{\eta} \int_t^{t_j} \sqrt{r(s)} \, dW(s) - \left( \frac{\rho^2}{2\eta^2} + 1 \right) \int_t^{t_j} r(s) \, ds \right\} \right| \mathcal{F}_t
\]

with the expectation taken with respect to \( P \), the probability measure under which \( W \) is a standard brownian motion. To derive (5.1.4) from \( \pi_t(Z_j) = b_j(t)E^P[Z_j|\mathcal{F}_t] \), observe that

\[
E^P[Z_j|\mathcal{F}_t] = E \left[ Z_j \exp \left\{ -\int_t^{t_j} q_j(s) \, dW(s) - \frac{1}{2} \int_t^{t_j} q_j^2(s) \, ds \right\} \right| \mathcal{F}_t
\]

with \( q_j \) defined in condition C1 of Section 2. Relation (5.1.4) follows then upon substituting \( q_j(s) = \frac{\rho \sqrt{r}}{\eta} - \sigma_j(b_j,s) \) in the above expression, and noting that, by applying Ito's Lemma to \( \ln b_j(t) \) and exploiting the condition \( b_j(t_j) = 1 \), one has

\[
b_j(t) = \exp \left\{ -\int_t^{t_j} \left[ \mu_j(b_j,s) - \frac{1}{2} \sigma_j^2(b_j,s) \right] \, ds - \int_t^{t_j} \sigma_j(b_j,s) \, dW(s) \right\}.
\]

Coming now to the bounds supplied in Section 4, we observe that, in the CIR framework, the assumptions of Proposition 4.3 are met since zero-coupon bond prices are decreasing in \( r \). For suitable sets of parameters, moreover, for all \( k=1,\ldots,n-1 \) we have

\[
\sup_r \sum_{h>k} R_h \ b_h(r,t_k) > Q_k \quad \text{and} \quad \sup_r \sum_{h>k} \partial_h b_h(r,t_k) = 1,
\]

so that the continuity and monotonicity of \( b_h(r,t_k) \) in \( r \) imply that, for all \( k<n \), the equations \( \sum_{h>k} R_h \ b_h(r,t_k) = Q_k \) and \( \sum_{h>k} \partial_h b_h(r,t_k) = 1 \) admit solutions. Since in the CIR framework the prices of call and put options can be expressed in closed form, the bounds \( L_j(t) \) and \( U_j(t) \) supplied by (4.7) and (4.8) admit themselves a closed form. In this framework, indeed, the time \( t \)
price of a European call option with exercise price $K$ and expiring at $t_k > t$, written on a zero-coupon bond with maturity $t_h > t_k$, takes the form

$$c(t, t_k, t_h, K) = b_h(r, t) \chi^2 \left( 2r^* \left[ \frac{\phi + \psi + A(t_h - t_k)}{\eta^2} , \frac{2r\phi^2 \exp(\gamma(t_k - t))}{\phi + \psi + A(t_h - t_k)} \right] \right)$$

$$- K b_k(r, t) \chi^2 \left( 2r^* \left[ \frac{\phi + \psi}{\eta^2} , \frac{2r\phi^2 \exp(\gamma(t_k - t))}{\phi + \psi} \right] \right)$$

where $\phi = \frac{2\gamma}{\eta^2 \left[ \exp(\gamma(t_k - t)) - 1 \right]}$, $\psi = \frac{a + \rho + \gamma}{\eta^2}$, $r^* = \frac{B(t_h - t_k) - \ln K}{A(t_h - t_k)}$ is solution to the equation $b_h(r, t_k) = K$ and $\chi^2(x, y, z)$ denotes the value in $x$ of the distribution function of a noncentral chi-square variate with $y$ degrees of freedom and noncentrality parameter $z$.

The price of the corresponding put is obtained from (5.1.5) by put-call parity.

5.2 Numerical results

We are now ready to present some numerical results for the sinking-fund bond price and the lower and upper bounds in the CIR framework. The valuation of the sinking-fund bond by means of the recursive procedure described in Section 3 requires, in principle, the computation of nested expectations, each one taking the form described by relation (5.1.4) of this section, with $Z_j = L_j + BSF(t_j)$. To estimate such expectations we follow a Monte Carlo approach, which requires the simulation, on a discrete but sufficiently fine grid, of the increments $dW(s)$, from which discrete paths for the state variable $r$ are obtained. In such a procedure, a crucial problem is to establish how fine should the discrete grid be, and how many discrete paths for $r$ should be simulated. A sensible answer to this problem is, in our context, to choose in such a way that the accuracy of the estimate for $BSF(t)$ via the Monte Carlo approach is better than the one obtained by estimating $BSF(t)$ with the middle point of the interval $[L_j(t), U_j(t)]$. More precisely, given a probability $\beta$, we choose the grid and the number of simulated paths in such a way that the interval in which $BSF(t)$ lies with probability $1 - \beta$ has length smaller than $U_j(t) - L_j(t)$, and its intersection with $[L_j(t), U_j(t)]$ is not empty. If these conditions are not met, the estimate of $BSF(t)$ given by $\frac{1}{2} [L_j(t) + U_j(t)]$, with accuracy $\frac{1}{2} [U_j(t) - L_j(t)]$, is to be preferred to the one obtained via Monte Carlo simulation.

The numerical results that follow are obtained letting $n = 3$, $t_j = j$ for $j = 0, \ldots, 3$, unit outstanding capital at issuance, and flat amortization schedule, i.e. $Q_0 = 1$ and $C_j = \frac{1}{3}$. In
particular, we estimate the sinking-fund bond price at its time of issuance, \( t=0 \). To reduce the variance of the estimator obtained by the simulation approach, we employ relation (4.1) of Section 4, from which, under our parametrization, and since \( p_{SF}(0,2,Q_{2}) = \frac{e^{\delta}}{3} p_{SF}(0,2,3,e^{-\delta}) \), we have

\[
(5.2.1) \quad B^{SF}(0) = B^{S}(0) - \frac{1}{2} p^{SF}(0,1,\frac{2}{3}) - \frac{e^{\delta}}{3} p_{SF}(0,2,3,e^{-\delta})
\]

with \( B^{S}(0) = \left( e^{\delta} - \frac{2}{3} \right) b_{1}(r(0),0) + \left( \frac{2e^{\delta}}{3} - \frac{1}{3} \right) b_{2}(r(0),0) + \frac{e^{\delta}}{3} b_{3}(r(0),0) \). Indeed, since \( b_{1}(r(0),0) \) is obtained from (5.1.3) and \( p(0,2,3,e^{-\delta}) \) from (5.1.5) via put-call parity, the only quantity in (5.2.1) that needs to be simulated is \( p^{SF}(0,1,\frac{2}{3}) \), and to do so we let \( t=0 \), \( t_{j}=1 \) and \( Z_{j} = \max\left\{ \frac{2}{3} - B^{SF}(1), 0 \right\} \), with \( B^{SF}(1) = \left( \frac{2e^{\delta}}{3} - \frac{1}{3} \right) b_{2}(r(1),1) + \frac{e^{\delta}}{3} b_{3}(r(1),1) - \frac{e^{\delta}}{3} p_{SF}(1,2,3,e^{-\delta}) \), in relation (5.1.4). A large amount of numerical experiments, performed with different sets of parameters, showed that, in order to get an estimate of \( B^{SF}(0) \) that improves the one obtained by considering the middle point of \([L_{1}(0), U_{1}(0)]\), at least 10,000 paths for \( r \) on a grid with step \( 10^{-2} \) need to be simulated. To get an insight of some of the comparative static properties of \( B^{SF}(0) \), and of the bounds \( L_{1}(0) \) and \( U_{1}(0) \), we fix the following set of "basic" values for the parameters of the process driving \( r \):

\[
\begin{align*}
    r(0) &= 0.08, \quad \eta = 0.06, \quad \alpha = 0.3, \quad \theta = 0.1, \quad \rho = 0,
\end{align*}
\]

while we set the coupon rate at \( \delta = 0.09 \). We let then the current instantaneous spot rate \( r(0) \), the volatility coefficient \( \eta \), and the mean reversion coefficient \( \alpha \) vary one at a time, while leaving the remaining parameters at their basic value.

Figure 1 reports a histogram in which the values of the sinking-fund bond at issuance, and of the corresponding lower and upper bounds, are presented for levels of the spot rate \( r(0) \) varying between 0.06 and 0.12, with step 0.002. For the same values of \( r(0) \), Figure 2 displays the upper bound \( U_{1}(0) \), and the prices \( B^{S}(0) \) and \( B^{CP}(0) \) of the corresponding serial and coupon bonds.
All the variables in Figures 1 and 2 present, as expected, a trend decreasing with \( r(0) \). In Figure 1, moreover, \( B_{SP}(0) \) practically coincides with the upper bound, and is hardly distinguishable from the lower bound as well. The difference \( U_1(0) - L_1(0) \) varies from 1.51bp when \( r(0) = 12\% \) to 3.85bp when \( r(0) \) is about 8\%, while the 95\%-confidence interval obtained via simulation reaches its minimum width, 0.535bp, when \( r(0) = 6\% \) and its maximum one, 2.69bp, when \( r(0) = 12\% \). Recall moreover that, by taking into account the arbitrage-based bounds, the interval in which the sinking-fund bond price lies with a 95\% probability is obtained by intersecting \([L_1(0), U_1(0)]\) with the 95\% confidence interval. For the parametrization under scrutiny, such intersection has minimum width, 0.171bp, when \( r(0) = 6\% \), and maximum one, 2.36bp, when \( r(0) = 10.8\% \). Figure 2 shows that the difference between \( \min\{B_S(0), B_{CF}(0)\} \) and the upper bound \( U_1(0) \) reaches its minimum, 4.75bp, when \( r(0) = 12\% \), and its maximum, 45.05bp, when \( r(0) = 8.2\% \), i.e. \( U_1(0) \) is an upper bound strictly tighter than \( \min\{B_S(0), B_{CF}(0)\} \).

Figures 3 and 4 report the results obtained when the volatility parameter \( \eta \) varies between 0.01 and 0.1 with step 0.003.

**Figure 3**

![Figure 3](image-url)
These figures display a negative trend for $B_{SF}(0)$ and the corresponding bounds, and a positive one for $B_S(0)$ and $B_{CP}(0)$. Moreover, the sinking-fund bond price is significantly different from its lower bound and, in contrast to the comparative static with respect to $r(0)$, here a difference between $B_{SF}(0)$ and $U_1(0)$ is perceptible as well. The difference between $U_1(0)$ and $L_1(0)$ reaches its minimum (1.08bp) when $\eta=1\%$ and its maximum (6.28bp) when $\eta=10\%$. The 95%-confidence interval has a width varying from 0.158bp when $\eta=1\%$ to 2.46bp when $\eta=10\%$, while the intersection between this interval and $[L_1(0),U_1(0)]$ has minimum width (0.104bp) when $\eta=1\%$, and maximum one (1.96bp) when $\eta=10\%$. Eventually, the difference between $\min\{B_S(0),B_{CP}(0)\}$ and $U_1(0)$ is minimum (5.57bp) when $\eta=1\%$, and maximum (67.12bp) when $\eta=10\%$.

Figure 5 reports the simulated value of the sinking-fund bond and the corresponding lower and upper bounds when the mean reversion coefficient $a$ varies between 0.1 and 1 with step 0.03, while Figure 6 reports the values of $U_1(0)$, $B_S(0)$ and $B_{CP}(0)$.
Figure 5

Figure 6
From Figures 5 and 6 we observe a negative trend for all the variables, and a value of the sinking-fund bond significantly different from its lower bound, but, once again, almost indistinguishable from \( U_1(0) \). The difference \( U_1(0) - L_1(0) \) varies between 0.825bp when \( a=100\% \) and 3.86bp when \( a \) is in a neighborhood of 25\%. The 95%-confidence interval reaches its minimum width, 1.08bp, when \( a=100\% \), and its maximum one, 1.5bp, when \( a=10\% \). The width of the intersection between the two intervals is minimum (0.735bp) when \( a=100\% \), and maximum (1.02bp) when \( a=10\% \). To conclude, the minimum difference between \( \min\{B^S(0), B^C(0)\} \) and \( U_1(0) \) is 2.81bp, reached when \( a=100\% \), while its maximum, 41.93bp, is attained for \( a=34\% \).

When more than two sinking-fund dates remain before maturity, i.e. \( n>3 \), some preliminary experiments seem to indicate that to obtain a sufficiently accurate estimate for \( B^{SF}(0) \), via Monte Carlo simulation, a very large amount of computation time is required, hence highlighting the operational usefulness of the arbitrage-based bounds produced in Proposition 4.3 of Section 4. Even for very large values of \( n \), indeed, the CPU time required to compute the bounds is negligible (at least on a DEC 7000 mainframe).

Maintaining the assumptions \( \tau_j = j \) for \( j=0, \ldots, n \), \( Q_0 = 1 \) and \( C_j = \frac{1}{n} \), we conclude this section by presenting, in Figure 7, the values assumed by \( L_1(0) \) and \( U_1(0) \) when all the parameters intervening in the valuation are set equal to the basic values previously introduced, while \( n \) varies between 3 and 30. In Figure 8, instead, we plot the differences \( U_1(0) - L_1(0) \) against \( n \).
Figure 7 shows that both \( L_1(0) \) and \( U_1(0) \) decrease with the time to maturity of the sinking-fund bond, hence suggesting that, for the parameters under scrutiny, \( B^{SF}(0) \) decreases with maturity. The difference \( U_1(0) - L_1(0) \) in Figure 8 is instead humped, reaching its maximum value, 31.1bp, when \( n=11 \), and then decreasing steadily to 10.6bp when \( n=30 \), a figure anyway above the minimum value, 3.83bp, attained for \( n=3 \).

6. Concluding remarks

We have discussed in this paper the problem of pricing a sinking-fund bond which allows its issuer to recurrently retire portions of the bond by either a lottery call at par, or by an open market operation. We have assumed the sinking-fund bond to be default-risk-free, and that the unique source of uncertainty is constituted by a stochastic term structure of interest rates, which we have modeled by formalizing directly the dynamics of zero-coupon bonds. By exploiting the martingale methodology for pricing contingent claims under no arbitrage, we have supplied a recursive valuation procedure for pricing a sinking-fund bond with a generic number of sinking-fund dates before maturity.
Observing that, even for relatively simple parametrization of the term structure, our valuation procedure does not yield a closed form solution, and that a numerical approach is cursed by a complexity exponentially increasing with the number of sinking-fund dates before maturity, we have produced arbitrage-based lower and upper bounds for the sinking-fund bond price. When zero-coupon bond prices are monotonic functions of a single state variable, our bounds have the appealing property of depending on zero-coupon bonds and options on them only. In particular, when the single state variable is the spot rate of interest and follows the CIR model, our bounds are in closed form and hence their computation is practically effortless. Some numerical experiments, moreover, have shown that these bounds provide a good first-hand approximation to the sinking-fund bond price.

A problem which has not been dealt with in this paper is that of merging interest rate risk with the risk of default, i.e. with the possibility that the issuer may default on its obligations. The application of the martingale methodology for pricing a sinking-fund bond issue when these risks are both taken into account constitutes an interesting topic for future research.

References


