Abstract.

The classical portfolio problem with two risky assets for strictly risk averse decision makers is considered. The necessary and sufficient conditions by Wright (1987), under which complete diversification is optimal, are recalled. Our contribution concerns the comparative static question. It consists to find conditions of relative riskiness between two pairs of risky returns, which guarantee that the corresponding optimal portfolios are ordered in an unambiguous way. The recent result by Eeckhoudt and Gollier (1995) for the standard portfolio problem with one risky asset and a risk-free asset is extended to the more general bivariate setting, and interpreted in terms of partial orders of riskiness. The special situation of bivariate normal random returns is studied in details. With equal means of returns, one shows an ambiguous comparative static effect for the correlation order of riskiness by equal marginals of the pairs of returns, which can be viewed as a bivariate version of the earlier univariate result by Fishburn and Porter (1976). With unequal means and under an exponential expected utility model, necessary and sufficient conditions for the comparative static property are also derived. However, even in the simple case of equal marginal means and variances of the returns, there exists for sufficiently high coefficients of risk aversion an ambiguous comparative static effect for the correlation order of riskiness as well as the mean-variance criterion for excess returns. For the standard portfolio problem with lognormal random returns, the diversification condition yields a new argument in favour of risk-neutral valuation in the sense of Cox and Ross (1976). The same condition shows also that no financial risk premium is offered on efficient portfolios with one riskless asset and a risky asset with lognormal return, where efficiency is understood in the sense of Merton (1990).

Keywords: portfolio selection, diversification, comparative static, monotone probability ratio order, correlation order, monotone expectation dependence order

Let \( R, S \) be the random returns on two risky assets, and let \( \lambda \) be the fraction of a unit invested in \( S \). Under the assumption that investors are strictly risk averse, their risk preferences are modelled by twice differentiable strictly increasing and concave utility functions \( u(x) \) such that \( u'(x) > 0, \quad u''(x) < 0 \). Denote by 
\[
\varphi(\lambda) = E\left[u(\lambda S + (1 - \lambda)R)\right]
\]
the expected utility of a diversified investment in the two risky assets \( R \) and \( S \). The classical portfolio problem consists to find an optimal portfolio \( \lambda^* \), which maximizes the expected utility such that \( \varphi(\lambda^*) = \max_{\lambda} \{ \varphi(\lambda) \} \). It will be assumed that \( \varphi(\lambda) \) is differentiable and concave, and that 
\[
\varphi'(\lambda) = E\left[X \cdot u'(R + \lambda X)\right], \quad X = S - R \text{ is the excess return.}
\]
Several questions are of interest.

First, one wants to find conditions under which \( \lambda^* > 0 \) (some \( S \) will be purchased) and \( 0 < \lambda^* < 1 \) (some of both \( R \) and \( S \) will be purchased, that is complete diversification is optimal). This well-known existence problem has been studied by many authors from Samuelson (1967) to Wright (1987). In particular, the last author shows that, under a non-negative expected excess return \( E[X] \geq 0 \), one has \( \lambda^* > 0 \) provided \( S \) is strictly relatively negative expectation dependent on \( R \), written \( \text{RNED}(S|R) \), which means that
\[
(1.1) \quad \text{Cov}[S - R, g(R)] < 0
\]
for every increasing function \( g(x) \) for which the covariance exists, or equivalently
\[
(1.2) \quad E[\text{S|R} \leq r] - E[\text{S}] < E[\text{R|R} \leq r] - E[\text{R}]
\]
for all \( r \) such that the conditional means exist. Furthermore, a general proof that complete diversification pays, first proposed by Samuelson (1967), can only be established with equal means, that is \( E[X] = 0 \). In this situation, one has \( 0 < \lambda^* < 1 \) if, and only if, one has strict \( \text{RNED}(S|R) \) and strict \( \text{RNED}(R|S) \). In the normally distributed case, this necessary and sufficient condition holds exactly when 
\[
\text{Cov}[X, Y] < \min\{\text{Var}[X], \text{Var}[Y]\}. \quad \text{This situation allows for positively correlated returns, which are typically observed in financial markets, and whose financial risk cannot be eliminated through diversification (e.g. Sharpe (1985), Section 6, p. 130). On the other hand, when \( u(x) \) is a quadratic utility function, the same condition on the covariance must hold in order that \( 0 < \lambda^* < 1 \). Replacing the expected utility model by the mean-variance model such that \( \lambda^* \) is optimal provided \( \sigma(\lambda^*) = \min_{\lambda} \{ \sigma(\lambda) \} \) with \( \sigma(\lambda) = \text{Var}[\lambda S + (1 - \lambda)R] \), one finds in case \( E[X] = 0 \) that \( 0 < \lambda^* < 1 \) if, and only if, one has \( \text{Cov}[X, Y] < \min\{\text{Var}[X], \text{Var}[Y]\} \). Therefore, both models yield the same answer in the two mean-variance legitimated cases (normal distributions or quadratic utility).
In the special case of the standard portfolio problem, for which \( R \) is a risk-free return with deterministic return \( r_f \), one knows that \( \lambda^* > 0 \) if, and only if, the expected excess return is strictly positive, that is \( \mathbb{E}[X] = \mathbb{E}[S] - r_f > 0 \) (theorem of Arrow(1971)).

Another important question, for which no similar satisfactory answers are known to the author, is the analysis of the comparative static of the classical portfolio problem, whose study has been suggested to us by Wright(1987). The problem consists to find conditions of relative riskiness between pairs of returns \((R_1, S_1)\) and \((R_2, S_2)\), which guarantee that the optimal portfolios are ordered, that is \( \lambda_1^* \leq \lambda_2^* \), when \((R_1, S_1)\) is less risky than \((R_2, S_2)\).

For the standard portfolio problem, that is \( R_1 = R_2 = r_f \) is the risk-free return, the comparative static question has been studied by Fishburn and Porter(1976), and more recently by Landsberger and Meilijson(1990) as well as by Eeckhoudt and Gollier(1995). Contrary to common intuition, if \( S_1 \) precedes \( S_2 \) in the usual stochastic order or stochastic dominance order of first order, that is \( S_1 \preceq S_2 \), risk-averse decision makers will not always invest more in the risky asset. This ambiguous effect on the optimal risk exposure, that is counterexamples with \( \lambda_1^* > \lambda_2^* \), has been first noticed by Fishburn and Porter(1976), who also proposed a sufficient condition, which implies the desired comparative static property \( \lambda_1^* \leq \lambda_2^* \). It is known that the stronger forms of monotone likelihood ratio order, written \( S_1 \preceq_c S_2 \), and the monotone probability ratio order, written \( S_1 \preceq_p S_2 \), imply that \( \lambda_1^* \leq \lambda_2^* \), as shown respectively by Landsberger and Meilijson(1990) and Eeckhoudt and Gollier(1995). These results remain valid for the higher dimensional standard portfolio problem with one riskless asset provided the utility function satisfies the two fund monetary separation property characterized by Cass and Stiglitz(1970) (e.g. Huang and Litzenberger(1988), Section 1.27).

In the next Section, we present an extension of the Eeckhoudt-Gollier results to the more general bivariate random situation, providing thus sufficient conditions for the desired comparative static effect for the classical portfolio problem.

Finally, another problem of comparative static consists to find conditions on the utility functions of two decision makers, say \( u_1(x) \) is in some sense more risk averse than \( u_2(x) \), which guarantee that \( \lambda_1^* \leq \lambda_2^* \) for the standard portfolio problem. That is, decision maker 1 will never invest more in the risky asset than decision maker 2. If decision maker 1 is more risk averse than decision maker 2 in the sense of Arrow(1971) and Pratt(1964), that is \( R_\lambda^1(x) = -\frac{u_1(x)}{u_1'(x)} \geq R_\lambda^2(x) = -\frac{u_2(x)}{u_2'(x)} \), then \( \lambda_1^* \leq \lambda_2^* \), as shown in Huang and Litzenberger(1988), Section 1.26, p. 51. A generalization to a two risky asset portfolio problem, which is based on a strongly more risk averse comparison by Ross(1981), is also presented in Huang and Litzenberger(1988), Section 2.14.
2. Sufficient conditions for the comparative static property.

For $i = 1, 2$, let $(R_i, S_i)$ be pairs of random returns, $X_i = S_i - R_i$, the excess returns, $\varphi_i(\lambda_i) = E[u(R_i + \lambda_i X_i)]$ the expected utilities, assumed differentiable and concave, and $\lambda_i^* > 0$ the corresponding optimal portfolios under the assumptions $E[X_i] \geq 0$ and strict RNED($S_i|R_i$). Let us state our main result, which will be interpreted in terms of partial orders of riskiness in Section 3.

**Theorem 2.1** *(Bivariate conditions for the comparative static property)* Let $(R_i, S_i)$, $i = 1, 2$, be pairs of random returns, which satisfy the assumptions made. Set $X_i = S_i - R_i$, and for each $\lambda_i > 0$, set $Y_i(\lambda) = R_i + \lambda X_i$, $i = 1, 2$. It is assumed that the supports of $X_i, Y_i(\lambda)$ are contained in some interval $[a, b]$ with $-\infty < a \leq 0 \leq b < \infty$. Suppose the following conditions hold for $\lambda_i = \lambda_i^*$, $\lambda_i > 0$ the optimal portfolio for the pair $(R_i, S_i)$:

1. $E[X_2] \geq E[X_1] \geq 0$ (the expected excess return of the second pair is higher than the one of the first pair, and both are non-negative)

2. $Y_i(\lambda) \preceq_p Y_i(\lambda)$:
   - there exists $c_\lambda \in [a, b]$ such that the distributions $F_i(x)$ of $Y_i(\lambda)$
   - satisfy $F_i(x) = 0$ for $x < c_\lambda$, and $-F'_i(x)$ is decreasing for $x \geq c_\lambda$
   (the return on the first portfolio precedes in monotone probability ratio order the one of the second portfolio)

3. $E[X_i|Y_i(\lambda) \leq y] \leq E[X_2|Y_2(\lambda) \leq y]$ for all $y$ such that the conditional means exist
   (the conditional mean of the excess return on the first portfolio given its return lies below some fixed amount is less than the corresponding quantity on the second portfolio for all possible such fixed amounts)

Then the optimal portfolios $\lambda_i^*, i = 1, 2$, satisfy the comparative static property $\lambda_i^* \preceq \lambda_2^*$.

**Proof.** Setting $\lambda = \lambda_1^* > 0$, one has by definition of the optimal portfolio:

$$\varphi_1(\lambda) = E[X_1 \cdot u'(Y_1(\lambda))] = 0.$$ 

Since the optimizing function is concave, a proof of $\varphi_1(\lambda) \geq 0$ implies immediately that $\lambda_2^* \geq \lambda = \lambda_1^*$. Therefore, it suffices to show that, under the constraint (2.1), one has always
(2.2) \( \varphi_2(\lambda) = E[X_2 \cdot u'(Y_2(\lambda))] - E[X_1 \cdot u'(Y_1(\lambda))] \geq 0, \text{ for all } \lambda > 0. \)

In the following, let us suppress the index \( \lambda \). From a well-known formula of Hoeffding(1940) (see also Lehmann(1966), Lemma 2), one derives, setting \( X = X_i, Y = Y_i, i = 1,2 \), the covariance expressions:

\[
\text{Cov}[X, u'(Y)] = \int \left\{ F_{X,Y}(x,y) - F_X(x)F_Y(y) \right\} u''(y)dy
\]

(2.3)
\[
= \int F_Y(x) \left\{ F_X(x|Y \leq y) - F_X(x) \right\} u''(y)dy
\]
\[
= \int F_Y(x) \left\{ E[X] - E[X|Y \leq y] \right\} u''(y)dy.
\]

A partial integration shows that

(2.4) \( E[u'(Y_i)] = u'(b) - \int_a^b u''(y)F_i(y)dy, \quad i = 1,2. \)

Combining (2.3) and (2.4), one has to show that

\[
\varphi_2(\lambda) = \left( E[X_2] \cdot E[u'(Y_2)] - E[X_1] \cdot E[u'(Y_1)] \right)
\]
\[
+ \left( \text{Cov}[X_2, u'(Y_2)] - \text{Cov}[X_1, u'(Y_1)] \right)
\]
\[
= u'(b) \cdot \left\{ E[X_2] - E[X_1] \right\}
\]
\[
+ \int_a^b (-u''(y)) \left\{ F_2(y)E[X_2|Y_2 \leq y] - F_1(y)E[X_1|Y_1 \leq y] \right\} dy \geq 0
\]

under the constraint

(2.6) \( \int_a^b u''(y) \cdot \mu_i(y)dy = u'(b) \cdot E[X_1] \geq 0, \quad \text{with } \mu_i(y) = F_i(y) \cdot E[X_i|Y_i \leq y]. \)

The assumptions (CS1) and \( u''(y) < 0 \) imply the existence of \( y_o \in (a,b) \) such that \( \mu_i(y) \leq 0 \) for \( y \leq y_o \) and \( \mu_i(y) \geq 0 \) for \( y \geq y_o \). The limiting case \( b = y_o = \infty \) implies as in (2.7) below that the defined function there is non-negative, that is \( \omega(y) \geq 0 \) for all \( y \geq a \), hence \( \varphi_2(\lambda) \geq 0. \) Therefore, one can assume \( y_o < \infty. \) By assumption (CS2), the probability ratio \( h(y) = \frac{F_2(y)}{F_1(y)} \) is increasing in \( y \), hence

\[
1 - h(y) \geq 1 - h(y_o) \quad \text{for } y \leq y_o \quad \text{and} \quad 1 - h(y) \leq 1 - h(y_o) \quad \text{for } y \geq y_o.
\]

Using these facts and assumption (CS3), one obtains for all \( y \in (a,b) : \)

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\[ \omega(y) := F_2(y)E[X_1 | Y_1 \leq y] - F_1(y)E[X_1 | Y_1 \leq y] \]
\[(2.7) \]
\[ \geq E[X_1 | Y_1 \leq y] \cdot \{F_2(y) - F_1(y)\} \]
\[ = -\mu_1(y) \cdot \{1 - h(y)\} \geq -\mu_1(y) \cdot \{1 - h(y_o)\} \]

Since \( u''(y) < 0 \) it follows by (2.5) and (2.6) that
\[ \varphi_1(\lambda) = \int_a^b (-u''(y))\omega(y)dy \]
\[(2.8) \]
\[ \geq \{1 - h(y_o)\} \cdot \int_a^b u''(y)\mu_1(y)dy = (1 - h(y_o)) \cdot u'(b) \cdot E[X_1] \geq 0, \]
which shows the desired property. \( \Box \)

Though the sufficient conditions (CS1)-(CS3) are quite general, they do not describe the comparative static properties of the classical portfolio problem completely, as follows from the detailed analysis in Section 4. However, as an important special case, one recovers the main comparative result by Eeckhoudt and Gollier (1995) for the standard portfolio problem with one risky asset and a risk-free asset.

**Corollary 2.1.** (Comparative static of the standard portfolio problem) Let \( S_1, S_2 \) be random returns on two risky assets, and let \( r_f \) be the risk-free return. Set \( X_i = S_i - r_f, \) \( i = 1,2, \) for the excess returns, whose supports are contained in some interval \( [a, b] \) with \( -\infty \leq a \leq 0 \leq b \leq \infty. \) Assume the expected excess returns are strictly positive, that is \( E[X_i] > 0 \) for \( i = 1,2, \) and that the excess returns are ordered in the monotone probability ratio, that is \( X_1 \leq_X X_2. \) Then the optimal standard portfolios \( \lambda_1^*, \lambda_2^* \) for the standard portfolio problem with one risky asset and a risk-free asset, satisfy the comparative static property \( \lambda_1^* \leq \lambda_2^*. \)

**Proof.** We show that the assumptions of Theorem 2.1 are fulfilled. Since \( E[X_i] > 0 \) one has \( \lambda_i^* > 0 \) by the theorem of Arrow (1971). Set \( \lambda = \lambda_1^* \) and \( Y_i(\lambda) = r_f + \lambda X_i, \) \( i = 1,2. \) Since \( X_1 \leq_X X_2 \) implies \( X_1 \leq_Y X_2, \) one has in particular \( E[X_2] \geq E[X_1] > 0, \) hence (CS1) holds. Since the monotone probability ratio order relation is invariant under a positive linear transformation, one has \( Y_1(\lambda) \leq_Y Y_2(\lambda), \) hence (CS2) holds. Further, by Lemma 3 in Eeckhoudt and Gollier (1995), one has \( E[X_i | X_i \leq x] \leq E[X_i | X_i \leq x] \) for all \( x \) such that the conditional means exist. One obtains
\[ E\left[ X_1 \left| X_1 \leq \frac{x - r_f}{\lambda} \right. \right] \leq E\left[ X_2 \left| X_2 \leq \frac{x - r_f}{\lambda} \right. \right] \] for all \( x \) such that these expressions are defined, which is equivalent with the condition (CS3). Therefore Corollary 2.1 follows from Theorem 2.1. \( \Box \)

In our short review of the classical portfolio problem in Section 1, we have seen that the existence of positive optimal portfolios depends upon the notion of relative negative expectation dependence as defined in (1.1) or equivalently (1.2). This notion is a slight modification of the negative expectation dependence structure, which is our key element for a risk theoretical interpretation of the main bivariate comparative static Theorem 2.1 in terms of partial orders of riskiness.

One says that a random variable $X$ is negative (positive) expectation dependent on $Y$, written $\text{NED}(X|Y)$ (\text{PED}(X|Y)) if one of the following equivalent inequalities (reversed inequalities) is fulfilled for all $x$, $y$ such that the given expressions are defined:

\begin{align*}
\text{NED}(X|Y) & \iff E[X|Y \leq y] \geq E[X] \iff E[X|Y > y] \leq E[X] \\
\text{PED}(X|Y) & \iff E[X|Y \leq y] \leq E[X] \iff E[X|Y > y] \geq E[X]
\end{align*}

The equivalence of the second and third condition follows from the identity

\begin{equation}
\bar{F}_Y(y) \cdot \left\{E[X|Y > y] - E[X]\right\} = F_Y(y) \cdot \left\{E[X] - E[X|Y \leq y]\right\}.
\end{equation}

This monotone dependence structure on bivariate random variables has been introduced by Yanagimoto(1973) and Kowalczyk and Pleszczynska(1977). Recall the intuitive meaning of a monotone dependence structure. By positive dependence, large values of $Y$ correspond stochastically to large values of $X$, while by negative dependence, large values of $Y$ correspond to small values of $X$. The most common monotone dependence structures include $\text{NQD}(X,Y)$ (\text{PQD}(X,Y)), that is negative (positive) quadrant dependence (Lehmann(1966)), and $\Lambda(X,Y)$, that is association (Esary et al.(1967)). These types of monotonic dependence satisfy the following well-known implications:

\begin{align*}
\Lambda(X,Y) & \Rightarrow \text{NQD}(X,Y) (\text{PQD}(X,Y)) \\
\Rightarrow \text{NED}(X|Y) (\text{PED}(X|Y)) & \Rightarrow \text{Cov}[X,Y] \leq 0 \ (\geq 0)
\end{align*}

In particular, the monotone expectation dependence structure is the weakest form among these types, which implies a negative (positive) covariance structure.

Consider now the set $\text{BD} = \text{BD}(F_x,F_y)$ of all bivariate random variables $(X,Y)$ with fixed marginal distributions $F_x(x), F_y(x)$. The subset of $\text{BD}$ of all those $(X,Y)$, which are negative (positive) expectation dependent, is denoted by $\text{NED}$ (\text{PED}). A partial order of bivariate riskiness for these subsets is defined as follows. For each $(X_i,Y_i) \in \text{NED} (\text{PED}), i=1,2$, one says that $(X_1,Y_1)$ is less negative (positive)
expectation dependent than \((X_2, Y_2)\), written \((X_1, Y_1) \leq_{\text{NED(PED)}} (X_2, Y_2)\), if one has the inequalities

\[
E[X_1 | Y_1 \leq y] \leq (\geq) E[X_2 | Y_2 \leq y]
\]

for all \(y\) such that the conditional expectations exist. An alternative definition can be derived without difficulty.

**Lemma 3.1.** One has \((X_1, Y_1) \leq_{\text{NED(PED)}} (X_2, Y_2)\) if, and only if, one has \(\text{Cov}[X_1, g(Y_1)] \geq (\leq) \text{Cov}[X_2, g(Y_2)]\) for all increasing functions \(g\) such that the covariances exist.

**Proof.** Making use of formulas similar to (2.3) and noting that \(X_i, Y_i\) have equal marginal distributions \(F_X(x), F_Y(x)\), one obtains the integral representation

\[
\text{Cov}[X_2, g(Y_2)] - \text{Cov}[X_1, g(Y_1)] = \int F_Y(y) \cdot \left[ E[X_1 | Y_1 \leq y] - E[X_2 | Y_2 \leq y] \right] dg(y).
\]

By (3.4) the covariance inequalities are necessary. Conversely, if the covariance inequalities hold for every increasing \(g\), then choose for arbitrary \(y_0\) the indicator function \(g(y) = 1_{\{y \leq y_0\}}\) to see that (3.4) must hold.

The defined partial orders of dependence are in some sense weaker than the more usual correlation order characterized as follows. For each \((X_i, Y_i) \in \text{BD}(F_X, F_Y), i=1,2,\) one says that \((X_i, Y_i)\) is less correlated than \((X_2, Y_2)\), written \((X_i, Y_i) \preceq (X_2, Y_2)\), if one of the following equivalent conditions is satisfied (e.g. Dhaene and Goovaerts(1996)):

\[
\text{Cov}[f(X_1), g(Y_1)] \leq \text{Cov}[f(X_2), g(Y_2)] \quad \text{for all increasing functions } f \text{ and } g \text{ such that the covariances exist (Barlow and Proschan(1975))}
\]

\[
\Pr(X_1 \leq x, Y_1 \leq y) \leq \Pr(X_2 \leq x, Y_2 \leq y) \quad \text{for all } x, y
\]


Economic and actuarial applications of the correlation order include Epstein and Tanny(1980), Aboudi and Thon(1993/95), and Dhaene and Goovaerts(1996). Taking \(f(x) = x\) in (3.6), one obtains from Lemma 3.1 the following implications:
This shows in which precise sense the correlation order is stronger than the negative (positive) expectation dependence order.

We are ready for an interpretation of the main bivariate comparative static result of Section 2 in terms of partial orders of riskiness. Suppose \((R_i, S_i), i=1,2\), are two pairs of random returns, which satisfy the assumptions preceding Theorem 2.1. If for the optimal portfolio on the first pair, that is \(\lambda = \lambda_1^*\), the marginal distributions of the excess returns \(X_i = S_i - R_i\), and the total returns \(Y_i = Y_i(\lambda) = R_i + \lambda X_i\), \(i=1,2\), are equal, and the orderings of risk \(Y_1 \preceq Y_2\) and \((X_1, Y_1) \succeq_{\text{NED}} (X_2, Y_2)\) are satisfied, then the optimal portfolios are ordered in an unambiguous way such that \(\lambda_1^* \leq \lambda_2^*\).

Finally, let us mention that the monotone expectation dependence orderings are equivalent to the Lorenz monotone dependence orderings, as well as to other monotone dependence conditions, considered recently in Muliere and Petrone (1993). In view of the abundant economic literature applying Lorenz curves, this connection might be fruitful for future research.

First, consider the generalized Lorenz curve of a transformed random variable \(g(Y)\), which is defined by (see Kakwani (1977))

\[
L_{g(Y)}(p) = \frac{1}{E[g(Y)]} \int_0^p g(t) dF_Y(t) = \frac{1}{E[g(Y)]} \int_0^p g(F_Y^{-1}(z)) dz, \quad 0 \leq p \leq 1, \quad y_p = F_Y^{-1}(p), \quad F_Y^{-1}(z) = \inf\{y : F_Y(y) \geq z\}, \quad 0 \leq z \leq 1.
\]

Second, for \((X, Y) \in BD = BD(F_X, F_Y)\) with finite means and continuous marginal distributions, specialize this notion to the regression function of \(X\) on \(Y\), that is set \(g(Y) = m(Y) = E[X|Y]\), to obtain the correlation curve of \(X\) on \(Y\) introduced by Blitz and Brittain (1964), which has been studied more specifically in Taguchi (1981). The generalized Lorenz curve \(L_{g[X|Y]}(p)\) leads to the following monotone dependence structure. A pair \((X, Y) \in BD\) is said to be negative (positive) Lorenz dependent, denoted by \(NL(X|Y)\) (\(PL(X|Y)\)), if one has \(L_{g[X|Y]}(p) \geq p\) (\(\leq p\)) for all \(p \in (0,1)\). The subset of \(BD\) of all pairs \((X, Y)\), which are \(NL(X|Y)\) (\(PL(X|Y)\)), is denoted by \(NL\) (\(PL\)). Third, order these subsets. For each \((X_1, Y_1) \in NL\) (\(PL\)), \(i=1,2\), one says that \((X_1, Y_1)\) is less negative (positive) Lorenz dependent than \((X_2, Y_2)\), written \((X_1, Y_1) \leq_{NL(PL)} (X_2, Y_2)\), if one has

\[
L_{g[X_1|Y_1]}(p) \leq (\geq) L_{g[X_2|Y_2]}(p) \quad \text{for all } \ p \in (0,1).
\]

Let us state two of the mentioned equivalent statements.
Lemma 3.2. For \((X_i, Y_i) \in \mathcal{NL} (\mathcal{PL})\), \(i=1,2\), the following statements are equivalent

\[(3.11) \quad (X_1, Y_1) \leq_{\mathcal{NL} (\mathcal{PL})} (X_2, Y_2)\]

\[(3.12) \quad (X_1, Y_1) \leq_{\mathcal{NEQ} (\mathcal{PED})} (X_2, Y_2)\]

\[(3.13) \quad \text{The elasticity } \eta_i(y) = \frac{y m_i'(y)}{m_i(y)} \text{ of } m_i(y) = \mathbb{E}[X_i | Y_i = y] \text{ is greater (less) than the elasticity } \eta_2(y) = \frac{y m_2'(y)}{m_2(y)} \text{ of } m_2(y) = \mathbb{E}[X_2 | Y_2 = y].\]

**Proof.** Interchanging \(X\) and \(Y\) this is found in Muliere and Petrone (1993) (Proposition 3.3 for (3.12) and Property 9 for (3.13), which goes back to Kakwani (1977)).

4. The classical portfolio problem for bivariate normal random returns.

For \(i=1,2\) let \((R_i, S_i)\) be two pairs of bivariate normal random returns with marginal means \(\mu_{1,R}, \mu_{1,S}\), variances \(\sigma_{1,R}^2, \sigma_{1,S}^2\) and correlation coefficient \(\rho_i\). We use the notations \(\alpha_i = \frac{\sigma_{1,R}}{\sigma_{1,S}}, X_i = S_i - R_i, Y_i(\lambda) = R_i + \lambda X_i, i=1,2.\)

The optimal portfolios \(\lambda^*_i, i=1,2\), are solutions of the first order conditions

\[(4.1) \quad \mathbb{E}[X_i \cdot u'(Y_i(\lambda^*_i))] = 0, \quad i=1,2.\]

From Wright (1987) one knows that if \(\mathbb{E}[X_i] \geq 0\) and \(\rho_i < \alpha_i\) then \(\lambda^*_i > 0\), and if \(\mathbb{E}[X_i] = 0\) then \(0 < \lambda^*_i < 1\) if, and only if, one has \(\rho_i < \min\left\{\alpha_i, \frac{1}{\alpha_i}\right\}\). Since \((X_i, Y_i)\) is bivariate normal, one obtains from Stein's Lemma (a self-contained proof is given in the Appendix) that

\[(4.2) \quad \text{Cov}[X_i, u'(Y_i)] = \mathbb{E}[u''(Y_i)] \text{ Cov}[X_i, Y_i], \text{ with}\]

\[(4.3) \quad \text{Cov}[X_i, Y_i] = (\lambda - \gamma_i) \cdot \text{Var}[X_i], \quad \gamma_i = \frac{\alpha_i - \rho_i}{(\alpha_i - \rho_i)^2 + \frac{1}{\alpha_i - \rho_i}}.\]

Inserting into (4.1), one obtains that the optimal portfolios are solutions of
A simple complete answer can be given in the special situation of vanishing expected excess returns, whose proof is obvious. It is remarkable that this solution does not depend on the utility functions of decision makers.

**Proposition 4.1.** Let \((R_i, S_i), i=1,2,\) be two pairs of bivariate normal random returns satisfying the equal means conditions \(E[X_i]=0,\) as well as the complete diversification conditions \(\rho_i < \min \left\{ \alpha_i, \frac{1}{\alpha_i} \right\}, \quad \alpha_i = \frac{\sigma_{i,R}}{\sigma_{i,S}}.\) The optimal portfolios are given by

\[
\lambda_i^* = \gamma_i = \frac{\alpha_i - \rho_i}{(\alpha_i - \rho_i) + \left( \frac{1}{\alpha_i} - \rho_i \right)} \in (0,1), \quad \rho_i = \frac{\text{Cov}[R_i, S_i]}{\sigma_{i,R} \cdot \sigma_{i,S}}.
\]

The comparative static property \(\lambda_1^* \leq \lambda_2^*\) holds if, and only if, the following inequality is satisfied:

\[
\left( \frac{\alpha_2 - \alpha_1}{\alpha_1} \right) + \left( \frac{1}{\alpha_1} - \frac{1}{\alpha_2} \right) \rho_2 - \left( \frac{1}{\alpha_2} \right) \rho_1 \geq 0.
\]

By equal standard deviation ratios \(\alpha = \alpha_1 = \alpha_2,\) one has

\[
\lambda_1^* < \lambda_2^* \iff \alpha > 1, \rho_2 > \rho_1 \quad \text{or} \quad \alpha < 1, \rho_2 < \rho_1,
\]

\[
\lambda_1^* = \lambda_2^* \iff \alpha = 1 \quad \text{or} \quad \rho_2 = \rho_1.
\]

**Example 4.1.**

Suppose that in Proposition 4.1 the pairs \((R_i, S_i)\) have equal marginals, in particular \(\alpha = \alpha_1 = \alpha_2.\) Now, if \(\alpha < 1, \rho_2 > \rho_1,\) which is compatible with the correlation order of riskiness (see Section 3), then \((4.7)\) implies \(\lambda_1^* > \lambda_2^*,\) which is an ambiguous comparative static effect, valid independently of the decision maker's strictly risk averse utility function.

In general, when \(E[X_i] > 0,\) the optimal portfolios depend by \((4.4)\) upon the utility functions of decision makers. The analysis simplifies considerably for the often encountered exponential utility model \(u(x) = \frac{1}{r} \left( 1 - e^{-rx} \right),\) where \(r = r(x) = -\frac{u''(x)}{u'(x)}\)
is the Arrow-Pratt coefficient of risk aversion. From (4.4) one obtains the optimal portfolios

\[ \lambda_i^* = \gamma_i + \frac{E[X_i]}{r \cdot \text{Var}[X_i]} \]

If \( E[X_i] > 0 \) one has \( \lambda_i^* > \gamma_i \), and the comparative static property \( \lambda_1^* \leq \lambda_2^* \) is fulfilled if, and only if, one has

\[ r \cdot (\gamma_2 - \gamma_1) + \frac{E[X_2]}{\text{Var}[X_2]} - \frac{E[X_1]}{\text{Var}[X_1]} \geq 0. \]

**Example 4.2.**

Suppose the pairs \((R_i, S_i)\) have the same marginal means \( \mu_R = \mu_{1,R} = \mu_{2,R}, \mu_S = \mu_{1,S} = \mu_{2,S} \), and variances \( \sigma_R^2 - \sigma_{1,R}^2 = \sigma_{2,R}^2, \sigma_S^2 - \sigma_{1,S}^2 = \sigma_{2,S}^2 \), but different correlation coefficients \( \rho_1 \neq \rho_2 \). Setting \( \alpha = \frac{\sigma_R}{\sigma_S} \) one has \( E[X_i] = \mu_S - \mu_R > 0, \)

\[ \text{Var}[X_i] = \sigma_R \sigma_S \cdot \left\{ (\alpha - \rho_i) + \left( \frac{1}{\alpha} - \rho_i \right) \right\}, i=1,2. \]

A straightforward calculation shows that (4.10) is satisfied if, and only if, one has

\[ \begin{aligned} & \left( \frac{\mu_S - \mu_R}{\sigma_R \sigma_S} \right) + \left( \frac{\sigma_R}{\sigma_S} - \rho_i \right) \cdot r \geq 0, \text{ or equivalently one of the following three conditions is fulfilled:} \\
& \sigma_R > \sigma_S, \quad \rho_2 > \rho_1 \quad (4.11) \\
& \sigma_R < \sigma_S, \quad r \leq 2 \left( \frac{\mu_S - \mu_R}{\sigma_S^2 - \sigma_R^2} \right), \quad \rho_2 > \rho_1 \quad (4.12) \\
& \sigma_R < \sigma_S, \quad r > 2 \left( \frac{\mu_S - \mu_R}{\sigma_S^2 - \sigma_R^2} \right), \quad \rho_2 < \rho_1 \quad (4.13) \\
\end{aligned} \]

In the first two situations, the pair \((R_2, S_2)\) is more risky than \((R_1, S_1)\) in the correlation order of riskiness. Also, the excess returns are more risky in the mean-variance sense because one has \( E[X_2] = E[X_1] \) and \( \text{Var}[X_2] > \text{Var}[X_1] \). However, if one supposes that \( \sigma_R < \sigma_S, \quad r > 2 \left( \frac{\mu_S - \mu_R}{\sigma_S^2 - \sigma_R^2} \right), \quad \rho_2 > \rho_1 \), which corresponds to an assumption of relatively high risk aversion, then an ambiguous comparative static effect for the correlation order of riskiness and the mean-variance criterion for excess returns is exhibited.

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5. The classical portfolio problem for lognormal random returns.

The discussion is separated in three distinct parts. We begin in Section 5.1 with a simple analytical solution to the (complete) diversification existence problem defined in Section 1. In contrast to the bivariate normal case, which remains quite tractable, one encounters in this important situation some computational difficulties, as pointed out in Section 5.2. Interestingly enough, Section 5.3 on the standard portfolio problem derives a new argument in favour of risk-neutral valuation for option-pricing in the sense first proposed by Cox and Ross(1976) (see e.g. Jarrow and Rudd(1983), p. 89). Also, it is shown that no financial risk premium is offered on efficient portfolios in this situation, where efficiency is understood in the sense of Merton(1990), Section 2.2.

5.1. Complete diversification for lognormal returns.

Suppose the random pair of returns \((R, S)\) is bivariate lognormally distributed such that \((\ln R, \ln S)\) is bivariate normal with means \(\mu_R, \mu_S\), variances \(\sigma_R^2, \sigma_S^2\), and correlation coefficient \(\rho\). The question of (complete) diversification, as explained in Section 1, is solved by the following analytical result.

**Proposition 5.1.** Let \((R, S)\) be a bivariate lognormal pair of random returns such that the expected excess return is non-negative, that is \(E[X] = E[S - R] \geq 0\). Then the following statements about the optimal portfolio \(\lambda^*\) are true:

(D) One has \(\lambda^* > 0\) (diversification holds) if, and only if, one has

\[
E[S] \cdot \left\{ N\left( \frac{\ln r - \mu_S}{\sigma_S} \right) - N\left( \frac{\ln r - \mu_R - \beta \sigma_S^2}{\sigma_S} \right) \right\} > E[R] \cdot \left\{ N\left( \frac{\ln r - \mu_S}{\sigma_S} \right) - N\left( \frac{\ln r - \mu_R - \sigma_S^2}{\sigma_S} \right) \right\}
\]

(5.1)

for all \(r > 0\), with \(\beta = \rho \frac{\sigma_R}{\sigma_S}\), and \(N(x)\) the standard normal distribution.

In particular, if \(\text{Cov}[\ln R, \ln S] < \text{Var}[\ln R]\), then \(\lambda^* > 0\).

(CD) If \(E[S] = E[R]\), one has \(0 < \lambda^* < 1\) (complete diversification holds) if, and only if, one has

\[
\text{Cov}[\ln R, \ln S] < \min\left\{ \text{Var}[\ln R], \text{Var}[\ln S] \right\}, \text{ or equivalently } \rho < \min\left\{ \frac{\sigma_R}{\sigma_S}, \frac{\sigma_S}{\sigma_R} \right\}
\]

(5.2)

(5.3)
Proof. From Section 1, we know that \( \lambda^* > 0 \) exactly when strict RNED(S|R) or (1.2) is satisfied, that is

\[
E[S|R \leq r] - E[S] < E[R|R \leq r] - E[R] \text{ for all } r > 0.
\]

Since \( (\ln S|\ln R = r) \) is normal with mean \( \mu(r) = \mu_S + \beta(\ln r - \mu_R) \) and variance \( \tilde{\sigma}^2_S = (1 - \rho^2)\sigma^2_S \), one has that \( (S|R = r) \) is lognormal with parameters \( \mu(r), \tilde{\sigma}_S \), and one gets

\[
E[S|R \leq r] = E[R^\theta|R \leq r] \cdot E[S] \cdot \exp\left\{-\beta\mu_R - \frac{1}{2}(\beta\sigma^2_R)\right\}.
\]

Using that \( R \) is lognormal with parameters \( \mu_R, \sigma_R \), one shows without difficulty that

\[
E[R^\theta|R \leq r] = \exp\left\{\beta\mu_R + \frac{1}{2}(\beta\sigma^2_R)\right\}.
\]

Inserting (5.6) into (5.5) and (5.4), one obtains condition (5.1). In particular, if \( \beta < 1 \), that is \( \text{Cov}[\ln R, \ln S] < \text{Var}[\ln R] \), then (5.1) is always fulfilled. By equal means of returns, one has \( 0 < \lambda^* < 1 \) exactly when (5.1) and its symmetric version obtained by interchanging \( R \) and \( S \) hold. But this is satisfied if, and only if, (5.2) or equivalently (5.3) is fulfilled. \( \Box \)

5.2. Evaluation of the optimal portfolio.

Under the assumptions of Section 5.1, one knows from Section 1 that the optimal portfolio \( \lambda^* \) is solution of the expected value equation

\[
E\left[S \cdot u'(\lambda^*S + (1 - \lambda^*)R)\right] = E\left[R \cdot u'((1 - \lambda^*)R + \lambda^*S)\right],
\]

where \( u'(x) \) is the marginal utility of a decision maker. Setting \( g(x) = u'(x) \), \( a = \lambda^* \), \( b = 1 - \lambda^* \), one must evaluate double integrals of the type

\[
E[S \cdot g(aS + bR)] = \int_0^\infty \int_0^\infty sg(as + br)f_{S,R}(s,r)dsdr,
\]

with the joint probability density \( f_{S,R}(s,r) = f_{S|R=r}(s) \cdot f_R(r) \). Using that \( (S|R = r) \) is lognormal with parameters \( \mu(r), \tilde{\sigma}_S \), as in the proof of Proposition 5.1, and \( R \) is lognormal with parameters \( \mu_R, \sigma_R \), and making the usual exponential transformations of variables, one must evaluate double normal integrals of the type
where \( \Phi(x) = N'(x) \) is the standard normal density. To overcome the computational difficulties encountered in the evaluation of (5.9), one must apply methods similar to those required for solving the problem posed by Krishnarao (1997).

5.3. The standard portfolio problem with a lognormal return.

It is interesting to investigate the standard portfolio problem with a lognormal return \( R \) with parameters \( \mu_R, \sigma_R \), and a riskless asset with deterministic return \( r_r \). A diversified optimal standard portfolio \( \lambda = \lambda^* > 0 \) will be solution of the expected value equation

\[
E[(R - r_r) \cdot u'(r_r + \lambda (R - r_r))] = 0,
\]

where \( u'(x) \) is the marginal utility of a decision maker. The analysis depends on a simple univariate lognormal analogue of Stein's Lemma.

**Lemma 5.1.** Let \( R \) be a lognormal random variable, \( a, b > 0 \), and set \( X = R - a \), a translated lognormal random variable. Then, for all monotone functions \( g(x) \), one has the covariance identity

\[
\text{Cov} [X, g(a + bX)] = -E[X] \cdot \left[ g((1-b)a) + E[g(a + bX)] \right].
\]

**Proof.** Setting \( W = a + bX \) and noting that \( W \) has support \( [(1-b)a, \infty) \), one obtains

\[
\text{Cov} [X, g(W)] = \int_{(1-b)a}^{\infty} F_W(w) \left\{ E[X|W \leq w] - E[X] \right\} dw.
\]

A routine calculation shows that

\[
E[X|W \leq w] = \frac{E[X]}{F_W(w)}.
\]

Using partial integration, it follows that

\[
\text{Cov} [X, g(W)]
= -E[X] \cdot \int_{(1-b)a}^{\infty} F_W(w) dg(w)
= -E[X] \cdot \left\{ g(w) \cdot F_W(w) \right\}_{(1-b)a}^{\infty} + \int_{(1-b)a}^{\infty} g(w) f_W(w) dw
= -E[X] \cdot \left\{ g((1-b)a) + E[g(W)] \right\}.
\]
Applied to (5.10), this condition is seen equivalent to

\[(5.14) \quad E\left[(R - r)\right] \cdot u'((1 - \lambda)r) = 0.\]

Now, a risk averse decision maker, who strictly prefers more to less, has a strictly increasing utility function. Thus, (5.14) implies \( E[R] = r \), or equivalently \( \mu = \ln(r) - \frac{1}{2} \sigma^2 \), which is nothing but the no-arbitrage condition in the option pricing model by Black and Scholes (1973). This justifies on the one side the risk-neutral argument by Cox and Ross (1976) for option pricing with lognormal returns. At the same time, it yields a remarkable elementary proof of a result, which otherwise follows from Ito's lemma in stochastic calculus applied to a geometric Brownian process for the risky asset return (see e.g. Hull (1989), Section 4.2). On the other hand, the same result shows that no financial risk premium is offered on efficient portfolios with one riskless asset and a risky asset with lognormal return, where an efficient portfolio is understood in the sense of Merton (1990), Section 2.2, p. 22.

**Appendix**: An elementary proof of Stein's Lemma

In the literature (e.g. Huang and Litzenberger (1988), p. 101) the following result is attributed to Stein (1972/81). Our elementary proof makes use of the stop-loss transform, which by the way appears to be of fundamental importance for both Actuarial Science and Finance.

**Stein's Lemma.** Let \((X,Y)\) be a bivariate normal random vector, \(g(x)\) a differentiable function such that the mean \(E[g'(Y)]\) exists. Then one has the covariance identity

\[(A.1) \quad \text{Cov}[X,g(Y)] = E[g'(Y)] \cdot \text{Cov}[X,Y].\]

**Proof.** The regression curve of \(X\) relatively to \(Y\) satisfies the identity (note the misprint in the formula for \(\beta\) in Wright (1987), p. 116):

\[(A.2) \quad E[X|Y = y] = E[X] + \beta \cdot (y - E[Y]), \quad \text{with} \quad \beta = \frac{\text{Cov}[X,Y]}{\text{Var}[Y]}.\]

Integrating both sides with respect to \(F_y(y|Y \leq y)\), one finds

\[(A.3) \quad E[X] - E[X|Y \leq y] = \beta \cdot \left[E[Y] - E[Y|Y \leq y]\right].\]

Applying the covariance formula by Hoeffding (1940) and Lehmann (1966), as in (2.3), one gets using (A.3) that
\( \text{Cov}[X, g(Y)] = \int \{ \mathbb{E}[X] - \mathbb{E}[X | Y \leq y] \} g'(y) F_Y(y) dy \)

\[ = \beta \int I(y) g'(y) dy, \quad I(y) = \{ \mathbb{E}[Y] - \mathbb{E}[Y | Y \leq y] \} F_Y(y). \]

Furthermore, one has using partial integration and making rearrangements that

\[ I(y) = F_Y(y) \cdot E[Y] - \int_{-\infty}^{y} y dF_Y(y) = F_Y(y) \cdot E[Y] - y F_Y(y) + E[(y - Y) -] \]

\[ = (y - E[Y]) \cdot F_Y(y) + E[(Y - y) -]. \]

But, for a normal random variable \( Y \) with mean \( \mu \) and variance \( \sigma^2 \), one has for its survival function and stop-loss transform the expressions

\[ \overline{F}_Y(y) = N\left( \frac{y - \mu}{\sigma} \right), \]

\[ E[(Y - y)_+] = \sigma \Phi\left( \frac{y - \mu}{\sigma} \right) - (y - \mu) \overline{N}\left( \frac{y - \mu}{\sigma} \right), \]

where \( N(x) \) is the standard normal distribution, \( \overline{N}(x) = 1 - N(x) \) and \( \Phi(x) = N'(x) \).

Inserting (A.6) into (A.5), it follows that

\[ I(y) = \sigma \Phi\left( \frac{y - \mu}{\sigma} \right). \]

Therefore (A.4) can be rewritten as

\[ \text{Cov}[X, g(Y)] = \beta \sigma^2 \cdot \int g'(y) \left( \frac{y - \mu}{\sigma} \right) dy = \mathbb{E}[g'(Y)] \cdot \text{Cov}[X, Y], \]

which is the desired covariance identity (A.1). 0

References.


