VALUATION OF LIFE INSURANCE USING DIFFUSION INTEREST RATE MODEL

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Abstract:
This paper presents the methodology how to use diffusion interest rate models, e.g. Hull – White or Vasicek model for discounting cash flows. The methodology is based on Markovian character of diffusion processes and is generally usable in actuarial computations. Described method is applied on endowment contract with profit sharing and interest rate guarantee. The paper is concerned mostly with liability valuation, guarantee valuation and also with profit, release from risk and income statement.

Keywords:
Diffusion process, Markovian character, Interest rate model, Valuation of life insurance liability, Interest rate guarantee, Income statement.
1 Introduction

The stochastic models become an important tool in practical actuarial modelling, focusing especially on guarantees pricing and liability valuation. The aim of the paper is to propose a methodology based on the Markovian character of diffusion processes, which can be used in practical actuarial computations for discounting cash flows. We will apply the method on endowment contract with profit sharing, concerning on liability valuation. The method will be illustrated by numerical examples which were calculated by a stochastic model created in MS Excel.

The current development on the field of international accounting and reporting standards with its open questions was also the motivation for the paper. We are concerned especially with activating acquisition expense, profit at issue, accounting about adjustments for risk and uncertainty and its changes and release from risk on insurance liabilities.

1.1 Risk approach

There are two main components of risk which are important for our approach. These are volatility and uncertainty. Volatility can be diversified, while risks are independent. Choice of a model and setting parameters constitute the uncertainty risk.

The main principles used for modelling risks are recommended by International Association of Actuaries (IAA) in [10]. The insurer is exposed to technical and investment risk. We will assume that the technical risk can be diversified. It is proper to use a stochastic model to allow for the volatility of investment risk.

International Association of Actuaries also declares the principle of parsimony, which consist of using rather simple models. These models can by easily interpreted and used for practical calculations.

1.2 Fair value principles

Our model will be based on fair value concept, which is also the part of IASB a FASB framework. The key elements are the time value of money and allowing for market value margins. We will use three main assumptions

- Risk free rate of interest is used for discounting cash flows.
The expected cash flows should be adjusted by market value margins in order to include the market value of risk connected with an adverse deviation of assumptions (referred also as adjustment for risk and uncertainty).

The adjustment should include imperfections of the market. This approach can be found in Towers Perrin’s study referred as [8].

2 Instantaneous rate of interest

Suppose the space \( \Omega = C^1_{[0,T]} \times C^2_{[0,T]} \) created by instantaneous trajectories of two Wiener processes with the real probability measure \( \sigma^1 \times \sigma^2 \).

For modelling the instantaneous rate of interest we will use a diffusion process \( \{r_t\} \), which can be expressed by the equation \( dr_t = \theta(t, r_t)dt + \delta(t, r_t)dW_t, \ t \in [0, T], \) where \( \theta \) and \( \delta \) are generally functions and \( W_t \) is Wiener process regarding to a filtration \( F = \{F_t, t \in [0, T]\} \).

The Markovian character of diffusion processes is very important for our method. It says that on condition that \( r_s = y \) there are \( \{r_t, t \geq s\} \) and \( \{r_u, u \leq s\} \) independent, as is shown in [6].

2.1 Risk free rate of interest

Denote \( I_k \) a random variable representing the risk free rate of interest in \( k \) th policy year. We assume that

\[
1 + I_k = e^{\int_{r_k}^{r_t} \text{d}t} = e^{(r_0 - r_{k-1})}
\]

and that \( 1 + I_k \sim \log N(\mu_k, \sigma_k^2) \), where parameters \( \mu_k \) and \( \sigma_k^2 \) depend on a model used for \( \{r_t\} \) and will be determined later.

We will use the Hull-White model \( dr_t = (\rho(t) + \delta \cdot \lambda_d - \kappa \cdot r_t)dt + \delta \cdot dW^1_t, \ t \in [0, T], \)

for modelling the instantaneous risk free rate of interest \( r_t \).

Let’s assume \( d\tilde{W}^{-1}_t = -\lambda_d dt = dW^1_t \), where \( \{\tilde{W}^{-1}_t\} \) is \( \tilde{\sigma}^1 \)-Wiener process (\( \tilde{\sigma}^1 \) is risk neutral probability measure). Thus Radon-Nikodym derivative is

\[
\frac{d\tilde{\sigma}^1}{\sigma^1} = e^{-\int_{\tilde{W}^{-1}_t}^{\tilde{W}^{-1}_t} \frac{1}{2} \text{d}W^1_t}.
\]
2.2 Discounting

According to principles of the fair value, we will use a deflator (stochastic discounting factor)

\[ D_k = e^{\int_{t}^{\infty} \sigma^2 du - \frac{1}{2} \sigma^2 k} \]

for discounting cash flows from the end of the period \( k \) to time 0. This stochastic discounting factor is the discounted value of the Radon-Nikodym derivative.

Using this deflator is the same as using \( v^{(k)} = e^{\int_{t}^{\infty} \sigma^2 du} \) with a probability measure \( \tilde{\sigma}_1 \times \sigma^2 \), while \( \{ r_i \} \) is the solution of the equation

\[ (2) \quad dr_i = (\rho(t) - \kappa \cdot r_i) dt + \delta_1 \ d\tilde{W}_i. \]

2.3 Parameters of the model

We will determine parameters of the model in compliance with the value of zero-coupon bond with maturity \( T \) expressed as (see [1])

\[ (3) \quad P(r, t, T) = E_{\tilde{\sigma}_1 \times \sigma^2} \left\{ e^{-\int_{r_i}^{T} dr} \right\}. \]

Let \( f^M(0, t) \) denote a market yield curve. Assuming \( f^M(0, t) = -\frac{\partial}{\partial t} \ln P(r, 0, t) \), we can express the parameter \( \rho(t) \) given by the equation \( dr_i = (\rho(t) - \kappa \cdot r_i) dt + \delta_1 \ d\tilde{W}_i \) as

\[ \rho(t) = f^M(0, t)' + \kappa \cdot f^M(0, t) + \frac{\delta_1^2}{2 \cdot \kappa^2} \left( 1 - e^{-2\kappa t} \right) \]

and \( r_0 = f^M(0, 0) \). This can be also found in [3].

It is useful to consider a simple function \( f^M(0, t) \) for further analytical calculation, for example

\[ f^M(0, t) = \alpha + \beta \left( 1 - e^{-\frac{t}{\gamma}} \right). \]

Thus \( \rho(t) = \left[ \kappa' \cdot (\alpha + \beta) + \frac{\delta_1^2}{2 \cdot \kappa} \right] + \left( \frac{\beta}{\gamma} - \kappa \cdot \beta \right) e^{-\frac{t}{\gamma}} - \frac{\delta_1^2}{2 \cdot \kappa} \cdot e^{-2\kappa t}. \)
2.4 Characteristics of $r_k$ a $R_k$

Solving the differential equation (2), the instantaneous rate of interest $r_k$ can be expressed as

\[ r_k = e^{-\kappa k} \cdot r_0 + \int_0^k e^{-\kappa (k-u)} \rho(u) du + \delta^1 \cdot \int_0^k e^{-\kappa (k-u)} d\widetilde{W}_u^1, \quad r_0 = s. \]

The variable $R_k$ equals to

\[ R_k = \int_0^k r_u du = \frac{1}{\kappa} \cdot (1 - e^{-\kappa k}) \cdot r_0 + \frac{1}{\kappa} \cdot \left( \int_0^k (1 - e^{-\kappa (k-u)}) \rho(u) du + \delta^1 \cdot \int_0^k (1 - e^{-\kappa (k-u)}) d\widetilde{W}_u^1 \right). \]

We need to determine

\[ E_s r_k = e^{-\kappa k} \cdot r_0 + \int_0^k e^{-\kappa (k-u)} \rho(u) du, \quad Var r_k = \delta^2 \cdot \int_0^k e^{-2\kappa u} du = \delta^2 \cdot \frac{1 - e^{-2\kappa k}}{2\kappa} \quad \text{and} \]

\[ Cov(r_k, R_k) = \frac{\delta^1}{\kappa} \cdot \int_0^k (1 - e^{-k-u}) \cdot e^{-\kappa u} du = \frac{\delta^2}{\kappa} \cdot \frac{1 - e^{-2\kappa k}}{2\kappa} \cdot \frac{(-1 + 2e^{\kappa k})}{2\kappa}. \]

where an index $s$ denotes the condition $r_0 = s$.

Finally we denote $\Delta R_k = R_k - R_{k-1}$. Assuming that $r_{k-1} = y$, we determine

\[ E_y(R_k - R_{k-1}) = E \int_{k-1}^k r_u du = \frac{1}{\kappa} \cdot (1 - e^{-k}) \cdot y + \frac{1}{\kappa} \cdot \left( \int_{k-1}^k (1 - e^{-k(k-z)}) \cdot \rho(z) dz \right). \]

\[ Var_y(R_k - R_{k-1}) = Var \int_{k-1}^k r_u du = \frac{\delta^2}{\kappa^2} \cdot \left( \int_{k-1}^k (1 - e^{-k(k-z)})^2 dz \right) = \frac{\delta^2}{\kappa^2} \cdot \left( \frac{3 - 4e^{-k} + e^{-2k} - 2k}{2k} \right). \]

\[ Cov_y(r_k, R_k - R_{k-1}) = E \left( \delta^1 \cdot \int_{k-1}^k e^{-k(k-z)} d\widetilde{W}_z^1 \right) \cdot \left( \delta^1 \cdot \int_{k-1}^k (1 - e^{-k(k-z)}) d\widetilde{W}_z^1 \right) = \]

\[ = \frac{\delta^2}{\kappa} \cdot \left( \frac{1}{2\kappa} - \frac{e^{-2\kappa} \cdot (-1 + 2e^{\kappa})}{2\kappa} \right). \]

2.5 Actually achieved rate of return

Denote $J_k$ a random variable representing the actually achieved rate of return in $k^{th}$ policy year. We assume that

\[ q \cdot \int_{k-1}^k r_u du + (1-q) \cdot \left[ + \delta^2 \left[ W_{k-1}^2 - W_{k-2}^2 \right] \right]. \]

\[ (5) \quad 1 + J_k = e^{r_{k-1}}. \]
$1 + J_k$ consists of the risk free yield, a risk premium $\nu$ (the value of risk) and a noise component, which can be modelled by increases of second Wiener process $\{W_t^2, t \geq 0\}$, independent of $\{W_t^1, t \geq 0\}$. We can use this component for modelling investment into shares or also imperfections of the market. We use a parameter $q$ for modelling of a mixed portfolio consisting of $q \cdot 100\%$ risk free instruments and $(1-q) \cdot 100\%$ risky instruments.

We assume that $q \neq 0$.

When we need to consider an adjustment by market value margin according to fair value principle, we replace $\nu$ with $\nu' = \nu - \tau$. Thus

$$1 + J'_k = e^{\int_{k-1}^k r_t \, dt + (1-q) \left[ \nu' + \delta_2 \left( W_{t+1}^2 - W_t^2 \right) \right]}.$$

The distribution of $(1 + J'_k)$ is $\log N(q \cdot \mu_k + (1-q) \cdot \nu', q^2 \cdot \sigma_k^2 + (1-q)^2 \cdot \delta_2^2)$, where $\mu_k = E\Delta R_k$, $\sigma_k^2 = Var \Delta R_k$.

### 3 Value of the liability from the policyholder’s point of view

We will apply the above-mentioned model to endowment insurance with profit sharing. Let $K$ denotes sum assured, $V_x$ reserve of premium for 1 unit at the end of $k$th policy year, $n$ duration of insurance, $i'$ technical interest rate, $s_{k,n}$ probability of surrender during $k$th policy year, $p_x = s_{k-1} \cdot (1-q_{s+k-1} - s_{k-1,n})$ probability, that the policy is in force at the and of $k$th policy year.

### 3.1 Valuation of the liability using a diffusion process

For a bonus reserve at the end of period $k$ we assume the formula

$$Q_k = 0.8 \cdot (J'_k - i') + \sum_{k=1}^n V_x + \left( 1 + i' + 0.8 \cdot (J'_k - i') \right) \cdot Q_{k-1},$$

for $k = 2, \ldots, n$ and $Q_1 = 0$.

The present value of expected benefits is
\[ C_{\text{Policysizer}} = -K \cdot \sum_{k=0}^{n-1} p_k \cdot q_{x+k} \cdot \left( E_{y}^{(k+1)} + E_{y}^{(k+1)} \cdot Q_k \right) - \]

\[ -K \cdot \sum_{k=0}^{n-1} p_k \cdot \left( E_{y}^{(k+1)} + 0.9 \cdot E_{y}^{(k+1)} Q_k \right) - K_n p_x \cdot \left( E_{y}^{(n)} + E_{y}^{(n)} \cdot Q_n \right). \]

It’s clear that \( E_{y}^{(k+1)} = e^{-E_{x}^{(k+1)} \cdot \frac{1}{2} \cdot \text{Var}_{k+1}} \) and to determine \( E_{y}^{(k+1)} \cdot Q_k \) we apply a recurrent formula

\[ E_{y}^{(k)} Q_k = 0.8 E_{x}^{(k-1)} \cdot E_{y}^{(k)} (J_k' - i')_+ \cdot (1 + I_k)^{-1} \cdot E_{v}^{(k-1)} + (1 + i') \cdot E_{y}^{(k)} Q_{k-1} + 0.8 \cdot E_{y}^{(k)} (J_k' - i')_+ \cdot (1 + I_k)^{-1} \cdot E_{v}^{(k-1)} Q_{k-1}, \]

where indices \( s \) and \( z \) denote conditions \( r_s = s, r_z = z \).

Utilizing the Markovian character of the diffusion process \( \{r_t\} \) and assuming \( r_{k-1} = y \) we can write

\[ E_{y}^{(k)} (J_k' - i')_+ (1 + I_k)^{-1} v^{(k-1)} = \int_{-\infty}^{\infty} E_{y}^{(k)} (J_k' - i')_+ (1 + I_k)^{-1} \cdot E_{y}^{(k-1)} \cdot \frac{1}{\sqrt{2\pi \cdot \text{Var}_{r_{k-1}}}} dy, \]

\[ E_{y}^{(k)} (J_k' - i')_+ (1 + I_k)^{-1} Q_{k-1} = \]

\[ \int_{-\infty}^{\infty} E_{y}^{(k)} (J_k' - i')_+ (1 + I_k)^{-1} \cdot E_{y}^{(k-1)} \cdot Q_{k-1} \cdot \frac{1}{\sqrt{2\pi \cdot \text{Var}_{r_{k-1}}}} dy, \]

\[ E_{y}^{(k)} Q_{k-1} = \int_{-\infty}^{\infty} E_{y}^{(k)} \cdot Q_{k-1} \cdot \frac{1}{\sqrt{2\pi \cdot \text{Var}_{r_{k-1}}}} dy. \]

For a calculation of integrals (8), (9) we need to determine

\[ E_{y}^{(k)} (J_k' - i')_+ (1 + I_k)^{-1} = E_{y}^{(k)} (1 + I_k)^{-1} \cdot \left( (1 + J_k') - (1 + i') \right)_+ = \]

\[ = E_{y}^{(k)} \int_{q+U + \ln(1+i')} e^{-x} \left( e^{q+U} - (1 + i') \right) \cdot \frac{1}{\sqrt{2\pi \cdot \sigma_k^2}} \cdot \left( \frac{x - \mu_k}{\sigma_k} \right)^2 \, dx, \]

where \( U = (1 - q) \cdot \left( \nu' + \delta_2 \cdot (W_k^2 - W_{k-1}^2) \right) \) has distribution \( N \left( (1 - q) \cdot \nu', (1 - q)^2 \cdot \delta_2^2 \right) \). It is

\[ \mu_k = E_{y}^{(k)} \cdot \Delta R_k + \frac{\text{Cov}(r_k, \Delta R_k)}{\text{Var} r_k} \cdot (z - E_{y} \cdot r_k) \quad \text{and} \quad \sigma_k^2 = \left( \frac{1}{\text{Var} r_k} \cdot \text{Var} \Delta R_k \right) \cdot \text{Var} \Delta R_k, \]

because the contingent distribution of \( R_k - R_{k-1} \) on conditions that \( r_{k-1} = y, r_k = z \) is
Let’s assume that $U = u$, then
\[
E_x^y(J'_k - i')_+ \cdot (1 + I_k)^{-1} = e^{\frac{(g_1 - 2\mu_x + \sigma^2(g_1 - 1))}{2}} \cdot \Phi\left(\frac{\ln(1 + i') - u - (\mu_k + \sigma^2(1 - q))}{\sigma_k}\right) - (1 + i') \cdot e^{-\mu_k \cdot \frac{\sigma_k^2}{2}} \cdot \Phi\left(\frac{\ln(1 + i') - u - (\mu_k - \sigma_k^2)}{\sigma_k}\right) = g_k(u).
\]

Finally we will integrate
\[
\int_{-\infty}^{\infty} g_k(u) \cdot e^{\frac{-(u-v)^2}{2\delta^2}} du.
\]

Similarly $E_x^y v^{(k-1)} = e^{-E_x^y R_{k-1} + \frac{1}{2} \text{Var}^y R_{k-1}}$, where
\[
E_x^y R_{k-1} = E_x R_{k-1} + \frac{\text{Cov}(r_{k-1}, R_{k-1})}{\text{Var} R_{k-1}} \cdot (y - E_x R_{k-1})
\]
and $\text{Var}^y R_{k-1} = \left(1 - \frac{\text{Cov}(r_{k-1}, R_{k-1})^2}{\text{Var} R_{k-1} \cdot \text{Var} R_{k-1}}\right) \cdot \text{Var} R_{k-1}$.

To calculate integrals (9) and (10) we need the value of $E_x^y v^{(k-1)} \cdot Q_{k-1}$, which is given by a table of results from the previous step of the recursion formula (7). We will need some numerical method to calculate these integrals, which will be shown in chapter 3.2.

Finally to calculate (6) we have
\[
E_x^y v^{(k+1)} \cdot Q_k = \int_{-\infty}^{\infty} e^{-E_x^y \Delta R_k + \frac{1}{2} \text{Var}^y \Delta R_k} E_x^y v^{(k)} \cdot Q_k \cdot e^{\frac{-(z-E_x \Delta R_k)^2}{2\text{Var} \Delta R_k}} dz.
\]

The method described above is based on Markovian character of diffusion processes and therefore it can be used also with another diffusion interest rate model, for example Vašíček or Ho-Lee model.

Especially Vašíček model is suitable for easier and simpler computations, because its parameters are constants. Its mean reversion is typical for interest rates behaviour. It can be described by the equation
(11) \[ d(r_t - \rho) = -\kappa(r_t - \rho) \cdot dt + \delta \cdot dW_t, \quad t \in [0,T]. \]

The method doesn’t use any approximations as well as it can be easily used for practical computation, for example using MS Excel. These can be considered as an important advantage of the method.

### 3.1.1 Market value margin from the policyholder’s point of view

To gain a conservative value of the liability from the policyholder’s point of view, we have to adjust \((1 + J_k)\) by replacing \(\nu\) with \(\nu' = \nu - \tau\). The value of the parameter \(\tau\) can be set from an equation \(C^{\text{Policyholder}} = K \cdot \sum_{k=0}^{n-1} p_x \cdot E_V^{(k)} \cdot \Pi_k\), where \(\Pi_k\) is market premium at the primary insurance market and \(C^{\text{Policyholder}}\) is a function of the variable \(\tau\). This adjustment represents the market value of risk, because policyholders with just this assessment of risk will buy the contract at the primary insurance marker for premium \(\Pi_k\).

### 3.2 Numerical valuation

We can use Gauss’s formula for a numerical integration. The MS Excel model used for numerical examples applies 5 points. It is possible to find more details about this method in [5]. Firstly we have to transform integrals \(\int_{-\infty}^{\infty} g(y) \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(y-\mu)^2}{2\sigma^2}} \, dy\) to integrals \(\int_{0}^{1} g(\Phi^{-1}(u) \cdot \sigma + \mu) \, du\) to solve the problem of integration interval \((-\infty, \infty)\). It is possible to find values of \(\Phi^{-1}(p) = -\Phi^{-1}(1-p)\) for particular parameters \(p \in (0,1)\) in literature. For other values of parameters \(p\) we can use Bessel interpolation.

### 4 Profit and release from risk

In next chapters we will concentrate on insurer’s point of view. Especially on value of interest rate guarantee, expected surplus and main items of income statement on insurance company.
4.1 Valuation and pricing assumptions

For deducing an expected surplus of life insurance we will follow up the method which is used particularly in Europe. The method is based on a difference between pricing (conservative) and valuation (best estimates) assumptions.

The provision for risk and uncertainty results from using pricing assumptions for determination of premium. This reserve is releasing during the insurance period, while the realization of assumptions is positive. We will designate pricing assumptions with comma as common.

4.2 Income statement

International Accounting Standards Board publicized in 2001 Insurance Contracts Draft [9] with a draft of income statement of insurance company. We will focus on main items for new business and for changes in estimates for previous year’s business.

New business

\[ A \] - Premium income,
\[ B \] - Reinsurance premium expense,
\[ C \] - Claims expense. For clarity we divide item \( C \) into following items \( C_1 \) - Maturity benefits, \( C_2 \) - Surrender claims, \( C_3 \) - Death benefits, \( C_4 \) - Value of an interest rate guarantee.
\[ D \] - Reinsurance recoveries,
\[ \bar{E} \] - Provision for risk and uncertainty\(^1\),
\[ F \] - Acquisition costs,
\[ G \] - Other operating costs.

Changes in estimates for previous year’s business

\[ H \] - Changes in estimates and assumptions,
\[ I \] - Release of risk on insurance liabilities,
\[ J \] - Changes to adjustments for risk and uncertainty.

Further items

\[ \bar{K} \] - Interest income\(^2\),
\[ R \] - Unwinding of discount rate,
\[ S \] - Changes in discount rate.

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\(^1\) We use a denotation with stripe to mark this item off from the denotation of mean value.
\(^2\) We again use a denotation with stripe to mark this item off from the denotation of sum assured.
Let’s assume issue date 1st January and a sequence \( V_0, V_1, \ldots, V_n \) specifying a method of reserving of insurance company. With regard to technical interest guarantee, we assume 
\[
V_x \cdot (1 + i') \leq V_{k+1} \quad \text{for} \quad k = 0, 1, \ldots, n - 1.
\]
While \( \alpha \) represents the percentage of sum assured standing for an acquisition cost at issue of a policy, we will account for \( F \) - acquisition costs as \( F = -\alpha \cdot K \).

Furthermore we account \( G \) - other operating costs
\[
G = -\sum_{k=0}^{n-1} K \cdot E^{V(k+1)} \cdot p_x \cdot (\beta + \gamma \cdot \Pi_k) \cdot (1 + I_{k+1}) = \n
= -\sum_{k=0}^{n-1} K \cdot E^{V(k)} \cdot p_x \cdot (\beta + \gamma \cdot \Pi_k).
\]

According to DSOP [9] acquisition costs don’t meet the definition of asset, consequently it is not recommended to defer and capitalize them as DAC (deferred acquisition costs). Regardless we will defer them respecting the matching principle, which says that revenues should be recognized in the same period in which the relevant costs are incurred. This approach can be found also in critical letters of three insurance associations to IASB.

### 4.2.1 Expected surplus

Principle of zero profit at issue can also be found in the mentioned critical letters. The profit should arise during the insurance period while releasing from risk. This can be managed by creating a provision for risk and uncertainty, as recommended in [7].

We assume that this provision arise from adjustments for risk and uncertainty to pricing assumptions, mentioned in paragraph 1.2. A profit is therefore given by releasing of these margins. We will denote \( S_{k+1} \) expected surplus at the end of year \( k + 1 \). The present value of these surpluses is
\[
\sum_{k=0}^{n-1} E^{V(k+1)} S_{k+1} =
\]

\( a \)
\[
+ \sum_{k=0}^{n-1} K E^{V(k+1)} \cdot p_x \cdot (1 + J_{k+1}) \cdot \Pi_k +
\]

\( b \)
\[
+ \sum_{k=0}^{n-1} K E^{V(k+1)} \cdot p_x \cdot (V_x + Q_x) \cdot (1 + J_{k+1}) -
\]

\( c \)
\[
- \sum_{k=0}^{n-1} K E^{V(k+1)} \cdot p_x \cdot (\beta + \gamma \cdot \Pi_k) \cdot (1 + I_{k+1}) -
\]

\( d \)
\[
- \sum_{k=0}^{n-1} K E^{V(k+1)} \cdot p_x \cdot p_{x+k} \cdot (V_x + Q_{k+1}) -
\]
(e) \[-\sum_{k=0}^{n-1} KE^{(k+1)}_k p_x \cdot s_{k+1,d} \cdot \left(\left(p_{x+x}^{\text{storno}} + 0,9 \cdot Q_k\right) - \sum_{k=0}^{n-1} KE^{(k+1)}_k p_x \cdot q_{x+k} \cdot (1 + Q_k)\right)\]

(f) \[-\sum_{k=0}^{n-1} KE^{(k+1)}_k p_x \cdot \Pi_k \]

The first term (a) will be partly charge to account \(A\) - Premium income, amounting to \(\sum_{k=0}^{n-1} KE^{(k+1)}_k p_x \cdot \Pi_k\) and partly to account \(K\) - Interest income, amounting to \(\sum_{k=0}^{n-1} KE^{(k+1)}_k p_x \cdot J_{k+1} \cdot \Pi_k\). The term (c) will be charge to \(G\) - other operating costs, term (e) to \(C_2\) - surrender claims and term (f) to \(C_3\) - death benefit. We use the following formula for modification of lines (b) and (d)

\[
\left(1 + J_{k+1}\right) \cdot \left(V_x + Q_k\right) = +\left(1 + i'\right) \cdot \left(V_x + Q_k\right) -
\left(i' - J_{k+1}\right) \cdot \left(V_x + Q_k\right) +
0,8 \cdot \left(J_{k+1} - i\right) \cdot \left(V_x + Q_k\right) +
0,2 \cdot \left(J_{k+1} - i\right) \cdot \left(V_x + Q_k\right),
\]

where the first term represents guaranteed interest on reserves, the second one represents a supplement needed in case of \(J_{k+1} < i'\), which can be interpreted as value of an interest rate guarantee. The third one is profit sharing belonging to a policyholder and the last term represents profit sharing retained by an insurer.

Sum of terms (b) and (d) equals to

\[
-\sum_{k=0}^{n-1} KE^{(k+1)}_k p_x \cdot \left[p_{x+k} \cdot (V_x + Q_k) - (V_x + Q_k) \cdot (1 + i')\right] -
\sum_{k=0}^{n-1} KE^{(k+1)}_k p_x \cdot \left(V_x + Q_k\right) \cdot (i' - J_{k+1}) +
\sum_{k=0}^{n-1} KE^{(k+1)}_k p_x \cdot 0,8 \cdot (V_x + Q_k) \cdot (J_{k+1} - i'),
\]

\[
+\sum_{k=0}^{n-1} KE^{(k+1)}_k p_x \cdot 0,2 \cdot (V_x + Q_k) \cdot (J_{k+1} - i'),
\]

where the second term of the formula (12) represents \(C_4\) - value of interest guarantee. The third plus fourth term of (12) will be charged to account \(\overline{K}\) - interest income. We will convert the first term of (12) to
\[- \sum_{k=0}^{n-1} K E v^{(k+1)} \cdot p_x \cdot \left( (x_{k+1} V_x + Q_{k+1}) - (V_x + Q_k) \cdot (1 + i') \right) = \]

\[
= - \sum_{k=0}^{n-1} K E v^{(k+1)} \cdot _k p_x \cdot \left( (x_{k+1} V_x + Q_{k+1}) + \sum_{k=0}^{n-1} K E v^{(k)} \cdot _k p_x \cdot (V_x + Q_k) + \right.

\left. + \sum_{k=0}^{n-1} K E v^{(k+1)} \cdot _k p_x \cdot (V_x + Q_k) \cdot (i' - I_{k+1}) = \right]

\[
= - K E v^{(n)} \cdot _n p_x \cdot (n \cdot V_x + Q_n) + \sum_{k=0}^{n-1} K E v^{(k+1)} \cdot _k p_x \cdot (V_x + Q_k) \cdot (i' - I_{k+1}). \]

The term \(- K E v^{(n)} \cdot _n p_x \cdot (n \cdot V_x + Q_n)\) represents \(C_1\) - maturity benefits and the last term \(\sum_{k=0}^{n-1} K E v^{(k+1)} \cdot _k p_x \cdot (V_x + Q_k) \cdot (i' - I_{k+1})\) can be charged to account \(S\) - changes in discount rate. To sum up, we can write

\[(13) \quad \sum_{k=0}^{n-1} E v^{(k+1)} S_{k+1} = A + C_1 + C_2 + C_3 + C_4 + G + \overline{K} + S. \]

### 4.3 Reserve for risk and uncertainty

The provision for risk and uncertainty \(\overline{E}\) is created at issue amounting to \(\overline{E} = \sum_{k=0}^{n-1} E \overline{E}_{k+1} \cdot v^{(k+1)}\). Lets denote \(P_0 = \sum_{k=0}^{n-1} E \left( S_{k+1} - \overline{E}_{k+1} \right) \cdot v^{(k+1)}\), then the expected profit at issue including the provision for risk and uncertainty is

\[P_0 - \alpha \cdot K = \sum_{k=0}^{n-1} E \left( S_{k+1} - \overline{E}_{k+1} \right) \cdot v^{(k+1)} - \alpha \cdot K. \]

If we set \(\overline{E}_{k+1} = S_{k+1} - \frac{1}{n \cdot p_x} \cdot \frac{\alpha \cdot K}{\sum_{k=0}^{n-1} E \cdot k p_x \cdot v^{(k)}} \cdot (1 + I_{k+1})\), the principle of zero profit at issue \(P_0 + F = 0\) is fulfilled, we assume \(\overline{E}_{k+1} \geq 0\).

However at the end of the first year, we have

\[(14) \quad P_1 = \sum_{k=1}^{n-1} E \left( S_{k+1} - \overline{E}_{k+1} \right) \cdot \cdot \cdot v^{(k)},\]

where the index 1 denotes best estimate assumptions based on experience from the first year.

For discounting we use

\[v^{(k)}_1 = \frac{1}{(1 + I_2) \cdot (1 + I_3) \cdots (1 + I_{k+1})}. \]
The realized surplus in the first year is \( S_1^a \) and realized acquisition cost to end of the first year is \( F_1^a \). This means that the asset arise by

\[
P_1 + S_1^a + F_1^a = S_1^a + \left( K_1 \cdot \alpha \cdot \sum_{k=0}^{n-2} E_k p_{x+1} \cdot 2v^{(k)}_1 \right) \cdot \left( 1 + I_1 \right) + F_1^a,
\]

where \( K_1 \) denotes the real sum assured in force at the end of the first year.

We should account \( P_1 - P_0 + S_1^a - (F - F_1^a) \) at the end of the first year, which can be modified as

\[
\sum_{k=2}^{n} E \left( S_k^1 - \bar{E}_k \right) \cdot 2v^{(k-1)}_1 + S_1^a - \sum_{k=2}^{n} E \left( S_k - \bar{E}_k \right) \cdot 2v^{(k-1)}_1 - (S_1 - \bar{E}_1) + I_1 \cdot P_0 - (F - F_1^a).
\]

Particular elements of income statement will be:

Changes in estimates and assumptions

\[
H = S_1^a - S_1 + \sum_{k=2}^{n} E \left( S_k^1 \cdot 2v^{(k-1)}_1 - S_k \cdot 2v^{(k-1)}_1 \right) + \left( F_1^a - F \cdot (1 + I_1) \right),
\]

Release from risk on insurance liabilities \( I = E_1 \),

Changes to adjustment for risk and uncertainty \( J = -\sum_{k=2}^{n} E \left( E_k^1 \cdot 2v^{(k-1)}_1 - \bar{E}_k \cdot 2v^{(k-1)}_1 \right) \) and

Unwinding of discount rate \( R = I_1 \cdot (P_0 + F) \).

So far we assumed that \( \bar{E}_{k+1} \geq 0 \). In case of \( \bar{E} = \sum_{k=0}^{n-1} E \bar{E}_{k+1} \cdot v^{(k+1)} \) < 0, the insufficiency of premium is indicated and the provision for risk and uncertainty should be accounted amounting to \( \bar{E} = 0 \). At the same time we establish a deficiency reserve \( -\sum_{k=0}^{n-1} E \bar{E}_{k+1} \cdot v^{(k+1)} \)

which will be a part of costs included in \( S_1^a \).

There will be \( \sum_{k=0}^{n-1} E \bar{E}_{k+1}^1 \cdot 2v^{(k)}_1 \) representing an expected change in the reserve in \( k^{th} \) year in the formula (14).
5 Value of the liability from the insurer’s point of view

Valuating a liability from the insurer’s point of view, we have to consider the value of a technical interest guarantee. We assume that the technical interest on a bonus reserve $Q_k$ is guaranteed as well as on reserve of premium $kV_x$. The expected present value of a liability can be expressed as

$$
C^{\text{Insurer}} = -K \cdot p_x \cdot E_t^{(n)} (1 + Q_n) +
- K \cdot \sum_{k=0}^{\infty} k \cdot p_x \cdot s_{k+1,a} \cdot E_t^{(k+1)} \cdot (E_t^{(k)} + 0.9 \cdot Q_k) -
- K \cdot \sum_{k=0}^{\infty} k \cdot q_x \cdot E_t^{(k+1)} \cdot (1 + Q_k) -
- K \cdot \sum_{k=0}^{\infty} k \cdot E_t^{(k+1)} \cdot (i' - J'_{k+1}) \cdot (kV_x + Q_k).
$$

5.1 Valuation of liability using diffusion process

The value of $E_t^{(k+1)}$ a $E_t^{(k+1)}Q_k$ will be calculated using the same method as in paragraph 3.1. Additionally we have to calculate

$$
E_t^{(k+1)}(i' - J'_{k+1}) =
\int_{-\infty}^{\infty} E_t^{(k)} \cdot E_t (1 + I_{k+1})^{-1} \cdot (i' - J'_{k+1}) \cdot \frac{1}{\sqrt{2\pi \text{Var} r_k}} e^{\frac{(z - E_t r_k)^2}{2\text{Var} r_k}} dz,
$$

$$
E_t^{(k+1)}(i' - J'_{k+1}) \cdot Q_k =
\int_{-\infty}^{\infty} E_t^{(k)} Q_k \cdot E_t (1 + I_{k+1})^{-1} \cdot (i' - J'_{k+1}) \cdot \frac{1}{\sqrt{2\pi \text{Var} r_k}} e^{\frac{(z - E_t r_k)^2}{2\text{Var} r_k}} dz,
$$

using the Markovian character of diffusion process $\{r_t\}$. To calculate integrals (16) and (17) we need to determine

$$
E_t (1 + I_{k+1})^{-1} \cdot (1 + i') - (1 + J'_{k+1}) =
E \int_{q \cdot x + U < \ln(1 + i')} e^{-x} \cdot (1 + i') - e^{q \cdot x + U} \cdot \frac{1}{\sqrt{2\pi \cdot \sigma_k}} \cdot e^{\frac{(x - \mu_k)^2}{2\sigma_k^2}} dx,
$$

where $U = (1 - q) \cdot (v' + \delta \cdot (W_x^2 - W_{k-1}^2))$ has a distribution $N((1 - q) \cdot v', (1 - q)^2 \cdot \delta^2)$.
Assuming \( U = u \), we can write
\[
E_z (1 + I_{k+1})^{-1} (i' - J_{k+1}')_+ = \\
= (1 + i') \cdot e^{-\mu_k + \sigma_k^2/2} \cdot \Phi \left( \frac{\ln(1 + i') - u - (\mu_k - \sigma_k^2)}{\sigma_k} \right) - \\
- e^{-\mu_k (q-1)(2\mu_k + \sigma_k^2(q-1))} \cdot \Phi \left( \frac{\ln(1 + i') - u - (\mu_k + \sigma_k^2(1 - q))}{\sigma_k} \right) = m_k(u),
\]
where \( \mu_k = E_z(R_{k+1} - R_k) \) and \( \sigma_k^2 = Var(R_{k+1} - R_k) \). Finally we integrate
\[
\int_{-\infty}^{\infty} m_k(u) \cdot \frac{1}{\sqrt{2\pi \cdot (1 - q) \cdot \delta_2}} \cdot e^{\frac{-(u-(1-q)\nu')^2}{2(1-q)\delta_2^2}} \, du \text{ using the Gauss’s formula.}
\]

### 5.1.1 Market value margin from the insurer’s point of view

According to the principle from the paragraph 1.2 we will adjust the variable \( J_k \) in the expression \((V_x + Q_k)(i' - J_{k+1}')_+\), replacing \( \nu \) by \( \nu' = \nu - \tau \) to get more conservative result. It is more complicated to adjust \( J_k \) in the expression \((J_{k+1} - i')_+ \cdot (V_x + Q_k)\) because it is not clear which adjustment (\( \nu' = \nu - \tau \) or \( \nu' = \nu - \tau \)) will lead to the more conservative result, so we keep \((J_{k+1} - i')_+ \cdot (V_x + Q_k)\) without any adjustment in our model. In practice this can be solved according to actuary’s best estimate of future development of variable \( J_k \). If \( J_k \) is above \( i' \) in a long term, the higher value of \( J_k \) is positive for the insurer. But in case of decreasing \( J_k \) below \( i' \), the insurer guarantees the technical interest on the higher bonus reserve and subsidizes this guarantee from other sources.

The value of the parameter \( \tau \) can be deducted form the equation
\[
C^{\text{Insurer}} = K \cdot \sum_{k=0}^{n-1} P_x \cdot E\nu^{(k)} \cdot \Pi_k, \text{ where } \Pi_k \text{ is premium at the primary market.}
\]

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5.2 Valuation of life insurance

Exploiting formulas derived in the paragraph 4.2.1 and using all interrelated cash flows, as premium, interest on reserves, changes in reserves, expense, benefits including surrenders and bonuses we can determine value of insurance as

\[
H_{\text{Insurer}} = - K \cdot p_x \cdot Ev^{(a)} \cdot (1 + Q) + \\
- K \cdot \sum_{k=0}^{n-1} p_x \cdot Ev^{(k+1)} \cdot (i' - J_{k+1}) \cdot (iV_x + Q) - \\
- K \cdot \sum_{k=0}^{n-1} p_x \cdot s_{k+1,n} \cdot Ev^{(k+1)} \cdot \left( kV_x + Q \right) - \\
- K \cdot \sum_{k=0}^{n-1} p_x \cdot q_{x+k} \cdot Ev^{(k+1)} \cdot (1 + Q) - \\
- K \cdot \sum_{k=0}^{n-1} p_x \cdot Ev^{(k)} \cdot (\beta + \gamma \cdot \Pi_k) - \alpha \cdot K + \\
+ K \cdot \sum_{k=0}^{n-1} p_x \cdot Ev^{(k+1)} \cdot (1 + J_{k+1} \cdot \Pi_k) + \\
+ K \cdot \sum_{k=0}^{n-1} p_x \cdot Ev^{(k+1)} \cdot (J_{k+1} - i'') \cdot (iV_x + Q) + \\
+ K \cdot \sum_{k=0}^{n-1} p_x \cdot Ev^{(k+1)} \cdot (i' - I_{k+1}) \cdot (iV_x + Q).
\]

We are able to calculate the value of \( H_{\text{Insurer}} \) using the method and formulas from paragraphs 3.1 and 5.1.

5.3 Example

It is quite easy to construct a computing model in MS Excel, which is a significant advantage of the method described above. In the following tables, we can find results for endowment with profit sharing insurance, with sum assured 100 000 and duration 10 years. Parameters of Hull-White model in this example are \( \kappa = 0.95, \delta_1 = 0.015, r_0 = 0.028 \). Furthermore we assume \( \nu = \ln(1 + 0.08) \) and \( \delta_2 = 0.12 \).

Firstly we will show how the value of the liability \( C_{\text{Insurer}} \) is affected by the structure of underlying portfolio. We will change the value of parameter \( q \), which represents the portion of risk free financial instruments in the portfolio. The technical interest rate is \( i' = 4\% \).
We can see that value of the liability decrease with higher portion of risk free financial instruments in the portfolio, especially because of decreasing value of profit sharing belonging to policyholders. Concurrently the higher portion of risky instruments with its volatility stands for higher value of the interest rate guarantee, as can be seen from the table below.

<table>
<thead>
<tr>
<th>k</th>
<th>q = 10%</th>
<th>q = 20%</th>
<th>q = 50%</th>
<th>q = 80%</th>
<th>q = 90%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>758</td>
<td>758</td>
<td>758</td>
<td>758</td>
<td>758</td>
</tr>
<tr>
<td>2</td>
<td>1396</td>
<td>1381</td>
<td>1336</td>
<td>1317</td>
<td>1331</td>
</tr>
<tr>
<td>3</td>
<td>1886</td>
<td>1852</td>
<td>1749</td>
<td>1685</td>
<td>1695</td>
</tr>
<tr>
<td>4</td>
<td>2244</td>
<td>2188</td>
<td>2023</td>
<td>1904</td>
<td>1898</td>
</tr>
<tr>
<td>5</td>
<td>2495</td>
<td>2417</td>
<td>2190</td>
<td>2011</td>
<td>1987</td>
</tr>
<tr>
<td>6</td>
<td>2665</td>
<td>2565</td>
<td>2277</td>
<td>2040</td>
<td>1998</td>
</tr>
<tr>
<td>7</td>
<td>2770</td>
<td>2648</td>
<td>2302</td>
<td>2013</td>
<td>1954</td>
</tr>
<tr>
<td>8</td>
<td>2820</td>
<td>2678</td>
<td>2278</td>
<td>1942</td>
<td>1868</td>
</tr>
<tr>
<td>9</td>
<td>2827</td>
<td>2665</td>
<td>2215</td>
<td>1838</td>
<td>1751</td>
</tr>
<tr>
<td>10</td>
<td>47709</td>
<td>46033</td>
<td>41498</td>
<td>37652</td>
<td>36589</td>
</tr>
</tbody>
</table>

Table No. 1: Value of the liability from the insurer’s point of view in dependence on portion q of risk free financial instruments in portfolio.

<table>
<thead>
<tr>
<th>k</th>
<th>q = 10%</th>
<th>q = 20%</th>
<th>q = 50%</th>
<th>q = 80%</th>
<th>q = 90%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>204</td>
<td>189</td>
<td>144</td>
<td>124</td>
<td>138</td>
</tr>
<tr>
<td>3</td>
<td>380</td>
<td>349</td>
<td>256</td>
<td>200</td>
<td>211</td>
</tr>
<tr>
<td>4</td>
<td>533</td>
<td>486</td>
<td>345</td>
<td>246</td>
<td>247</td>
</tr>
<tr>
<td>5</td>
<td>666</td>
<td>602</td>
<td>416</td>
<td>274</td>
<td>260</td>
</tr>
<tr>
<td>6</td>
<td>783</td>
<td>704</td>
<td>475</td>
<td>291</td>
<td>263</td>
</tr>
<tr>
<td>7</td>
<td>887</td>
<td>793</td>
<td>523</td>
<td>302</td>
<td>261</td>
</tr>
<tr>
<td>8</td>
<td>982</td>
<td>873</td>
<td>565</td>
<td>309</td>
<td>257</td>
</tr>
<tr>
<td>9</td>
<td>1070</td>
<td>946</td>
<td>601</td>
<td>315</td>
<td>253</td>
</tr>
<tr>
<td>10</td>
<td>1153</td>
<td>1015</td>
<td>634</td>
<td>320</td>
<td>250</td>
</tr>
</tbody>
</table>

Table No. 2: Value of the interest rate guarantee in dependence on portion q of risk free financial instruments in portfolio.

In the next example we will assume \( q = 85\% \), because the highest value of parameter \( q \) will be more realistic. Let’s take a look, how can be the liability from the insurer’s point of view affected by the volatility of risky assets.
Table No. 3: Influence of volatility of risky asset.

<table>
<thead>
<tr>
<th>Standard deviation of risky asset</th>
<th>Value of interest rate guarantee</th>
<th>Maturity benefits</th>
<th>Surrender claims</th>
<th>Death benefits</th>
<th>Value of the liability C_{Insurer}</th>
</tr>
</thead>
<tbody>
<tr>
<td>4%</td>
<td>1 482</td>
<td>34 867</td>
<td>13 472</td>
<td>1 230</td>
<td>51 051</td>
</tr>
<tr>
<td>6%</td>
<td>1 630</td>
<td>34 986</td>
<td>13 487</td>
<td>1 231</td>
<td>51 334</td>
</tr>
<tr>
<td>8%</td>
<td>1 809</td>
<td>35 124</td>
<td>13 505</td>
<td>1 232</td>
<td>51 671</td>
</tr>
<tr>
<td>10%</td>
<td>2 007</td>
<td>35 274</td>
<td>13 524</td>
<td>1 234</td>
<td>52 039</td>
</tr>
<tr>
<td>12%</td>
<td>2 216</td>
<td>35 433</td>
<td>13 544</td>
<td>1 235</td>
<td>52 428</td>
</tr>
<tr>
<td>14%</td>
<td>2 433</td>
<td>35 597</td>
<td>13 564</td>
<td>1 237</td>
<td>52 831</td>
</tr>
<tr>
<td>16%</td>
<td>2 656</td>
<td>35 764</td>
<td>13 585</td>
<td>1 238</td>
<td>53 243</td>
</tr>
</tbody>
</table>

Now we demonstrate how the value of the liability from insurer’s point of view is affected by the guaranteed technical interest rate $i'$. Firstly the value of $C_{Insurer}$ decreases with increasing $i'$. The reason is decreasing value of profit sharing. Afterwards the value of $C_{Insurer}$ grows with increasing value of the guarantee.

Graph No.1: Value of liability (axis y) in dependence on technical interest rate (axis x).

Table No. 4: Table of values from graph No 1.

<table>
<thead>
<tr>
<th>$i'$</th>
<th>$C_{Insurer}$</th>
<th>$i'$</th>
<th>$C_{Insurer}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2,0%</td>
<td>53 739</td>
<td>5,0%</td>
<td>52 711</td>
</tr>
<tr>
<td>2,5%</td>
<td>53 181</td>
<td>5,5%</td>
<td>53 052</td>
</tr>
<tr>
<td>3,0%</td>
<td>52 768</td>
<td>6,0%</td>
<td>53 490</td>
</tr>
<tr>
<td>3,5%</td>
<td>52 515</td>
<td>6,5%</td>
<td>54 002</td>
</tr>
<tr>
<td>4,0%</td>
<td>52 428</td>
<td>7,0%</td>
<td>54 569</td>
</tr>
<tr>
<td>4,5%</td>
<td>52 497</td>
<td>7,5%</td>
<td>55 173</td>
</tr>
</tbody>
</table>
6 Conclusion

The paper describes the method, how to use a diffusion model of instantaneous interest rate for discounting cash flows, using the Markovian character of diffusion processes without any approximation. The described model is used for calculation of life insurance liability arising from endowment with profit sharing. The paper is also concerned with a numerical computation which could be practically exercisable. The method is universal and can be easily used for practical computation even in MS Excel.

7 Bibliographies