Risk Management for a bond using bond put options

Griselda Deelstra (ULB)
Dries Heyman (Ghent University)
Michèle Vanmaele (Ghent University)

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Introduction: problem description and motivation

• Setup similar to Ahn et al., Journal of Finance, 1999
• Classical hedging example: hedging exposure to price risk of an asset
• e.g. Currency, oil, gold
• Ahn et al. (1999): share as underlying asset
• Minimize VaR of position in share by using put options
• Optimal strike price of put option?
• Here:
  – underlying asset: bond
  – VaR and TVaR
Introduction: problem description and motivation

- Buy a zero-coupon bond with maturity $S$, $Y(0, S)$
- Sell it again at time $T$, with $T < S$, at price $Y(T, S)$
- No hedging: exposed to interest rate movements
- Hedging using bond put options
- Determine strike price such that risk is minimized, using budget $C$
- Risk measures: Value-at-Risk (VaR) and Tail Value-at-Risk (TVaR)
Risk measurement

- VaR$\alpha,T$ satisfies:

$$\Pr(L_0 \geq \text{VaR}_\alpha,T) = \alpha.$$ 

- VaR$\alpha,T$ is the loss of the worst case scenario on the investment at a $(1 - \alpha)$ confidence level during the period $[0, T]$.

- TVaR$\alpha,T$, is defined as follows:

$$\text{TVaR}_\alpha,T = \frac{1}{\alpha} \int_{1-\alpha}^{1} \text{VaR}_{1-\beta,T} \, d\beta.$$ 

- TVaR$\alpha,T$ is the arithmetic average of the quantiles of our loss, from $1 - \alpha$ to 1 on.
Hull-White model

• Advantages:
  – Perfect fit with an initial given term structure
  – Analytic solutions for zero-coupon bond and European options

• Disadvantage: Possibility of negative interest rates

\[
dr(t) = (\theta(t) - \gamma(t)r(t))dt + \sigma(t)dZ(t)
\]

• \(Z(t)\) a standard Brownian motion under the risk-neutral measure \(Q\)
• \(\theta(t)\): time dependent long-term average level of the spot interest rate
• \(\gamma(t)\): the mean-reversion speed
• \(\sigma(t)\): the volatility
• \(\gamma\) and \(\sigma\) constant
Hull-White model

\[ E[r(t)] = m = r(0)e^{-\gamma t} + a(t) - a(0)e^{-\gamma t} \]

\[ Var[r(t)] = s^2 = \frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma t}) \]

- \( a(t) = F^M(0, t) + \frac{\sigma^2}{2} \left( \frac{1 - e^{-\gamma(S-t)}}{\gamma} \right)^2 \) with \( F^M(0, t) \) the instantaneous forward rate observed in the market on time zero with maturity \( t \).
Bond valuation in the Hull-White model

Price of a zero-coupon bond at time $t$ with maturity $S$:

$$Y(t, S) = A(t, S)e^{-B(t, S)r(t)}$$

- $B(t, S) = \frac{1 - e^{-\gamma(S-t)}}{\gamma}$

- $A(t, S) = \frac{Y^M(0, S)}{Y^M(0, t)}e^{\frac{\sigma^2}{4\gamma}(1-e^{-2\gamma t})B^2(t,T)}$

with $Y^M(0, t)$ the price of a zero coupon bond with maturity $t$ observed in the market at time $0$.

- $Y(t, S)$ is lognormally distributed with

  mean $\Pi(t, S) = \ln A(t, S) - B(t, S)m$, and

  variance $\Sigma(t, S)^2 = B(t, S)^2s^2$
Bond option valuation in the Hull-White model

Price of a bond put option:

\[ P(0, T, S, X) = -KY(0, S)\Phi(-d_1) + XY(0, T)\Phi(-d_2) \]

with \( K \) the principal of the bond and \( X \) the strike price of the option

- \( d_1 = \frac{1}{\sigma_p} \log \left( \frac{KY(0, S)}{XY(0, T)} \right) + \frac{\sigma_p}{2} \)
- \( d_2 = d_1 - \sigma_p \)
- \( \sigma_p^2 = \frac{\sigma^2}{2\gamma^3}(1 - e^{-2\gamma T})(1 - e^{-\gamma(S-T)})^2 \)
Risk minimization

• fixed budget $C$ for hedging.
  – If $C$ insufficient for buying entire option, buy portion $h$ of option.
  – No overhedging

• Formally: $C = hP(0, T, S, X)$ and $h \in (0, 1)$

• Payoff of bond + put option at $T = \max(hX + (1 - h)Y(T, S), Y(T, S))$

• Discounted payoff $= ((1 - h)Y(T, S) + hX)Y(0, T)$

• $L_0 = Y(0, S) + C - ((1 - h)Y(T, S) + hX)Y(0, T)$

• Minimize $\text{VaR}_{\alpha, T}$ and $\text{TVaR}_{\alpha, T}$ of $L_0$, taking into account the budget constraint
• VaR$_{\alpha,T}$ minimization:

$$e^{\Pi(T,S)+\Sigma(T,S)(c(\alpha))} = \frac{Y(0, S)\Phi(-d_1)}{Y(0, T)\Phi(-d_2)}$$

with $c(\alpha)$ the percentile of the standard normal distribution, i.e. $\Pr(z \leq c(\alpha)) = \alpha$

• TVaR$_{\alpha,T}$ minimization:

$$\frac{1}{\alpha}e^{\Pi(T,S)+\frac{1}{2}\Sigma^2(T,S)\Phi(c(\alpha) - \Sigma(T, S'))} = \frac{Y(0, S)\Phi(-d_1)}{Y(0, T)\Phi(-d_2)}$$

• Optimal strike price is implicitly defined (cfr. $d_1$)

• Optimal strike price is independent of hedging expenditure $C$ !!
Hull-White model calibration

- Determine parameters of model in a credible way
- Suppose we have $M$ market prices of traded instruments
  \[ \min_{\gamma, \sigma} \sqrt{\sum_{i=1}^{M} \left( \frac{\text{model}_i - \text{market}_i}{\text{market}_i} \right)^2} \]
- Market instruments: caps
  - provide protection against a specified interest rate (e.g. the three month LIBOR, $R_L$) rising above a specified level (the cap rate, $R_C$)
  - Cap = series of caplets = series of zero-coupon bond put options
- Link: Zero-coupon bond put options can be valued by Hull-White model
- $\gamma = 0.31621 \quad \sigma = 0.011631$
Parameters: $T = 1$, $S = 5$, $R(0) = 0.0213$, $C = 0.0043$, $\alpha = 0.05$

Figure 1: Bond value at $T = 1$

- Minimum bond price: 0.8159
• No hedging: large potential losses
• Potential discounted loss: 0.0608
• Avoid this by hedging!
**Numerical illustration**

**VaR**$_{\alpha,T}$ minimization:

- Optimal strike price: 0.8915
- buy 26.23% of an option
- Second peak.
- Shorter tail.

Figure 3: Loss of the hedged position at $T = 1$
Numerical illustration

TVaR_{\alpha,T} minimization:

- Lower optimal strike price: 0.87698
- Buy 59.35\% of an option.
- Second peak more to right
- Shorter tail

Figure 4: Loss of the hedged position at T = 1
Numerical illustration

\( \text{VaR}_{\alpha,T} \) minimization:

\[ \text{Strike price} \]

\[ \text{VaR} \quad \text{TVaR} \]

Figure 5: VaR and TVaR in function of distance from optimal strike price
TVaR$_{\alpha,T}$ minimization:

Figure 6: VaR and TVaR in function of distance from optimal strike price
Conclusions and further research

- VaR and TVaR Minimization of position in bond
- Formula for determining optimal strike price for put option used for hedging
- Model for instantaneous interest rate: Hull-White
- Extend to other interest rate models?
- Extend to other hedging instruments