Development and Pricing of a New Participating Contract

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Outline of the Talk

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   “Constant rate guarantee Participating LIC”
3. Building of New Contracts
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5. Conclusion
Bibliography

Brennan and Schwartz [1976]
Briys and de Varenne [1993, 1997]
Grosen and Jørgensen [1997, 2000, 2002]
Jørgensen [...], Ballotta [...], Bacinello [...]
Bernard, Le Courtois and Quittard-Pinon [2005]
Heath, Jarrow and Morton [1992]
Longstaff and Schwartz [1995]
Collin-Dufresne and Goldstein [2001]
Jeanblanc, Yor and Chesney [2005]
Presentation and Pricing of Standard Contracts
### Standard Contracts
(the Company)

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_0$</td>
<td>$E_0 = (1 - \alpha)A_0$</td>
</tr>
<tr>
<td></td>
<td>$L_0 = \alpha A_0$</td>
</tr>
</tbody>
</table>

- $E_0 = \text{initial equity value}$
- $L_0 = \text{initial policyholders’ investment}$
Standard Contracts (Minimum Guarantee)

Existence of a minimum guaranteed rate $r_g$:

$$L^g_T = L_0 e^{r_g T} \quad \text{at } T$$

- **Solvency at time** $T$ : $A_T \geq L^g_T$
  
  Policyholders receive $L^g_T$

- **Default at time** $T$ : $A_T < L^g_T$
  
  Policyholders receive $A_T$
Policyholders are given a part $\delta$ of the benefits when

$$A_T > \frac{L_T^g}{\alpha} \quad \text{where} \quad \alpha < 1.$$ 

Policyholders now receive at $T$:

$$\Theta_L(T) = \begin{cases} 
A_T & \text{if } A_T < L_T^g \\
L_T^g & \text{if } L_T^g \leq A_T \leq \frac{L_T^g}{\alpha} \\
L_T^g + \delta(\alpha A_T - L_T^g) & \text{if } A_T > \frac{L_T^g}{\alpha}
\end{cases}$$
Standard Contracts
(Early Default)

The firm pursues its activities until $T$ if:

$$\forall t \in [0, T], \quad A_t > L_0 e^{rgt}$$

Let $\tau$ be the default time

$$\tau = \inf \{ t \in [0, T] \mid A_t < L_0 e^{rgt} \}$$

In case of prior insolvency, policyholders receive:

$$\Theta_L(\tau) = L_0 e^{rg\tau}$$
With a Constant Minimum Guaranteed Rate,
a Participation in the Assets Performance,
and the Possibility of an Early Default, we have:

\[
V_1 = \mathbb{E}_Q \left( e^{-\int_0^T r_s ds} \left[ L_T^g + \delta (\alpha A_T - L_T^g)^+ - (L_T^g - A_T)^+ \right] 1_{\tau \geq T} + e^{-\int_0^\tau r_s ds} \left[ L_0 e^{rg}\tau \right] 1_{\tau < T} \right)
\]

\[
= \mathbb{E}_Q \left( e^{-\int_0^T r_s ds} \left[ L_T^g + \delta (\alpha A_T - L_T^g)^+ - (L_T^g - A_T)^+ \right] 1_{\tau \geq T} + e^{-\int_0^\tau r_s ds} \left[ L_0 e^{rg}\tau \right] 1_{\tau < T} \right)
\]
The dynamics under $Q$ of the ZC bonds $P(t, T)$ are:

\[
\frac{dP(t, T)}{P(t, T)} = r_t dt - \sigma_P(t, T) dZ^Q_1(t)
\]

whilst the assets dynamics under $Q$ are:

\[
\frac{dA_t}{A_t} = r_t dt + \sigma dZ^Q(t)
\]

where $Z^Q$ and $Z^Q_1$ are correlated $Q$-Brownian motions.

\[
(dZ^Q \cdot dZ^Q_1 = \rho dt).
\]
Standard Contracts
(Pricing Methodology)

To price this contract:

1. decorrelation of the assets and interest rate risks
2. introducing the forward-neutral measure.

- a 2D problem in $(r, \tau)$
- an extension of Collin-Dufresne and Goldstein [2001] solves the problem in terms of a recurrence equation
- Yet, No Closed-Form Formulae Can Be Obtained
Building of New Contracts

Obtaining Closed-Form Formulae
Introducing...

... A New Contract

which is:

- only very slightly different from the previous one
- in a totally identical framework for $A$ and $r$
- where only the Guaranteed Amount is modified
  and Indexed on Government Zero-Coupon Bonds
The New Contract’s Guaranteed Amount

The Minimum Guaranteed Rate is proportional to the yield of a Government Zero-Coupon Bond.

The Guarantee is the one of an equivalent position in \( \frac{\beta L_0}{P(0,T)} \) Government ZC Bonds Maturing at time \( T \). Indeed:

- At time 0, this Guarantee is worth \( \beta L_0 \),
- At time \( t \), it is worth \( l_t^g = \frac{\beta L_0}{P(0,T)} P(t, T) \),
- At time \( T \), it is worth \( l_T^g = \frac{\beta L_0}{P(0,T)} \).
The New Contract’s Guaranteed Amount

What is the main Implication of Introducing a Guarantee such as the one Defined in the Previous Slide?

Indeed the default time becomes:

\[ \tau = \inf \left\{ s < T / A_s < l_s^g = l_T^g P(s, T) \right\} \]

and this allows pricing in closed-form:

\[
V_2 = \mathbb{E}_Q \left[ e^{-\int_0^T r_s ds} \left( l_T^g + \delta (\alpha A_T - l_T^g)^+ - (l_T^g - A_T)^+ \right) 1_{\tau \geq T} \\
+ e^{-\int_0^T r_s ds} l_T^g 1_{\tau < T} \right] !!!!
\]
The New Contract
(Valuation)

So, how can $V_2$ be priced in closed-form?

We Illustrate our Approach by Computing the Forward-Neutral Ruin Probability

$$Q_T(\tau < T) = Q_T\left(\inf_{u \in [0,T]} \left( \frac{A_u}{P(u,T)} \right) < I_T^g \right)$$
The New Contract
(Valuation)

The solution of our problem lies in the fact that:

$$\frac{A_u}{P(u, T)} = \frac{A_0}{P(0, T)} e^{N_u - \frac{1}{2} \xi(u)}$$

where the differential of the martingale $N$ is defined by:

$$dN_s = (\sigma_P(s, T) + \rho \sigma) dZ_{1}^{QT}(s) + \sigma \sqrt{1 - \rho^2} dZ_{2}^{QT}(s)$$

and the quadratic variation of $N$ is:

$$\xi(u) = <N>_u = \int_{0}^{u} [(\sigma_P(s, T) + \rho \sigma)^2 + \sigma^2(1 - \rho^2)] ds$$
Thanks to Dubins-Schwarz Theorem

\[ FNRP = Q_T \left\{ \min_{s \in [0, \xi(T)]} \left( B_s - \frac{1}{2} s \right) < \ln (\beta \alpha) \right\} \]

Therefore the contract can be priced as a linear combination of Gaussian functions (and so the pricing is instantaneous).

As the Minimum Guarantee moves with the interest rates some of the Participating Contract’s Characteristics become very interesting for the company...
Numerical Analysis

We set our parameter range according as:

<table>
<thead>
<tr>
<th>$A_0$</th>
<th>$\alpha$</th>
<th>$\rho$</th>
<th>$\sigma$</th>
<th>$T$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.85</td>
<td>0.2</td>
<td>10%</td>
<td>10</td>
<td>90%</td>
</tr>
</tbody>
</table>

$L_0 = \alpha A_0 = 85$

Contract Maturity : 10 years
Ruin Probability $E_1$ w.r.t. $r_g$

![Graph showing ruin probability $E_1$ with two curves labeled $V_2$: New and $V_1$: Standard.](image)
Contract Value w.r.t. $\rho$
(Correlation $A/r$)
General Conclusion

A Proposal for New Participating Contracts.

Instead of guaranteeing at time $t$ a constant minimum rate, guarantee a yield proportional to the one of a ZC bond.

In such a setting, closed-form formulae can be obtained.

This method also allows to price exotic options (sharks options for instance).