The hidden cost of delay in a credit loan portfolio

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Overview

- Accumulated/discounted aggregate losses of a credit loan portfolio
- Shot noise process
- Delay between default occurrence and partial (or full) payment
- Hidden cost of delay in a portfolio and its illustration
Accumulated aggregate losses

• Accumulated value of aggregate losses up to time $t$ is

$$L(t) = \sum_{i=1}^{N(t)} X_i e^{\delta(t-s_i)}$$

where $X_i, i = 1, 2, \cdots$, is the amount of money that banks suffers when $i$-th borrowers defaults, which are assumed to be independent and identically distributed with the distribution function $H(x) \ (0 < x < U)$, $U$ is the maximum amount of loan to which banks allow borrowers to have their loans. $\delta$ is the instantaneous rate of interest, $s_i$’s are time points at which losses occur ($s_i < t < \infty$) and $N(t)$ is the number of losses up to time $t$, which follows the Poisson process with arrival rate $\rho$. 
Accumulated aggregate losses with delay

- In practice, the case that the loss amounts would be paid immediately after default occurrences is highly rare. Considering the delay between default occurrence and final settlement, denoted by $\tau$ ($0 \leq \tau < \infty$), which is independent of $s$, accumulated value of aggregate losses paid with delay is given by

$$L_\tau(t) = \sum_{i=1}^{N(t)} X_i \delta(t-s_i-\tau_i).$$

- And its expectation is given by $E \{L_\tau(t)\} = E_\tau [E \{L(t-\tau)\}]$. 
Accumulated aggregate losses with delay and recovery

- Also in reality, the eventual loss amounts paid at settlement can be different from the loss amounts at default. As the worst case, the final payment could be 0 but in most case, the bank can recover the part (or whole) of losses after the liquidation of borrowers’ assets. So if we consider the recovery rate, $\kappa_i$ which is an independent, identical random variable, accumulated aggregate losses paid with delay and recovery is given by

$$L_{\tau, \kappa}(t) = \sum_{i=1}^{N(t)} \kappa_i X_i e^{\delta(t-s_i-\tau_i)}.$$  

- And its expectation is given by $E \{ L_{\tau, \kappa}(t) \} = E_{\tau} [ E \{ L_{\kappa}(t - \tau) \} ]$. 
Discounted aggregate losses with delay and recovery

- Multiply $e^{-\delta t}$ both sides in the above equation, it becomes the discounted value of aggregate losses up to time $t$, denoted by $L_{\tau,\kappa}^0(t) = e^{-\delta t}L_{\tau,\kappa}(t)$ and its expectation is given by

$$E \left\{ L_{\tau,\kappa}^0(t) \right\}.$$
We will adopt the shot noise process used by Cox & Isham (1980):

$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{\text{all } i \atop s_i \leq t} y_i e^{-\delta (t-s_i)}$$

where:

- $\lambda_0$ initial value of $\lambda$
- $y_i$ jump size of primary event, where $E(y_i) < \infty$
- $s_i$ time at which primary event $i$ occurs, where $s_i < t < \infty$
- $\delta$ exponential decay
- $\rho$ the rate of primary event arrival.
Graph illustrating shot noise process
Duality of the accumulated losses processes and shot noise

- Set $-\delta$ to $\delta$ and substitute $y$ with $X$ then it becomes $\lambda(t) = \lambda(0)e^{\delta t} + \sum X_i e^{\delta(t-s_i)}$ and assume that $\lambda(0)$, that can be considered as the total losses up to present time 0, is 0, then interestingly, we can see that it is equivalent to $L_t$, that is the accumulated value of aggregate losses, i.e.

$$L(t) = \sum_{i=1}^{N_t} X_i e^{\delta(t-s_i)}$$

- The decay rate $\delta$ is now is the instantaneous rate of interest.
The generator of the process \((L(t), t)\)

- The generator of the process \((L(t), t)\) acting on a function \(f(l, t)\) belonging to its domain is given by

\[
A \ f(l, t) = \lim_{dt \downarrow 0} \frac{E[f(L(t + dt), t + dt) | L(t) = l] - f(l, t)}{dt} = \frac{\partial f}{\partial t} + \delta l \frac{\partial f}{\partial l} + \rho \left[ \int_0^U f(l + x, , t) \, dH(x) - f(l, t) \right].
\]
Loss size, delay and recovery distributions

• Let us assume that the loss size distribution follows truncated exponential, i.e. 
  \[ h_U(x) = \left( \frac{1}{1-e^{-\alpha U}} \right) \alpha e^{-\alpha x}, \quad 0 < x < U \quad \text{and} \quad \alpha > 0 \]

• and the delay between default occurrence and final settlement follows exponential distribution, i.e. 
  \[ j(\tau) = \beta e^{-\beta \tau}, \quad \tau > 0 \quad \text{and} \quad \beta > 0 \]

• and the recovery rate follows Beta distribution, i.e.

  \[ u(\kappa) = \frac{\Gamma (\gamma + \xi)}{\Gamma (\gamma) \Gamma (\xi)} \kappa^{\gamma-1} (1 - \kappa)^{\xi-1}, \quad 0 < \kappa < 1, \quad \gamma > 0, \quad \xi > 0. \]
The expectation of discounted aggregate losses

The expectation of discounted value of aggregate losses is given by

\[ E\{L^0(t)\} = \left(\frac{\rho}{1 - e^{-\alpha U}}\right) \left\{ \frac{1 - e^{-\alpha U}(1 + \alpha U)}{\alpha} \right\} \bar{a}_t \]

where \( \bar{a}_t = \frac{1 - e^{-\delta t}}{\delta} \).
The expectation of discounted aggregate losses with delay and full recovery

- The expectation of discounted value of aggregate losses paid fully with delay between default occurrence and final settlement is given by

$$E\left\{ L^0_\tau(t) \right\} = \left( \frac{\rho}{1 - e^{-\alpha U}} \right) \left\{ \frac{1 - e^{-\alpha U} (1 + \alpha U)}{\alpha} \right\} \left( \frac{\beta}{\beta + \delta} \right) \tilde{a}_t \left| \right. $$

$$- \left( \frac{\rho}{1 - e^{-\alpha U}} \right) \left\{ \frac{1 - e^{-\alpha U} (1 + \alpha U)}{\alpha} \right\} \frac{1}{(\beta + \delta)} e^{-\delta t}$$

where $\beta \geq \frac{1}{t}$. 
The expectation of discounted aggregate losses with delay and partial recovery

- The expectation of discounted value of aggregate losses paid partially with delay between default occurrence and final settlement is given by

\[
E \left\{ L^0_{\tau, \kappa}(t) \right\} = \left( \frac{\rho}{1 - e^{-\alpha U}} \right) \left\{ \frac{1 - e^{-\alpha U} (1 + \alpha U)}{\alpha} \right\} \left( \frac{\gamma}{\gamma + \xi} \right) \left( \frac{\beta}{\beta + \delta} \right) \bar{a}_t | - \left( \frac{\rho}{1 - e^{-\alpha U}} \right) \left\{ \frac{1 - e^{-\alpha U} (1 + \alpha U)}{\alpha} \right\} \left( \frac{\gamma}{\gamma + \xi} \right) \frac{1}{(\beta + \delta)} e^{-\delta t}.
\]
The predictor of the hidden cost of delay with full recovery

• Let us define the predictor of the hidden cost of delay with full recovery as

\[
E \{ L^0(t) \} - E \{ L^0(t) \} = \left( \frac{\rho}{1 - e^{-\alpha U}} \right) \left\{ \frac{1 - e^{-\alpha U}(1 + \alpha U)}{\alpha} \right\} \times \left[ \left\{ 1 - \left( \frac{\beta}{\beta + \delta} \right) \right\} \bar{a}_t + \frac{1}{(\beta + \delta)} e^{-\delta t} \right].
\]
The predictor of the hidden cost of delay with partial recovery

- And define the predictor of the hidden cost of delay with partial recovery as

\[
E \{ L^0(t) \} - E \{ L^0_{\tau, \kappa}(t) \} = \left( \frac{\rho}{1 - e^{-\alpha U}} \right) \left\{ \frac{1 - e^{-\alpha U}(1 + \alpha U)}{\alpha} \right\} \\
\times \left[ \left\{ 1 - \left( \frac{\gamma}{\gamma + \xi} \right) \left( \frac{\beta}{\beta + \delta} \right) \right\} \bar{a}_t \right] + \left( \frac{\gamma}{\gamma + \xi} \right) \frac{1}{(\beta + \delta)} e^{-\delta t} .
\]
Now let us illustrate the calculations of hidden cost of delay, assuming that $\alpha = 0.00001$, $\beta = 3$, $\gamma = 10$, $\xi = 10$, $\delta = 0.05$, $\rho = 5$, $U = 1,000,000$ and $t = 1$. 
Example 1

- The calculations of the predictors of the hidden cost of delay are shown in Table 1:

### Table 1

<table>
<thead>
<tr>
<th>Expression</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E \left{ L^0(t) \right} - E \left{ L^0_{\tau}(t) \right} = \left( \frac{\rho}{1 - e^{-\alpha U}} \right) \left{ \frac{1 - e^{-\alpha U}(1 + \alpha U)}{\alpha} \right} \times \left{ 1 - \left( \frac{\beta}{\beta + \delta} \right) \right} \tilde{a}_t \mid + \frac{1}{(\beta + \delta)}e^{-\delta t} \right}</td>
<td>163,860</td>
</tr>
<tr>
<td>$E \left{ L^0(t) \right} - E \left{ L^0_{\tau,\kappa}(t) \right} = \left( \frac{\rho}{1 - e^{-\alpha U}} \right) \left{ \frac{1 - e^{-\alpha U}(1 + \alpha U)}{\alpha} \right} \times \left{ 1 - \left( \frac{\gamma}{\gamma + \xi} \right) \left( \frac{\beta}{\beta + \delta} \right) \right} \tilde{a}_t \mid + \left( \frac{\gamma}{\gamma + \xi} \right) \frac{1}{(\beta + \delta)}e^{-\delta t} \right}</td>
<td>325,670</td>
</tr>
</tbody>
</table>
Example 2

The calculations of the predictor of the hidden cost of delay at each value of $\delta$ and $\beta$ with partial recovery are shown in Table 2:

Table 2

<table>
<thead>
<tr>
<th></th>
<th>$\delta = 0.03$</th>
<th>$\delta = 0.05$</th>
<th>$\delta = 0.07$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta = 10$</td>
<td>271,090</td>
<td>268,610</td>
<td>266,200</td>
</tr>
<tr>
<td>$\beta = 3$</td>
<td>328,650</td>
<td>325,670</td>
<td>322,740</td>
</tr>
<tr>
<td>$\beta = 1$</td>
<td>488,780</td>
<td>481,730</td>
<td>474,880</td>
</tr>
</tbody>
</table>
Example 3

- The calculations of the predictor of the hidden cost of delay at each value of $\gamma$ and $\xi$ with partial recovery are shown in Table 3:

Table 3

<table>
<thead>
<tr>
<th>$\gamma$ = 1.5 and $\xi$ = 28.5</th>
<th>471,300</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$ = 10 and $\xi$ = 10</td>
<td>325,670</td>
</tr>
<tr>
<td>$\gamma$ = 14 and $\xi$ = 6</td>
<td>260,950</td>
</tr>
</tbody>
</table>