A Continuous-Time Model for Reinvestment Risk in Bond Markets

Mikkel Dahl
Laboratory of Actuarial Mathematics
University of Copenhagen
Denmark
dahl@math.ku.dk
Outline of talk

• Motivation
• The model
• Hedging
• Conclusion
Danish life insurance companies have to value liabilities at market value

\[ \rightsquigarrow \] Increased focus on correct valuation and risk management

Special features concerning life insurance contracts

- Depends on death/survival of the insured
  - Unsystematic mortality risk (randomness of deaths in a portfolio with known mortality intensity)
  - Systematic mortality risk (stochastic mortality intensity)

- Very long term contracts \[ \rightsquigarrow \] no liquid market for sufficiently long bonds
  - Reinvestment risk (focus of this talk)
Example

• Assume the time to maturity of the longest bond traded is 10 years and new bonds are each year

• We have sold a claim of 1 with time of maturity 30 years

• What is the price? (Not unique)

• How to hedge? And what is the risk associated with the hedge?
The model

Today

Issue of new bonds with maturity $\tau \in (\tilde{T}, T_1 + \tilde{T}]$

Maturity of longest bond traded today

Issue of new bonds with maturity $\tau \in (T_1 + \tilde{T}, T_2 + \tilde{T}]$

Maturity of longest bond issued at $T_1$

Time of payment

$T_0 = 0$ $T_1$ $\tilde{T}$ $T_2$ $T_1 + \tilde{T}$ $T = T_3$
Notation

- Price at time $t$ of a zero coupon bond maturing at time $\tau$:
  \[ P(t, \tau) \]

- Forward rates are given by
  \[ P(t, \tau) = e^{-\int_t^\tau f(t,u)du} \]
  The forward rate $f(t, \tau)$ can be interpreted as the riskfree rate of interest, contracted at time $t$ over the infinitesimal interval $[\tau, \tau + d\tau)$.

- More convenient to model forward rates than bond prices
Forward rate curve at time 0

\[ 0 \quad T_1 \quad \tilde{T} \quad T_2 \quad T_1 + \tilde{T} \quad T = T_3 \]
Forward rate curve at time $T_1$ prior to issue of new bonds

\[ 0 < T_1 < \tilde{T} < T_2 < T_1 + \tilde{T} < T = T_3 \]
Examples of forward rate curves at time $T_1$ after issue of new bonds
The model 6/7

- \( P \)-dynamics of forward rates between issue of new bonds

\[
df(t, \tau) = \alpha(t, \tau)dt + \sigma(t, \tau)dW_t
\]

- Initial value of new forward rates at time \( T_i, i = 1, \ldots, n \)

\[
f(T_i, \tau) = f(T_i, T_{i-1} + \tilde{T}) + \int_{T_{i-1} + \tilde{T}}^{T} \gamma^i(u)du
\]

\( \gamma^i \) depends on r.v. \( Y_i \) independent of \( W \).

A possible choice of \( \gamma^i \)

\[
\gamma^i(u) = \frac{1}{T_i - T_{i-1}} (k_1 Y_i + k_2 (1 - Y_i))
\]

\( Y_i \in \{0, 1\}, k_1 \) and \( k_2 \) constants.

\( \sim \) Extension is a straight line with slope \( k_j/(T_i - T_{i-1}), j \in \{1, 2\} \).
• Equivalent martingale measures (EMM’s)
  
  – Unique dynamics of forward rates under EMM
  
  – No assets depending on $Y = (Y_1, \ldots, Y_n)$ are traded prior to observation
  $$\leadsto$$ No unique distribution of $Y$ under EMM
  $$\leadsto$$ Infinitely many equivalent martingale measures
  $$\leadsto$$ Incomplete model
Hedging

Controlling the reinvestment risk

• Short term contracts (depends on bonds with maturity before time $\tilde{T}$)
  $\leadsto$ Attainable
  $\leadsto$ Perfect hedge for short term claims
  $\leadsto$ No risk

• Long term contracts (depends on bonds with maturity after time $\tilde{T}$)
  $\leadsto$ In general: Unattainable
  $\leadsto$ In general: No perfect hedge
  $\leadsto$ Minimize risk associated with hedge
  $\leadsto$ Choose criterion measuring risk
  $\leadsto$ Considered risk-minimization (in the paper)
Conclusion

• Main contribution (and focus of this talk): Proposed a bond market model including reinvestment risk

• Applied criterion of risk-minimization
  – Risk-minimizing strategy
  – Quantify reinvestment risk

• A possible implementation

• Currently working on applications in insurance