1 One period Model

In this section, we remind "no arbitrage", the basic idea of mathematical finance. First, we think of the simplest model, one period model. Let us think of the following setting.

There are only two dates, date 0 (the present date) and date 1 (the future).

- There are only two securities, a risk free bond and a stock.
- The price of the bond at date 0 is $B_0$, and the price of the bond at date 1 is $B_1$. So the interest rate $R$ is $(B_1 - B_0)/B_0$.
- The price of the stock at the present date is $S_0$.

Let us assume that there will be only two possibilities on the price of the stock at date 1.

○ Scenario 1 ($\omega_1$) The price of the stock at date 1 is $S_1$.
○ Scenario 2 ($\omega_2$) The price of the stock at date 1 is $S_2$.

Then the rate of return of the Stock will be $Q_1 = (S_1 - S_0)/S_0$ at Scenario 1 and will be $Q_2 = (S_2 - S_0)/S_0$ at Scenario 2. We assume that $Q_1 < R < Q_2$.

Now let us think of the following derivative. That is, the derivative holder will take $Z_1$ yen at date 1 if the Scenario 1 takes place, and will take $Z_2$ yen at date 2 if the Scenario 2 takes place. Then the payoff $Z$ is a function of Scenarios, i.e.

$$Z(\omega_1) = Z_1, \quad Z(\omega_2) = Z_2$$

We may regard the set of scenarios as a set of events mathematically. So we may regard $Z$ as a random variable. Our main problem is to price this derivative.
Now suppose that we take the following portfolio at date 0 such that we hold the bond by amount of $x$ yen and hold the stock by amount of $y$ yen. Here we assume that there is no restriction on short sale. Therefore $x$ or $y$ can be negative. The cost to take this portfolio is $x + y$ yen, of course. The return will be

$$(1 + R)x + (1 + Q_1)y, \text{ if Scenario 1 takes place, and}$$

$$(1 + R)x + (1 + Q_2)y, \text{ if Scenario 2 takes place.}$$

Now let us think of the following linear equation.

$$\begin{align*}
(1 + R)x + (1 + Q_1)y &= Z_1 \\
(1 + R)x + (1 + Q_2)y &= Z_2
\end{align*}$$

(1)

This equation can be rewritten as follows:

$$\begin{align*}
x + (1 + R)^{-1}(1 + Q_1)y &= (1 + R)^{-1}Z_1 \\
x + (1 + R)^{-1}(1 + Q_2)y &= (1 + R)^{-1}Z_2
\end{align*}$$

(2)

One can easily see that there exists a unique solution $(x, y)$ to Equation (1). Therefore we see that if we pay $x + y$ yen at date 0 and take a portfolio strategy $(x, y)$, we can replicate the same payoff at date 1 as same as the derivative given by $Z$. This cost $x + y$ is called the replication cost of the derivative. By "no free lunch" argument, we may conclude that the price of derivative is equal to the replication cost $x + y$.

Moreover, we have the following. Let $\pi_1, \pi_2$ be given by

$$\pi_1 = \frac{Q_2 - R}{Q_2 - Q_1}, \quad \pi_2 = \frac{R - Q_1}{Q_2 - Q_1}.$$

Then we easily obtain

$$x + y = (1 + R)^{-1}Z_1 \pi_1 + (1 + R)^{-1}Z_2 \pi_2$$

(3)

Obviously we have

$$\pi_1 + \pi_2 = 1, \quad \pi_1, \pi_2 > 0$$

and also we have

$$(1 + R)^{-1}S_1 \pi_1 + (1 + R)^{-1}S_2 \pi_2 = S_0$$

(4)

One may think that $\pi_1$, and $\pi_2$ are probability of Events $\omega_1$ and $\omega_2$, respectively. This probability is called risk neutral probability. Equation (3) shows that the price of the derivative at date 0 is given by the expectation of discounted payoff at date 1 under risk neutral probability. Equation (4) shows that the expectation of discounted price of the stock at date 1 under risk neutral probability is equal to the price of the stock at date 0. One should note that the risk neutral probability is different from subjective probability.

Let us think of a little more general model. In the previous model, we think of only two scenarios. What does happen if we think of three scenarios?
We think of the following setting. The interest rate is \( R \). The rate of return of the Stock will be \( Q_i, i = 1, 2, 3 \), if Scenario \( i \) takes place. The payoff of the derivative at date 1 is \( Z_i, i = 1, 2, 3 \), if Scenario \( i \) takes place. Then we have the following linear equation.

\[
(1 + R)x + (1 + Q_1)y = Z_1 \\
(1 + R)x + (1 + Q_2)y = Z_2 \\
(1 + R)x + (1 + Q_3)y = Z_3
\]

This equation does not have a solution in general, and so we cannot determine the price of the derivative.

Since we want to think of various scenarios, we have to think of a multi-period model or a continuous time model. By using a stochastic differential equation, one can generate an infinitely many scenarios.

2 Continuous time model: Black-Sholes model

Let \((\Omega, \mathcal{F}, P)\) be a probability space. Let \( \{W_t\}_{t \in [0, \infty)} \) be a 1-dimensional Brownian motion starting from the origin, and let us think of a filtration \( \mathcal{F}_t = \sigma\{W_s; s \leq t\}, t \geq 0 \). Also, let \( r > 0, \sigma > 0, \mu \in \mathbb{R} \). We assume that there are two securities, Bond and Stock and the information up to time \( t \) is given by \( \sigma \)-algebra \( \mathcal{F}_t \).

Let \( S_t \) be the price of the Stock at time \( t \) and let \( B_t \) be the price of the Bond at time \( t \). We assume that the price processes \( S_t \) and \( B_t \) are adapted and they satisfy the following SDE.

\[
dB_t = B_t r dt, \\
dS_t = S_t (\sigma dW_t + \mu dt).
\]

Then we see that

\[
B_t = B_0 \exp(rt),  \\
S_t = S_0 \exp(\sigma B_t + (\mu - \frac{\sigma^2}{2})t).
\]

So the interest rate per unit time is \( r \). We think of a continuous trading portfolio strategy such that we hold the Bond by amount of \( \eta_t B_t \) yen and hold the Stock by amount of \( \xi_t S_t \) yen at time \( t \). The portfolio process \((\eta_t, \xi_t)\) has to be adapted. If the strategy is self-financing, we have the following equation.

\[
\exp(-rt) (\eta_t B_t + \xi_t S_t) = (\eta_0 B_0 + \xi_0 S_0) + \int_0^t \xi_s d\tilde{S}_s. \tag{5}
\]

Here \( \tilde{S}_t = \exp(-rt) S_t \) (discounted stock price). The integral appeared here is Ito integral.
Now let us think of an European type derivative such that the payoff at the maturity $T$ is $Z(\omega)$, $\omega \in \Omega$, if the state is $\omega$. Then the random variable $Z$ should be $\mathcal{F}_T$ measurable. For example, an European call option of the exercise price $a$ is given by $Z = \max\{S_T - a, 0\}$.

We need the following theorem.

**Theorem 1 (Ito’s representation theorem)** If $Z$ is a good random variable, then there are $c \in \mathbb{R}$ and a good adapted process $\{\xi_t\}_{t \in [0,T]}$ such that

$$\exp(-rT)Z = c + \int_0^T \xi_t d\tilde{S}_t$$

(6)

By virtue of this theorem, we see that there is a good self-financing trading strategy $(\eta_t, \xi_t)$ such that

$$\eta_0 B_0 + \xi_0 S_0 = c,$$

and

$$Z = \eta_T B_T + \xi_T S_T = e^{rT}(c + \int_0^T \xi_t d\tilde{S}_t).$$

These imply that one can replicate the derivative $Z$ with an initial cost $c$. The following theorem is important to compute the replication cost $c$.

**Theorem 2 (Cameron-Martin-Maruyama-Girsanov)** There exists a probability measure $Q$ in $(\Omega, \mathcal{F})$ equivalent to the probability measure $P$ such that $\tilde{S}_t = \exp(-rt)S_t$, $t \in [0,T]$, is a martingale under $Q$.

By the property of stochastic integral, we see that

$$E^Q[\int_0^T \xi_t d\tilde{S}_t] = 0.$$ 

So we have

$$c = \exp(-rT)E^Q[Z].$$

Suppose that the payoff $Z$ is given by $Z = f(S_T)$ for some continuous function $f$ of polynomial order growth. If a function $u$ is a nice function defined in $[0,T] \times \mathbb{R}$, by Ito’s lemma we have

$$\exp(-rT)u(T,S_T) = u(0,S_0) + \int_0^T \frac{\partial u}{\partial x}(t,S_t)d\tilde{S}_t + \int_0^T \exp(-rt)(\frac{\partial u}{\partial t}(t,S_t) + Lu(t,S_t))dt,$$

where

$$Lu(t,x) = \frac{\sigma^2}{2}x^2 \frac{\partial^2 u}{\partial x^2}(t,x) + r \frac{\partial u}{\partial x}(t,x) - ru(t,x)$$

So if $u$ is a solution to the PDE

$$\frac{\partial u}{\partial t}(t,x) + \frac{\sigma^2}{2}x^2 \frac{\partial^2 u}{\partial x^2}(t,x) + r \frac{\partial u}{\partial x}(t,x) - ru(t,x) = 0 \quad (t,x) \in [0,T] \times \mathbb{R},$$

$$u(T,x) = f(x),$$
we see that
\[ \exp(-rT)u(T, S_T) = u(0, S_0) + \int_0^T \frac{\partial u}{\partial x}(t, S_t) dS_t \]
and so we have
\[
u(0, S_0) = EQ[\exp(-rT)f(S_T)] = \exp(-rT) \int_{-\infty}^{\infty} f(S_0 \exp(\sigma z + (r - \frac{\sigma^2}{2})T)) \left( \frac{1}{2\pi T} \right)^{\frac{1}{2}} e^{-\frac{z^2}{2T}} dz.
\]
Moreover, we see that the hedging strategy is given by \( \frac{\partial u}{\partial x}(t, S_t) \) (Delta hedge).

In the case of European call option we have
\[
c = \exp(-rT)EQ[\max\{S_T - a, 0\}] = \exp(-rT) \int_{-\infty}^{\infty} \max\{S_0 \exp(\sigma z + (r - \frac{\sigma^2}{2})T) - a, 0\} \left( \frac{1}{2\pi T} \right)^{\frac{1}{2}} e^{-\frac{z^2}{2T}} dz
\]
\[= S_0 N(d_1) - Ke^{-rT} N(d_2). \]

Here
\[N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy\]
and
\[d_1 = \frac{\log(S_0/a) + (r + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}}, \quad d_2 = d_1 - \sigma \sqrt{T}.
\]
This is the Black-Scholes formula.

3 Historical Remark on Stochastic Analysis

We see that Stochastic Analysis plays very important role in Mathematical Finance. In particular, the following are basic tools.

(1) Brownian motion and additive processes (noise and innovation)

(2) Stochastic Integral, Stochastic Differential Equation and Ito’s lemma

(3) Martingale Representation Theorem

(4) Transformation of Measure (Girsanov Transformation)

We will review the following historical papers, the origin of Stochastic Analysis.

(1) Brownian motion etc.

Bachelier, M.L., Théorie de la Speculation, Ann. de l’École norm. 17(1900) 21-86

Wiener, N., Differential space, J.Math. and Physics 2(1923), 131-174

Lévy, P., Théorie de l’addition des variables aléatoires, Gauthier-Villoars, Paris, 1937

(2) SDE etc.


Itô, K.(伊藤清), Markoff processyorunメル微分方程式, 全国著者数学討話会 1077(1942)

Differential equations determining a Markoff process, Zenkoku Sizyo Sugaku Danwakaisi
(English translation is in 'Kiyosi Itô Selected papers', Springer 1987)


Itô, K., On a formula concerning stochastic differentials, Nagoya Math. Journ. 3(1951), 55-65

Itô, K., On stochastic differential equations, Mem. Amer. Math. Soc. 4(1951), 1-51


Doob, J.L., Stochastic Process, John Wiley and Sons, New York, 1953


(3) Martingale Representation

N.Wiener, The homogeneous chaos, Amer.J.Math. 60(1938), 897-936


(4) Girsanov Transformation

Cameron, R.H., and Martin, W.T., Transformation of Wiener integrals under translations, Ann. Math. 45(1944), 386-396


Girsanov, I.V., On transforming a certain class of stochastic processes by absolutely continuous substitution of measures, Theory Prob. Appl. 5(1960), 285-301
THÉORIE DE L'ADDITION

VARIABLES ALÉATOIRES

Paul Lévy

FASCICULE I

MONOGRAPHIES DES PROBABILITÉS

PARIS
GUTHIER-VILLARS, ÉDITEUR

1934
1077. Markov過程の定数微分方程式

伊藤 清（内閣調査官）

ハシガキ

(1) 有限個 n 可能な場合 \( a_1, a_2, \ldots, a_m \) あり、自然数 n 進数で示す simple Markov process \( x_1, x_2, \ldots \)

等の形で考えると、Markov process の条件が得られること。例として、

\[ x_{n+1} = a_{i_n} \text{ なる条件} \]

下に列挙される。\( x_1 = a_{i_1}, \ldots, x_n = a_{i_n} \) の条件が得られる

\[ x_{n+1} = a_{i_{n+1}} \text{ なる確率} \]

等が求まる。シガマ変数の例を結び

\[ x_{n+1} = a_{i_{n+1}} \text{ なる確率} \]

\( (i=0, 1, 2, \ldots, m) \) が限界分布。コルモゴロフ 本(1) に書かれている。以後コルモゴロフ

の基本的 n 進移動確率を呼

べる。

更に可能 n 個がある場合には、例に数実数 2 以下

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併シナガウ標識が自然数デナクテ 実数 n 場合即フ continuous parameter = 定数 n = Markov process =

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ハナイ。

更に一般 n 可能 n 場合が実数 = ヨリ標識付ケッレ、且ツ

continuous parameter = 定数 n = simple Markov

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