Consequences of Differing Actuarial Interest Rates on the Profitability of an Endowment Policy: An Analysis in Consideration of Stochastic Investment Income and a Simple Profit Participation System

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Summary
In this paper, we analyse the consequences resulting from differing actuarial interest rates. The profitability of an endowment policy is considered in the case of the interest rate used for premium calculation $i_p$ being higher than the interest rate used to calculate the net premium reserve $i_n$. From this results a positive net premium reserve at the beginning of the contract period whereas no premium income has been realized so far. Therefore, the change in the capital resources is negative and this deficit must be compensated by investment income. In order to build a realistic model, investment income must be treated as a random variable. Once the initial deficit has disappeared, a profit participation is paid that is used to buy additional insurance coverage.

Résumé
Dans cette contribution, nous analysons les conséquences résultant des taux d'intérêt différents dans l'assurance-vie. La profitabilité d'un produit d'assurance-vie est considérée dans le cas où le taux d'intérêt utilisé pour le calcul de la prime $i_p$ est supérieur au taux d'intérêt utilisé pour le calcul de la réserve actuarielle $i_n$. Par conséquent, il résulte une réserve actuarielle positive au début du contrat bien que la première prime ne soit pas encore payée. C'est pourquoi, la variation du capital propre est negative et ce déficit doit être compensé par le rendement des investissements. Pour la construction d'un modèle réaliste, il faut traiter le rendement des investissements comme variable aléatoire. Dès que le déficit initial a disparu, une participation du profit est payée qui est utilisé pour l'achat du protection d'assurance supplémentaire.

Keywords
Profitability, risk analysis, differing actuarial interest rates, profit participation.

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1. Introduction

Since July 29, 1994, German life insurance companies have the permission to calculate premiums and net premium reserves with differing interest rates. As German accounting principles require the prospective method to determine the amount of the net premium reserve of a life insurance contract, a temporary deficit at the beginning of the contract period will result in case the interest rate used for premium calculation $i_p$ is higher than the interest rate used for reserving $i_r$. This can be seen by the fact that according to the equivalence principle expected discounted premiums must equal expected discounted payments to the person insured at the beginning of the contract period. As the prospective reserve at any point in time is defined as the difference between expected future premiums and expected future payments, both discounted at $i_r$, and as premiums are the lower the higher the interest rate $i_p$ is chosen, expected future premiums are not sufficient to offset expected future payments to the person insured. Therefore, the net premium reserve is positive at the beginning of the contract period though no premium income has been realized so far. German regulation demands that assets are to be held in the amount of the net premium reserve. In case of $i_p > i_r$, the necessary amount of assets cannot be financed, at least at the beginning of the life insurance contract, by premium and resulting investment income alone so that additional amounts of assets must be provided in order to offset this deficiency. These assets can only be financed by the capital resources of the insurance company, and for that reason such a deficiency has a major impact on the profitability of the life insurance company. The focus of our discussion therefore is on the change in the insurer’s capital resources induced by the life insurance contract in question.

As premiums are invested in the financial market and the resulting investment income of rate $i_o$ is likely to be higher than the relevant actuarial interest rate $i_p$ which has to be chosen carefully according to the German Insurance Supervision Act, the initial deficit can be reduced by the surplus earned from investment income over the increase in the net premium reserve. Once the sum of premiums paid to the life insurer and the proceeds of interim investments is higher than the net premium reserve, an actuarial
surplus exists which must be distributed between the insured and the company. We assume that profit participation is used to buy additional insurance coverage so that the sum insured grows the longer the contract is valid (so called bonus system). The additional sum insured can be calculated using the premium interest rate $i_p$ or the reserve interest rate $i_r$. As we will see, the results differ dramatically. All numerical examples are based on an endowment policy, which is the most important product in the German life insurance market, and a male person aged $x = 30$ years at the beginning of the contract period. The duration of the life insurance contract is also fixed at $n = 30$ years.

As we cannot know the income earned from investing premiums in the financial market at the time the contract is settled, the resulting interest rate $i_o$ must be modelled as a random variable. Therefore, the amount of assets attributable to the life insurance contract is a random variable, too. As the profit participation granted to the life insured is dependent on the amount of assets available and as in addition profit participation itself determines the total amount of the net premium reserves, all these quantities must be treated as random variables relative to the investment income. The increase in the assets and liabilities for that reason is of stochastic nature. As the insurer guarantees a minimum return on the net premium reserve and as the total net premium reserve grows with profit participation each period an actuarial surplus has been realized, an adverse investment result may cause a severe financial crisis for the life insurer because the amount of assets necessary to cover the increased liability must be financed in advance by its capital resources if assets resulting from the contract are insufficient to cover it.

For the purposes of this paper, the risk associated with a life insurance contract is defined as the danger that liabilities are greater than the assets induced by the life insurance contract; that means the risk is conceptualized as the danger that the change in the capital resources of the insurer becomes negative. Assets and liabilities are stochastic relative to the investment income, and from this the change of the capital resources is a random variable, too. The problem of attaining a positive change of the
capital resources is complicated in the context of this paper because the initial net premium reserve is positive at the beginning of the contract period and therefore we start with a negative change in the capital reserve.

The rest of the paper is organized as follows. Chapter 2 introduces the actuarial model of surplus generation, the development of the change in the capital resources and the participation of the person insured as well as a simple stochastic investment model in which the natural logarithm of the accumulation factor \((1 + i_n)\) is treated as an autoregressive moving average process of the order \((1,1)\). As assets and liabilities are stochastic in nature because of the uncertainty of the investment result and as the complexity of the problem with multiple truncated cash flow relations between insurer and contract holder is very high, the probability distribution of the change in the capital resources cannot be derived analytically. We use a Monte Carlo approach instead. Chapter 3 shows the simulation results for different parameter constellations of the ARMA \((1,1)\)-process and for different calculation systems for the additional sum insured generated by profit participation. Chapter 4 concludes the paper.

2. The Model

2.1 Modelling the Participation System and the Change in the Capital Resources of the Insurance Company

Let \(TSI(m)\) be the total sum insured which is attained by the contractholder after \(m\) periods including profit participation. The initial sum insured is fixed at \(SI(0) = TSI(0) = 1000\) and the additional sum insured created by an actuarial surplus leading to profit participation in period \(j\) will be given by \(SI(j)\).

After the first period the net premium reserve attributable to the contract in question before profit participation \(V_x(i_p,i)\) comes to\(^2\)
As the interest rate for premium calculation $i_p$ is higher than the interest rate used for reserving $i_r$, the initial value of the net premium reserve $V_x(i_p, i_r)$ is positive. If $i_r$ was higher than $i_p$, the initial reserve would be negative, and in case of $i_p = i_r$ the initial reserve would be zero. This was the state of regulation in Germany before July 1994.

The premium is invested in the financial market. In consideration of the interest income earned for the first period $i_{w,1}$, the expected amount of assets (relative to mortality) attributable to the contract in question is given by

$$ W_x(i_p, i_r, i_{w,1}) = \frac{1}{P_x} \pi(i_p) (1+i_{w,1}) - \frac{q_x}{P_x} TSI(0). $$

From the perspective of the beginning of the contract period, this expected value (relative to mortality) is a random variable relative to investment income. At time one, the assets (premium paid by the contractholder plus investment income) and liabilities (initial value of the net premium reserve plus guaranteed increase in the net premium reserve) incurred by the insurance company are compared. Suppose there is a surplus of assets over liabilities, the contractholder will obtain a participation of rate $\psi(1)$%. In case of a deficit, the contractholder will not be affected. If the interest income that is necessary to finance the growth of the reserve cannot be attained in the financial market, there must be a subsidy financed by the insurance company's capital resources. The amount of profit participation is for that reason given by

$$ PP_x(i_p, i_r, i_{w,1}) = \psi(1) \max \{ W_x(i_p, i_r, i_{w,1}) - V_x(i_p, i_r); 0 \} $$

where $PP_x(i_p, i_r, i_{w,1})$ symbolizes the profit participation after one period that is dependent as well on the interest rate for premium and reserve calculation as on the interest rate earned in the financial market. Therefore, it has also to be treated as a random variable.

Interpreting profit participation of the first period as a single premium of a new endowment policy with duration $n-1$, the resulting additional sum insured runs up to
\[ SI(1) = \frac{PP_x(i_p, i_r, i_u)}{A_{x+1,n-1}(i_p)}, \]  

where \( A_{x+1,n-1}(i_p) \) is the single premium of an endowment policy for a person aged \( x+1 \) and duration \( n-1 \) calculated with the interest rate \( i_p \). Thus the total sum insured for the second period is given by

\[ TSI(1) = TSI(0) + SI(1) = SI(0) + SI(1). \]  

The amount to be reserved for the additional insurance coverage attributed to the contract by profit participation comes to

\[ VPP_x(i_p, i_r, i_u) = W_x(i_r, i_u) A_{x+1,n-1}(i_p). \]  

Hence, the total reserve after one period and in consideration of the profit participation mechanism amounts to

\[ V_x^*(i_p, i_r, i_u) = V_x^*(i_p, i_r) + VPP_x(i_p, i_r, i_u). \]  

The resulting reserve from profit participation equals the amount of profit participation if and only if the \( i_p = i_r \). In our case \( i_p > i_r \), the life insurance company is confronted with the same problem as for the initial sum insured: There will not be sufficient assets financed by the profit participation in order to cover all liabilities resulting from the contract. As the interest rate earned in the financial market \( i_u \) is modelled as a stochastic variable, the total reserve after profit participation \( V_x^*(i_p, i_r, i_u) \) is of stochastic nature as well.

At this point, we have to differentiate whether the expected amount of assets resulting from the life insurance contract after two periods is higher than the net premium reserve after profit participation or not. If \( W_x(i_p, i_u) > V_x^*(i_p, i_r, i_u) \), the excess amount is transferred to the capital resources of the insurance company and the amount of assets attributable to the contract is reduced to the amount of the total reserve. The change in the capital resources induced by the life insurance contract in question which is relevant for the development of the insurer's profit and risk situation after one period is modelled in the following way:
\( VCR(i_p,i_r,i_w) = \max \{ 0, VCR(i_p,i_r,i_w); 0 \} (1+i_w) + W_s(i_p,i_r,i_w) - V_s(i_p,i_r,i_w). \) (8)

The assets resulting from the existence of the contract which are not necessary to cover the reserve after profit participation are due to the life insurer. If the assets resulting from the contract are not sufficient to cover the actuarial liabilities, a subsidy of the assets financed by the insurer's capital resources is necessary. Therefore, the amount of assets induced by the contract after two periods is to be modelled as

\[ W_s(i_p,i_r,i_w) = \frac{1}{p_{x+1}} \left[ \max \{ 0; \min \{ W_s(i_p,i_r,i_w), V_s(i_p,i_r,i_w) \} \} + \pi(i_p) \right] (1+i_w) - \frac{q_{x+1}}{p_{x+1}} TSI(1). \] (9)

We are now able to give a general formulation of the profit generation and capital accumulation mechanism. After \( m-1 \) periods, the total sum insured up to this point in time (before profit participation at time \( m \), but after profit participation at time \( m-1 \)) amounts to

\[ TSI(m-1) = SI(0) + SI(1) + \ldots + SI(m-1) = SI(0) + \sum_{j=1}^{m-1} \frac{PP_s(i_p,i_r,i_w)}{A_{x+j,n-j}(i_p)}. \] (10)

The total net premium reserve at time \( m \) (before profit participation at time \( m \), but after profit participation at time \( m-1 \)) is given by the recursion

\[ V_s(i_p,i_r,i_w) = \frac{1}{p_{x+m-1}} \left[ m-1 V_s(i_p,i_r,i_w) + \pi(i_p) \right] (1+i_r) - \frac{q_{x+m-1}}{p_{x+m-1}} TSI(m-1), \] (11)

whereas the amount of assets resulting from the contract runs up to

\[ W_s(i_p,i_r,i_w) = \frac{1}{p_{x+m-1}} \left[ \max \{ 0; \min \{ W_s(i_p,i_r,i_w), V_s(i_p,i_r,i_w) \} \} + \pi(i_p) \right] (1+i_w) - \frac{q_{x+m-1}}{p_{x+m-1}} TSI(m-1). \] (12)

A new comparison of assets and liabilities enables us to express the profit participation after \( m \) periods through application of the profit participation factor \( \psi(m) \) as
\[ m PP_x(i_p, i, i_u) = \psi(m) \max \{ m W_x(i_p, i, i_u) - m V_x(i_p, i, i_u); 0 \}. \] (13)

The additional sum insured amounts to
\[
SI(m) = \frac{m PP_x(i_p, i, i_u)}{A_{x+m,n-m}(i_p)}. \] (14)

For this profit participation, an initial reserve in the amount of
\[
m VPP_x(i_p, i, i_u) = SI(m) A_{x+m,n-m}(i_p) = m PP_x(i_p, i, i_u) \frac{A_{x+m,n-m}(i_p)}{A_{x+m,n-m}(i_p)} \] (15)

is necessary. For \( i_p > i_r \), the initial amount to be reserved is greater than the profit participation itself. Therefore, it may be the case that a positive change in the capital resources before profit participation which is the reason for an actuarial surplus and the participation of the insured turns into a negative change after profit participation. If the investment income of the next period is high enough to finance the deficiency and the growth of the net premium reserve, a new profit participation can be effected.

The total net premium reserve at time \( m \) after profit participation is then by
\[
m V_x(i_p, i, i_u) = m V_x(i_p, i, i_u) + m VPP_x(i_p, i, i_u), \] (16)

and the total sum insured at this point in time after profit participation reaches
\[
TSI(m) = TSI(m-1) + SI(m) = \sum_{j=0}^{m} SI(j). \] (17)

The change in the capital resources then amounts to
\[
m VCR_x(i_p, i, i_u) = \max \{ m-1 VCR_x(i_p, i, i_u); 0 \} (1 + i_{w,m})
+ m-1 W_x(i_p, i, i_u) - m-1 V_x(i_p, i, i_u). \] (18)

The following paragraph concentrates on a simple stochastic model of the investment income earned by the life insurer through investing premium income.
2.2 Modelling the Logarithm of the Accumulation Factor \((1+i,n)\) as an Autoregressive Moving Average Process of the Order \((1,1)\)

In order to model the interest income earned on the life insurer's assets, we assume that the natural logarithm of the accumulation factor \(\delta_i := \ln (1+i,n)\) follows an autoregressive moving average process of the order \((1,1)\).\(^3\)\(^4\) This choice guarantees a fairly general stochastic model that is able to characterize a great variety of empirical time series adequately. We assume that the natural logarithm of the accumulation factor \(\delta_i\) is given by the recursive equation

\[ \delta_i = \theta + \phi [\delta_{i-1} - \theta] + U_i - \beta U_{i-1}, \]

where we further assume that the noise variable \(U_i\) is independent and identically normally distributed with expected value zero and variance \(\gamma^2\) in each period. Beyond that, the parameters \(\beta\) and \(\phi\) are supposed to be less than one in absolute value. Defining additionally \(Y_i := \delta_i - \theta\), the centered process is given by

\[ Y_i = \phi Y_{i-1} + U_i - \beta U_{i-1}. \]

When applying the recursion repeatedly, we arrive at \((t-\tau)\) and the process can then be expressed as

\[ Y_t = \phi^\tau Y_{t-\tau} + U_t + (\phi - \beta) \sum_{j=1}^{\tau-1} U_{t-j} - \phi^{\tau-1} U_{t-\tau}. \]

For \(\tau \to \infty\) the process has the representation of an moving average process of infinite order.\(^5\)

\[ Y_t = U_t + (\phi - \beta) \sum_{j=1}^{\infty} \phi^{j-1} U_{t-j}. \]

By this representation, it is immediately obvious that \(E(Y_t) = 0\) for all points in time \(t\) and therefore \(E(\delta_t) = \theta\). As the process is weakly stationary\(^6\), expected value and variance of the process are independent of \(t\). The autocovariances are only dependent on the lag, but not on the absolute position on the time axis. After some matrix algebra, the closed form solution of the covariance function with lag \(\tau\) for the process \(\{Y_t, t = 0, 1, \ldots\}\) is given by\(^7\)^8

\[ \text{cov}(Y_t, Y_{t-\tau}) = (\phi - \beta) \sum_{j=1}^{\infty} \phi^{j-1} \text{cov}(U_{t-j}, U_{t-\tau}). \]
The variance of \( Y_t \) will be abbreviated by \( \text{Var}(Y_t) = \sigma^2 \).

Due to the basic assumption concerning the noise variables, with \( \ln(1+i_{w,t}) \) we have a linear function of independent and identical normally distributed variables, so is the logarithm of the accumulation factor. The accumulation factor \( (1+i_{w,t}) \) itself in period \( t \) then follows a logarithmic normal distribution with parameters \( \theta \) und \( \sigma^2 \). The process is not only weakly stationary, but even stationary. As the residual variables are supposed to be identical normally distributed in each period and as the logarithmic normal distribution is completely determined by two parameters, the accumulation factor follows the same logarithmic normal distribution in each period. Hence, we have for expected value, variance and \((1-\epsilon)\) quantile of the distribution:

\[
\begin{align*}
E[(1+i_{w,t})] &= \exp \left( \theta + \frac{1}{2} \sigma^2 \right) \\
\text{Var}[(1+i_{w,t})] &= \exp(2\theta + \sigma^2) \left[ \exp(\sigma^2) - 1 \right] \\
F_{\epsilon}[(1+i_{w,t})] &= \exp[\theta + N_\epsilon \sigma].
\end{align*}
\]

The crucial factor in modelling the yield on the assets is the realistic representation of the investment behaviour of German life insurance companies by the parameters of the process. The process must be chosen in a way that excludes investment results which are highly improbable due to the asset allocation of German life insurance companies. Therefore, we choose the parameters of the process in a way that only 5\% of the realizations yield an investment return less than 4%; with other words, the 5\%-quantile of the distribution of the accumulation factor is set to 1.04 which can be interpreted as a cautious approximation of reality. By fixing the expected value and the 5\%-quantile, the whole distribution is unequivocally determined. Figure 1 illustrates the density function of the accumulation factor.
3. Simulation Results

3.1 Payments and Capital Resources when Calculating the Bonus Using $i_p$

The analysis of this paragraph aims at the explication of the danger that the assets induced by the life insurance contract are not sufficient to cover the total net premium reserve when using a higher interest rate for premium calculation than for the determination of the reserve in consideration of the profit participation system of § 2.1. The change of the capital resources of the insurer would then be negative; it’s now our task to quantify this danger. In order to quantify the danger, we have to choose a suitable risk measure.

The symmetry of the risk measure standard deviation assumes that deviations from the expected change in the capital resources to either side are equally dangerous to the insurance company. This is not an appropriate way to conceptualize risk as especially
the possibility of a negative change of the capital resources demands a corresponding subsidy. Therefore, asymmetric risk measures are more appropriate in order to represent the danger due to the stochasticity of the investment income that assets resulting from the life insurance contract are insufficient to cover the corresponding liabilities. We use as a general class of risk measures lower partial moments of the distribution of the change in the capital resources \( CCR \) at a special point in time \( t \). The \( k \)th lower partial moment \( \mu_k[CCR(t)] \) and a non-negative change in the capital resources as target yields\(^{12}\)

\[
\mu_k[CCR(t)] = (-1)^k \int_{-\infty}^{0} ccr^k(t) f(crr) \, dcr. 
\] (27)

As the change in the capital resources is a very complex function of assets and liabilities with multiple payment considerations, an analytic distribution of this random variable cannot be found. Therefore, we derive this unknown distribution by a Monte Carlo approach. In the context of the Monte Carlo approach, we approximate the integral through an appropriate estimation in the following way

\[
\hat{\mu}_k[CCR(t)] = \frac{1}{H} \sum_{h=1}^{H} ccr^k(h,t) I(h,t), 
\] (28)

where \( I(h,t) \) symbolizes an indicator variable that takes the value 1 if the change in the capital resources of the \( h \)th simulation run at time \( t \) is negative and the value 0, if we find a positive result. \( H \) is the number of simulation runs.\(^{13}\) In the following, we look for different points in time at the 0th lower partial moment which will be named shortfall probability and the first lower partial moment, the shortfall expectation.

The construction of the profit participation mechanism has a major effect on the development of the change in the capital resources if stochastic interest income is integrated in the analysis. Supposing that assets and liabilities attributable to the life insurance contract are equal at the beginning of the period considered and that the insurance company has realized a high investment performance in the that period which leads at the next point in time to a high profit participation, the financial consequences
are considerable if in the following period the minimum investment return required for
financing the assets to cover the necessarily higher reserves cannot be attained. In
order to compensate a negative change in the capital resources after a further period,
the investment yield earned on the assets induced by the contract must exceed the
guaranteed minimum investment return i,. The stochasticity of the investment return
implies that the probability for a negative change in the capital resources is strictly
positive.

For the concrete calculations we use a premium interest rate i, = 4.5% and a reserve
interest rate i, = 4%. For that reason the insurance company is confronted with an
initial deficit that must be equalized by net capital financed assets. To begin with, we
suppose that the expected value of the interest rate that is earned in the financial market
on the insurers’ assets amounts to 7%.

The profit participation starts after settlement of the initial deficit, and the additional
sum insured is calculated using the premium interest rate i,. Due to this, we are in the
same situation as for the initial sum insured: The profit participation is lower than the
corresponding reserve and the assets attributable to the profit participation are not
sufficient to cover the reserve. The life insurance company therefore is confronted with
the continuance of the demand of the capital resources. We assume that the periodic
participation rate is time independent and runs up to ψ = 95%.

As a result of the initial deficit caused by i, > i, and the non-development of an
actuarial surplus, the net premium reserve is independent of the investment income of
the insurance company during the first five periods. Beginning with this point in time,
the results of the simulation begin to diverge from each other. Numerical values for
selected points in time (t = 15, 20, 25 und 30) can be found in Table 1.
Values derived from the simulation for $V^*(5), V^*(10), V^*(15), V^*(20), V^*(25), V^*(30)$

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<tbody>
<tr>
<td>expected value [DM]</td>
<td>121,84</td>
<td>246,26</td>
<td>438,86</td>
<td>711,10</td>
<td>1,089,96</td>
<td>1,624,31</td>
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<td>standard deviation [DM]</td>
<td>0</td>
<td>9,91</td>
<td>33,92</td>
<td>69,35</td>
<td>122,52</td>
<td>202,40</td>
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<tr>
<td>5%-quantile [DM]</td>
<td>121,84</td>
<td>238,45</td>
<td>384,03</td>
<td>604,28</td>
<td>901,29</td>
<td>1,304,01</td>
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<tr>
<td>95%-quantile [DM]</td>
<td>121,84</td>
<td>267,18</td>
<td>496,53</td>
<td>827,82</td>
<td>1,304,51</td>
<td>1,963,38</td>
</tr>
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Table 1: Statistical index numbers for the process of the total net premium reserve with $E(i) = 7\%$ and 5%-quantile = 1,04

Figure 2 describes the development of the process of the net premium reserve and visualizes the growing volatility of the life insurer’s liability.

![Figure 2: Development of the total net premium reserve with stochastic interest income ($E(i) = 7\%$, 5%-quantile = 1,04)](image)

In the same way as the net premium reserve, after introduction of the profit participation mechanism and in consideration of the stochasticity of the investment income, the total sum insured is to be treated as a stochastic process from the perspective of the starting point of the contract. Table 2 shows relevant index numbers of the distribution derived by Monte Carlo simulation for selected points in time.
Table 2: Statistical index numbers for the process of the total sum insured with $E(i_n) = 7\%$ and 5%-quantile $= 1.04$

Figure 3 describes the stochastic process of the total sum insured derived from the Monte Carlo simulation by the expected value as well as the 5%-quantile and the 95%-quantile.

Once again it becomes evident that there is no expected profit participation within the first five periods. The final expected payment of 1.624,31 DM per 1.000 DM initial sum insured implies a yield of 6.52% if the life insurance contract is recognized as a
pure investment product. The estimated probability that the yield falls below 5.34% and exceeds 7.52% is 5%.14)

The counterpart to the net premium reserve and the total sum insured is the change in the insurer's capital resources. Because of the stochasticity of the investment income, the net capital is no more a one dimensional variable but stochastic in nature as well as the other variables connected to the contract in question. Figure 4 shows the development of the change in the capital resources in consideration of the profit participation mechanism as well as the stochasticity of the investment income.

Figure 4: Development of the capital resources with stochastic interest income ($E(i_{\pi}) = 7\%$, 5%-quantile = 1.04)

It is remarkable that the volatility of the change in the capital resources grows in the first instance up to the ninth period; afterwards, it remains fairly stable until the 20th period. Then the volatility increases until the end of the contract period. The reason for this surprising behaviour is due to the construction of the profit participation mechanism and the fact that the interest rate for premium calculation is higher than the interest rate used to calculate the net premium reserve. As $i_p > i_r$, we have an initial deficit, which leads to a negative change in the capital resources. In the first instance,
there is no profit participation of the person insured: Indeed with very high probability, the investment income is higher than the growth of the net premium reserve, but in consideration of the initial deficit, the investment income is likely to be not sufficient in order to overcompensate the liability. The volatility in this time period is only reflected in the development of the change in the capital resources. As there is a guarantee of minimum insurance coverage, the total sum insured and the net premium reserve do not react on different interest income situations. In the moment where the assets attributable to the contract in question offset the net premium reserve, an actuarial surplus is realized, and the person insured is paid a profit participation. Only from this time on, the stochasticity of the interest income begins to affect the total sum insured and the net premium reserve. As a profit participation is already paid if the assets compensate the liabilities without consideration whether profit participation results in a new deficit, the volatility of the change in the capital resources remains fairly constant until the it reaches a substantial positive value. Up to the end of the contract, the volatility grows continually.

The effect of perpetuation of the financing deficit resulting from calculating the additional sum insured with the (higher) premium interest rate \( i_p \) becomes apparent by the fact that the expected change in the capital resources stays negative up to the 23rd period. But in expected value, the profit participation begins already at the beginning of the seventh period. Nevertheless, it takes further 16 periods time until the change in the capital resources reaches a positive expected value that allows a corresponding investment of the capital resources gained by the contract in question. At the final date, we have an expected change in the capital resources of 12.60 DM per 1000 DM initial sum insured.

Some important index numbers of the distribution of the change in the capital resources derived from the Monte Carlo simulation are shown in Table 3 for selected points in time after 5, 10, 15, 20, 25 and 30 periods.
Values derived from the simulation for 

<table>
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<tr>
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<th>$CCR(5)$</th>
<th>$CCR(10)$</th>
<th>$CCR(15)$</th>
<th>$CCR(20)$</th>
<th>$CCR(25)$</th>
<th>$CCR(30)$</th>
</tr>
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<tbody>
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<td>expected value [DM]</td>
<td>-20.42</td>
<td>-3.89</td>
<td>-0.76</td>
<td>-0.24</td>
<td>2.72</td>
<td>12.60</td>
</tr>
<tr>
<td>standard deviation [DM]</td>
<td>3.43</td>
<td>6.39</td>
<td>2.83</td>
<td>2.18</td>
<td>2.76</td>
<td>8.17</td>
</tr>
<tr>
<td>shortfall probability</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.064</td>
<td>0.046</td>
<td>0.026</td>
</tr>
<tr>
<td>shortfall expectation [DM]</td>
<td>-20.42</td>
<td>-3.89</td>
<td>-0.76</td>
<td>-0.39</td>
<td>-0.30</td>
<td>-0.42</td>
</tr>
<tr>
<td>5%-quantile [DM]</td>
<td>-25.95</td>
<td>-18.85</td>
<td>-2.49</td>
<td>-1.53</td>
<td>0.10</td>
<td>2.04</td>
</tr>
<tr>
<td>95%-quantile [DM]</td>
<td>-14.67</td>
<td>-0.05</td>
<td>-0.04</td>
<td>0.29</td>
<td>5.63</td>
<td>23.06</td>
</tr>
</tbody>
</table>

Table 3: Statistical index numbers for the process of the change in the capital resources with $E(t_i) = 7\%$ and 5%-quantile = 1.04

In order to quantify risk, we are going to analyse the development of the 0th lower partial moment, the shortfall probability and the first lower partial moment, the shortfall expectation. Up to the 19th period, the shortfall probability amounts to 1 in each point in time; this means that in no case, accumulated assets are sufficient to offset liabilities from the contract. One period later, the shortfall probability falls to approximately 5% and further drops to ca. 2.5% at the end of the treaty. This amazing development results from the behaviour of the standard deviation of the change in the capital resources which is strongly dependent on the profit participation system. When the change in the capital resources grows to a positive value, the standard deviation is very small. The major part of the probability mass reaches a positive value within one period.

Up to period 29, the shortfall expectation quantifying the extent of the autonome financing necessity shows a behaviour that is more or less monotonically growing. Only in the last period this index falls sharply. The number of simulation runs resulting in a negative value of the change in the capital resources drops with increasing duration of the contract but the extent of negativity grows. In this case, the combination of the two effects leads to a higher shortfall expectation.
As there is a minimum rate of return $i$, for the net premium reserve guaranteed by the insurance company, the probability that the contract induced assets are not sufficient to cover the actuarial liabilities must always be positive. As we see, we have a simulated shortfall probability of 2.6% at the end of the contract. If the management thinks that this number is too high, a change in the profit participation system must be effected: It would be possible to retain parts of the periodic profit participation and to fix the rest in a special reserve. Another way consists in the determination of the additional sum insured which the person insured gets as profit participation with the lower interest rate for reserve calculation instead of the interest rate for premium calculation. Doing this, the financing problem only refers to the initial sum insured but no more to the additional sum insured caused by profit participation.

3.2 Variations in the Distribution of the Accumulation Factor

Now we are interested in the sensitivity of the simulation results for the change in the capital resources with regard to a change in the probability distribution of the accumulation factor. For that reason, the expected value of the distribution of the accumulation factor will be given by 1.06, 1.08, and 1.045 whereas the 5%-quantile of the distribution will always be 1.04. The parameters of the process $\phi$ and $\beta$ as well as the random numbers used for simulation are the same as in the case of an expected value of 1.07. As we are particularly interested in the change of the risk measures used in the analysis we do not analyse the change in the payment to the person insured nor the reaction of the change in the capital resources.  

Up to the 19th period, the shortfall probability as a function of the interest rate constellation and the number of survived periods is always one. In the third part of the contract the shortfall probability falls to 2.5 - 5%. Only if the interest rate earned in the financial market falls in expectation to the level of the interest rate used for premium calculation ($E(i_n) = i_0$), the reduction of the shortfall probability is significantly slower. The final value of the shortfall probability in this case at the end of the contract amounts to 17.5%. Figure 5 visualizes the analysis.
The second important index number within the scope of the risk analysis is the shortfall expectation which quantifies the absolute volume of the external financial needs resulting from a life insurance contract calculated with differing premium and reserve interest rate. As it was to be expected, the external financial needs are the lower the higher the expected interest income is. As the shortfall probability does not reach zero at the end of the contract, the shortfall expectation has a positive value, too. The development of this index number can be seen from Figure 6.
3.3 Effects of the Calculation of the Additional Sum Insured Using $i_p$

If the calculation of the additional sum insured generated by the profit participation mechanism is not effected using the (higher) interest rate for premium calculation $i_p$, but rather the (lower) interest rate used for reserving, the amount of profit participation and the corresponding initial reserve coincide. Therefore, there is no initial deficit to be financed by the insurer's capital resources. Under these circumstances, the payment to the person insured is naturally lower. Simulation results indicate that the expected payment to the person insured falls by 1.64% compared with the situation where the additional sum insured is also calculated using $i_p$.

This relatively small change in the final net premium reserve is reflected in a much higher increase in the change in the capital resources when changing the calculation mode to the reserve interest rate. On the basis of the expected value, the change in the capital resources increases from 12.60 DM when using the premium interest rate for
the determination of the additional sum insured to 36,72 DM per 1000 initial sum insured when using the reserve interest rate. This corresponds to an increase of 191.43%. The following Figure 7 traces the two stochastic processes of the change in the capital resources when calculating the additional sum insured using \( i_r \) or \( i_p \). In the case of \( i_r \), we can especially see the steady growth after redemption of the initial deficit, whereas the deficit is extended because a new deficit is caused by the calculation mode of the additional sum insured when using the premium interest rate \( i_p \). The volatility is always higher when calculating the bonus with \( i_r \).

![Figure 7: Expected values, 5%- and 95%-quantiles of the capital resources when calculating the additional sum insured using \( i_p \) bzw. \( i_r \) with \( E(i_r) = 7\% \)](image)

A relatively small reduction of the payment to the person insured enables the life insurance company to improve its profit situation dramatically; for that reason the use of the reserve interest rate for the determination of the bonus represents the better alternative.
4. Concluding Remarks

The analysis shows that an endowment policy calculated with differing actuarial interest rates generates a situation in which the life insurance company has to finance in advance the assets necessary to cover the corresponding actuarial liabilities by its capital resources as the net premium reserve is positive at the beginning of the contract period and is likely to grow faster within the first few periods than the assets induced by the life insurance contract. In this time period, no profit participation is paid to the contractholder.

Once the assets are higher than the actuarial liabilities, a profit participation is granted that is supposed to be used to buy additional insurance coverage. It is shown that the calculation mode of the additional sum insured due to the profit participation system has a major impact on the profit generated by the life insurance contract. By calculating the bonus with the (lower) reserve interest rate \( i \), instead of the (higher) premium interest rate \( i_p \), the profitability of the contract can be increased substantially.

When considering the effects generated by choosing a premium interest rate \( i_p \) being higher than the reserve interest rate \( i \), it becomes evident that the success of such a strategy depends crucially on the capital resources of the life insurance company as a substantial amount must be used in order to finance in advance assets that are necessary to cover incurred liabilities.

Endnotes

1) In fact, these quantities are not only random variables relative to the investment income but also relative to mortality. In this paper, we accept the life table as a cautious approximation to reality and reflect only on expected quantities relative to mortality.


3) Modelling the logarithm of the accumulation factor appears to be advantageous because negative yields between -100% and 0% are possible, too. This represents the typical behaviour of assets with a limited liability.
A general discussion of autoregressive moving average processes can be found in Box/Jenkins 1970, p. 73-84; Abraham/Ledolter 1983, p. 219-224; Lütkepohl 1991, chapter 6, generalized as vector autoregressive moving average processes (VARMA); Greene 1993, p. 550 ff.

This is possible as the condition of invertibility $|\beta| < 1$ is fulfilled; Abraham/Ledolter 1983, p. 221. We assure by a suitable parameter choice that the condition of invertibility is always valid in the Monte Carlo simulations performed in this paper.

A stochastic process fulfills the condition of weak stationarity if $|\phi| < 1$. For our purposes, we always assume that weak stationarity is valid. Abraham/Ledolter 1983, p. 221.

The variances and covariances of $Y$ and $\delta$ are identical as the parameter $\theta$ is not stochastic.

The parameters $\phi$ and $\beta$ are determined in a way so that the variance of the logarithmic accumulation factor correspond with the parameter of the underlying normal distribution.

Let $F$ be the cumulative distribution function of the accumulation factor, $N$, characterizes the $(1-\varepsilon)$-quantile of the standardized normal distribution function.

In this paper, we have $H = 1000$.

These values can be determined from the 5%-quantile (1.304,00) and the 95%-quantile (1.963,38) of the distribution of the final payment.

The payment to the person insured will be the higher the higher we choose the expected interest rate in the financial market as the profit participation takes a larger part of the total payment.
References


