A Multi-Period View on Actuarial and Financial Pricing for Guaranteed Minimum Death Benefits in Unit-Linked Life Insurance

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Secura, www.secura-re.com
Consider the following product:

- 1000 insured aged 50 invest $S_0$ in a risky asset \((S_t)_{t \geq 0}\).
- Guaranteed Minimum Death Benefit of $K$.
  
  If insured dies in month \((t, t + 1]\), GMDB leads to payment of
  
  \[
  (K - S_{t+1})_+ = \max\{0; K - S_{t+1}\}
  \]
  
  at time $t + 1$
  
  - No surrender
  
  - No guarantee at retirement (age 65).

- Possibility to invest in a risk-less bond process $B_t = B_0 e^{rt}$
Introduction

- Financial Reserving:
  - Under Black-Scholes market model and assuming mortality risk can be completely eliminated, a risk-less hedging strategy exists
  - Unhedged liability and additional costs due to:
    - Transaction costs
    - Discrete hedging intervals
    - Log-returns are not normally distributed
    - Mortality risk

- Actuarial Reserving: no hedging but capital allocation

- Multi-period setting \(\Rightarrow\) Future information about:
  - Underlying Asset
  - Mortality
Introduction

- How to come to a price in the different reserving strategies?
- Potential price differences?
- Impact of future information:
  - On capital and technical provisions?
  - In pricing?
  - In function of reserving strategy?
Financial Reserving

- Black-Scholes Price of GMDB at $t \in \{0, \ldots, T - 1\}$:

$$BSP_t = N_t \sum_{s=t}^{T-1} s_{-t} q_{s,t} P(t, s + 1),$$

where $P(t, s + 1)$ denotes the Black-Scholes price of a put with strike $K$ on underlying asset $(S_i)_{i \in [t, s+1]}$ and $N_t$ denotes the number of survivors at $t$

- Value of hedging portfolio from period $[t-1, t)$ at $t$:

$$V_t^- = N_{t-1}(\xi_{t-1} S_t + \beta_{t-1} B_t)$$

- Hedging error at $t \in \{1, \ldots, T\}$:

$$HE_t = BSP_t + (N_{t-1} - N_t)(K - S_t)_+ - V_t^-, \text{ with } BSP_T = 0$$

- Transaction costs at $t \in \{0, \ldots, T\}$: $TC_t = \tau S_t |\xi_t - \xi_{t-1}|, \xi_{-1} = 0 = \xi_T$
Actuarial versus Financial Reserving: Discounted Future Costs

- Financial Reserving:

\[
D_0^{(F)} = BSP_0 + \sum_{s=1}^{T} HE_s e^{-rs} + \sum_{s=0}^{T} TC_s e^{-rs},
\]

\[
D_t^{(F)} = \sum_{s=t+1}^{T} HE_s e^{-r(s-t)} + \sum_{s=t+1}^{T} TC_s e^{-r(s-t)}, \text{ for } t \in \{1, \ldots, T - 1\}
\]

- Actuarial Reserving:

\[
D_t^{(A)} = \sum_{s=t+1}^{T} (N_{s-1} - N_s)(K - S_s) e^{-r(s-t)}, \text{ for } t \in \{0, \ldots, T - 1\}
\]
Multi-Period Capital Allocation

- Notations:
  - $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$: filtration describing the information
  - $D_t$: discounted future costs at $t$
  - $P_t = $ provision, $K_t = $ capital and $TSL_t = P_t + K_t = $ total solvency level at $t$

- We compare 2 approaches:

**Not using future information**

$$P_t^{(N)} = E[D_t|F_0]$$
$$K_t^{(N)} = TVaR_{.99}[D_t|F_0] - P_t^{(N)}$$

- Asset $\downarrow \downarrow \Rightarrow$ Solvency $\downarrow$
- Asset $\uparrow \uparrow \Rightarrow$ Unnecessary capital costs
- Easy to calculate

**Using future information**

$$P_t^{(U)} = E[D_t|F_t]$$
$$K_t^{(U)} = TVaR_{.99}[D_t|F_t] - P_t^{(U)}$$

- More rational
- Future reserves and capitals are random variables as seen from 0
Cash-flow Model

• We look at product on a stand-alone basis
• Model average in-and outflows for shareholders:
  • Inflows:
    – Premium income
    – Net mean return on reserves
    – Net mean return on capital
  • Outflows:
    – Net mean claim payments
    – Net mean change in reserves
    – Net mean change in capital
• Discount average cash-flows at cost of capital


**Cash-flow Model**

**Technico-Financial Premium (TFP)**

- $TFP$ : premium which makes the sum of the discounted inflows equal to the sum of the discounted outflows

\[
TFP = P_0 + \sum_{t=0}^{T} e^{-tCOC} \left[ \Delta k_t - R_t(k) \right] \frac{1}{1 - \gamma}
\]

where:
- $P_0$ is the reserve taken at time 0
- $\gamma$ is the tax rate
- $\Delta k_t$ is the mean change in the capital
- $R_t(k)$ is the net mean return on the capital (after taxation)
Modelling Methods and Assumptions

- Monthly basis for simulation of asset and mortality
- Underlying risky asset:
  - Regime-Switching Log-Normal model
  - Parameters based upon maximum likelihood estimates for S&P500 from 1960 to 2003
  - Volatility for financial reserving: estimated on S&P500 from 1960 to 2003
- Mortality
  - Gompertz-Makeham approach
  - Parameters based upon table 197 Assuralia (Belgian Union of Insurance Companies)
## Future Information:
### Number of Survivors

<table>
<thead>
<tr>
<th>Year</th>
<th>Mean</th>
<th>StDev</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>996</td>
<td>1.91</td>
<td>-0.47</td>
<td>3.14</td>
</tr>
<tr>
<td>2</td>
<td>992</td>
<td>2.79</td>
<td>-0.32</td>
<td>3.05</td>
</tr>
<tr>
<td>5</td>
<td>976</td>
<td>4.73</td>
<td>-0.17</td>
<td>2.92</td>
</tr>
<tr>
<td>14</td>
<td>890</td>
<td>9.81</td>
<td>-0.10</td>
<td>3.00</td>
</tr>
</tbody>
</table>

- Volatility increases with time and is very small at initiation
- Distribution more and more symmetric with time
- Impact on provisions and $TSL$’s is linear
- Considering future information for pricing is not necessary
Future Information: Underlying Asset

- Actuarial approach:
  - Limited number of future values
  - Avoid resimulating

- Financial approach:
  - Same techniques
  - Many hedging errors are needed
    ⇒ Black-Scholes prices needed
  - Solution: reference values for which Black-Scholes prices and hedge are calculated beforehand

- Cfr. paper for more details
**Distribution of DFC’s at time 0**

\[ K = S_0 = 1 \]

- Dashed line: actuarial approach
- Full line: financial approach
- With a probability of 97.3%, actuarial reserving leads to less costs than financial reserving
- Volatility and tail are a lot more important in actuarial approach

<table>
<thead>
<tr>
<th>Approach</th>
<th>( BSP_0 )</th>
<th>Mean</th>
<th>StDev</th>
<th>Skewness</th>
<th>TVaR(_{0.95})</th>
<th>TVaR(_{0.99})</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_0^{(F)} )</td>
<td>4.24</td>
<td>4.68</td>
<td>1.11</td>
<td>1.98</td>
<td>8.07</td>
<td>9.88</td>
</tr>
<tr>
<td>( D_0^{(A)} )</td>
<td>-</td>
<td>0.83</td>
<td>2.91</td>
<td>7.07</td>
<td>10.79</td>
<td>23.30</td>
</tr>
</tbody>
</table>
Distribution of DFC’s at time 0

Impact of Ratio $S_0/K$

$S_0 = 0.5$ and $K = 1$

$S_0 = 1.5$ and $K = 1$

<table>
<thead>
<tr>
<th>Approach</th>
<th>$BSP_0$</th>
<th>Mean</th>
<th>StDev</th>
<th>Skewness</th>
<th>TVaR$_{0.95}$</th>
<th>TVaR$_{0.99}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_0^{(F)} + S_0 = 0.5$</td>
<td>26.14</td>
<td>26.58</td>
<td>2.09</td>
<td>0.41</td>
<td>31.64</td>
<td>33.63</td>
</tr>
<tr>
<td>$D_0^{(A)} + S_0 = 0.5$</td>
<td>-</td>
<td>10.33</td>
<td>9.06</td>
<td>2.00</td>
<td>38.70</td>
<td>50.29</td>
</tr>
<tr>
<td>$D_0^{(F)} + S_0 = 1.5$</td>
<td>0.68</td>
<td>0.84</td>
<td>0.51</td>
<td>4.56</td>
<td>2.66</td>
<td>4.28</td>
</tr>
<tr>
<td>$D_0^{(A)} + S_0 = 1.5$</td>
<td>-</td>
<td>0.11</td>
<td>1.09</td>
<td>17.00</td>
<td>2.22</td>
<td>8.62</td>
</tr>
</tbody>
</table>
Distribution of Future TSL’s

\[ K = S_0 = 1 \]

\[ TSL_t^{(F)} = TVaR_{0.99}[D_t^{(F)}|\mathcal{F}_t] \]

\[ TSL_t^{(A)} = TVaR_{0.99}[D_t^{(A)}|\mathcal{F}_t] \]

- Vertical line: initial TSL
- For other lines: the higher at the left, the further in time
- Both relative and absolute potential increases are smaller in financial approach
- Larger skewness in actuarial approach

S. Desmedt, 15th International AFIR Colloquium, Zurich, 6-9 September 2005 – p.15/20
Impact of Future Information in Pricing for $K = S_0 = 1$

$\text{TVaR}_{0.99}[D_t^{(F)}|S_0]$ (full line) vs. $E[\text{TVaR}_{0.99}[D_t^{(F)}|S_t]]$ (dashed line)

$\text{TVaR}_{0.99}[D_t^{(A)}|S_0]$ (full line) vs. $E[\text{TVaR}_{0.99}[D_t^{(A)}|S_t]]$ (dashed line)

- Relative and absolute differences are importantly larger in actuarial approach
- Average capital is substantially smaller in financial approach
- For lower security levels of the TVaR, the relative differences are smaller
Pricing

\[ K = S_0 = 1 \]

<table>
<thead>
<tr>
<th>Risk Measure + Approach</th>
<th>TVaR_{0.99}</th>
<th>TVaR_{0.95}</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Act</td>
<td>Fin</td>
</tr>
<tr>
<td>BSP</td>
<td>-</td>
<td>4.24</td>
</tr>
<tr>
<td>Average DFC at 0</td>
<td>0.83</td>
<td>4.68</td>
</tr>
<tr>
<td>TFP without future info</td>
<td>14.74</td>
<td>6.83</td>
</tr>
<tr>
<td>TFP with future info</td>
<td>7.01</td>
<td>6.42</td>
</tr>
</tbody>
</table>

- Cost of capital is assumed to be 10% both in actuarial and financial approach
- **Financial approach:**
  - Price is mainly determined by the Black-Scholes price of the option
  - Transaction costs + Hedging errors: on average about 10% of BSP
  - Rest: capital costs
  - Impact of future information on price is not very important
Pricing

\[ K = S_0 = 1 \]

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<th>Risk Measure + Approach</th>
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<tr>
<td>TFP with future info</td>
<td>7.01</td>
<td>6.42</td>
</tr>
</tbody>
</table>

- **Actuarial approach:**
  - No fixed cost of Black-Scholes price and no transaction costs
  - Impact of capital costs and corresponding security level is very important
  - Impact of using future information is very important
Influence of Ratio $K$ and $S_0$

<table>
<thead>
<tr>
<th>Approach</th>
<th>$BSP$</th>
<th>$E$</th>
<th>TVaR$_{0.95}$</th>
<th>TVaR$_{0.99}$</th>
<th>TFP$^U_{0.95}$</th>
<th>TFP$^U_{0.99}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(F')$ with $S_0 = 0.5$</td>
<td>26.14</td>
<td>26.58</td>
<td>31.64</td>
<td>33.63</td>
<td>28.80</td>
<td>29.89</td>
</tr>
<tr>
<td>$(A)$ with $S_0 = 0.5$</td>
<td>-</td>
<td>10.33</td>
<td>38.70</td>
<td>50.29</td>
<td>20.57</td>
<td>25.48</td>
</tr>
<tr>
<td>$(F')$ with $S_0 = 1$</td>
<td>4.24</td>
<td>4.68</td>
<td>8.07</td>
<td>9.88</td>
<td>5.71</td>
<td>6.42</td>
</tr>
<tr>
<td>$(A)$ with $S_0 = 1$</td>
<td>-</td>
<td>0.83</td>
<td>10.79</td>
<td>23.30</td>
<td>3.48</td>
<td>7.01</td>
</tr>
<tr>
<td>$(F')$ with $S_0 = 1.5$</td>
<td>0.68</td>
<td>0.84</td>
<td>2.66</td>
<td>4.28</td>
<td>1.28</td>
<td>1.75</td>
</tr>
<tr>
<td>$(A)$ with $S_0 = 1.5$</td>
<td>-</td>
<td>0.11</td>
<td>2.22</td>
<td>8.62</td>
<td>0.73</td>
<td>2.24</td>
</tr>
</tbody>
</table>

- $S_0 = 0.5$:  
  - Actuarial approach leads to lowest price under wider range of conditions  
  - Prices are less sensitive to level of security (distribution $DFC$'s less skewed)

- $S_0 = 1.5$:  
  - Financial approach leads to lowest price under a wider range of conditions  
  - Prices are more sensitive to level of security
Conclusion

- Actuarial reserving: future information has important impact
  - On price
  - On technical provisions and required solvency level

- Financial reserving: future information has
  - Smaller influence on price
  - Important impact on technical provisions and required solvency level

- Choice between actuarial or financial reserving depends on
  - Required security level
  - The portfolio where the GMDB is part of

- Important sensitivity to ratio of guarantee and initial investment
## Used Parameters

### Simulations

<table>
<thead>
<tr>
<th>Number of simulations</th>
<th>( N_S )</th>
<th>10000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of classes for using future information for the underlying asset</td>
<td>( N_A )</td>
<td>200 for ( A ), 100 for ( F )</td>
</tr>
</tbody>
</table>

### Contractual Parameters

<table>
<thead>
<tr>
<th>Portfolio composition</th>
<th>( { N_0, x, C } )</th>
<th>( { 1000, 50, 1 } )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial value underlying asset</td>
<td>( S_0 )</td>
<td>1</td>
</tr>
<tr>
<td>Guarantee at death</td>
<td>( K )</td>
<td>1</td>
</tr>
<tr>
<td>Age of retirement</td>
<td>( x )</td>
<td>65</td>
</tr>
</tbody>
</table>

### Mortality Parameters

<table>
<thead>
<tr>
<th>Gompertz-Makeham parameters</th>
<th>( \alpha )</th>
<th>0.000591</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_{65} )</td>
<td>0.00000738</td>
<td></td>
</tr>
<tr>
<td>( \gamma_{0-65} )</td>
<td>0.118</td>
<td></td>
</tr>
<tr>
<td>( \beta_{65-99} )</td>
<td>0.000619</td>
<td></td>
</tr>
<tr>
<td>( \gamma_{65-99} )</td>
<td>0.0532</td>
<td></td>
</tr>
</tbody>
</table>

### Financial Parameters

<table>
<thead>
<tr>
<th>Risk free rate</th>
<th>( r )</th>
<th>0.0035 (= 0.0425 yearly)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volatility underlying asset</td>
<td>( \sigma )</td>
<td>0.0432 (= 0.150 yearly)</td>
</tr>
<tr>
<td>Tax rate</td>
<td>( \gamma )</td>
<td>0.4</td>
</tr>
<tr>
<td>Average return on invested capital</td>
<td>( \delta )</td>
<td>0.00458 (= 0.055 yearly)</td>
</tr>
<tr>
<td>Cost of capital</td>
<td>( COC )</td>
<td>0.0083 (= 0.10 yearly)</td>
</tr>
<tr>
<td>Transaction costs</td>
<td>( \tau )</td>
<td>0.2%</td>
</tr>
</tbody>
</table>

### Parameters RSLN model (monthly)

| Average log-return in regime 1 | \( \mu_1 \) | 0.0135 |
| Average log-return in regime 2 | \( \mu_2 \) | -0.0109 |
| Volatility in regime 1 | \( \sigma_1 \) | 0.0344 |
| Volatility in regime 2 | \( \sigma_2 \) | 0.0645 |
| Probability to move from regime 1 to 1 | \( p_{11} \) | 0.0483 |
| Probability to move from regime 2 to 1 | \( p_{21} \) | 0.1985 |