Distribution of Returns Generated by Stochastic Exposure
An Application to VaR Calculation in the Futures Markets

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Abstract

Stochastic exposures are frequently encountered in the world of finance. For instance, corporate companies are faced with uncertain cash flows in a tendering situation. Active Investors also change their exposure to the market according to their anticipations. In other words, the directional views are captured by the changing weights over the period. The market timing ability is best rendered by the performance returns, which are the result of the by-product between the stochastic exposure and the market returns. Our goal here is not to forecast the excess returns generated by active timers but rather to highlight the commonality of uncertain exposure and its effect on Value at Risk calculations, both theoretically and empirically, in the Futures markets. Examples of stochastic weights are chosen from popular strategies used by traders. They encompass both discrete and continuous distributions of cash flows. Raw calculations using the absolute value of end of day positions grossly underestimate Value at Risk. The error is largest at the 99% confidence level, where it is the most needed, because of lack of historical information. Considering instead that profits and losses follow a normal distribution provides more accurate calculations. However, treating the cash flow as a stochastic variable has the potential to improve further Value at Risk calculations. In the case of continuous exposures, the improvement is marginal and subject to the perfect knowledge of the distribution of market returns. In the case of simple discrete exposure(s), the distribution of the uncertain cash flow can be worked out ex-ante using analytical results independent of the underlying markets. We show that incorporating such information drastically improves Value at Risk calculations.

Keywords: Stochastic Exposure, Value at Risk, performance returns, Commodity Trading Advisors
1) **Introduction**

When market risk is calculated, it gives the loss in value of a portfolio over a given holding period with a given confidence level. This calculation assumes that the composition of the portfolio does not change during the holding period. However, variable exposures are frequent in the world of finance and real life examples can be found within corporations, banking or asset management. Corporate companies are faced with uncertain cash flows in a tendering situation. Imagine a company that plans to make a bid of a specified amount of units in a foreign currency to acquire another firm domiciled in the foreign country. It may not be desirable for the takeover company to hedge the potential currency exposure. Indeed if the takeover is not accepted the optimal strategy retrospectively was to do nothing. However, if the takeover was certain full hedging should have been recommended. Takeover-contingent foreign exchange call options have been priced (Kwok, 1998: 104-107). A more general problem consists in modelling the uncertain cash flows such that an optimal hedging strategy can be designed ex-ante. Brown and Toft (2001) derive optimal hedging strategies using vanilla derivatives (forwards and options) and custom “exotic” derivative contracts for a value-maximizing firm that faces both price and quantity risks. They find that optimal hedges depend critically on price and quantity volatilities, the correlation between price and quantity, and profit margin.

Within banking, Jorion (2001) notes that traders change positions actively during the trading day whereas Value at Risk (VaR) is measured over a one-day horizon assuming that the current positions are “frozen” over that time span. Despite this, he observes that empirical results for eight large banks indicate that on a quarterly basis VaR measures offer strongly significant predictions of the variability of trading revenues. Berkowitz and O’Brien (2001) independently compare, for a sample of six large dealer banks, daily VaR data as reported to regulators against subsequent trading profits. They find that VaR estimates tend to be conservative; that is, too high. The problem acknowledged in both papers is that the profits and losses refer to broad trading income including both the revenues generated by market-making activities and proprietary trading. The income generated by the purchase and sales of trading instruments on behalf of clients tend to be smoother than proprietary directional bets. A good illustration is provided by the Bankers Trust 1994 Annual report reproduced in Chew (1996: p 210). On the one hand, the statistical significance of the trading profits reported in both papers, Jorion (2001) and Berkowitz and O’Brien (2001), tend to be extremely high as measured by the T-statistics which roughly vary between 3.8 and 22. On the other hand, a pure directional bet is considered as very successful when the T-statistic reaches 2. This can be seen by studying the performance of alternative investments over long periods of time (See Managed Account Reports”). Although interesting, studies on banking profits can be difficult to interpret. Indeed, a too high VaR estimate may reflect a change of management policy rather than a methodology issue. Implementing corrective actions when directional losses start to develop are not uncommon.
The purpose of our article is to highlight the direct effect of uncertain exposures on Value at Risk calculations both theoretically and empirically. Whereas corporations cash flows are difficult to analyse for confidentiality reasons and banking profits are unfiltered, we have chosen to concentrate our examples on popular strategies used by active investors and directional traders. They encompass both discrete and continuous distributions of cash flows. The market timing ability is best rendered by the performance returns, which are the result of the by-product between the stochastic exposure and the market returns. Deans (2000) provides numerous examples of profit and loss calculation for backtesting. He especially recommends the use of profit and loss histograms to detect if the distribution is approximately normal, skewed, fat-tailed or if it has other particular features. This is why Section 1 discusses the distribution of performance returns when the exposure is stochastic. Section 2 quantifies the implications for Value at Risk calculations. Section 3 illustrates trading returns in the Futures markets. Section 4 summarises our findings and proposes new avenues for future research.

2) Distribution of performance returns

It is clear that no money manager or trader has control over the markets returns denoted $X$. The best a trader can do is to time his entry and exit in the market via his exposure, labelled $B$ (long, squared or short). In other words, the directional views are captured by the changing weights over the period. The market timing ability is best rendered by the performance returns $Z=BX$. Our goal here is not to forecast the excess returns generated by active timers but rather to quantify the risk taken by active money managers under the random walk assumption. Then we will assume no forecasting ability, which implies that active timing either based on discretion or trading rules cannot generate profits above and beyond the buy-and-hold returns. In statistical terms, this means that there is independence between the exposure $B$ and the forthcoming returns $X$. Despite violating the inner purpose of using a forecasting strategy, the random walk assumption is nevertheless useful in giving us a proxy for VaR calculations. Indeed, it is critical for performance returns to include several different contributions other than those related to market risk measurement, namely leverage and timing.

We are interested in establishing the distribution of the performance returns resulting from the product of two independent random variables. $B$, the stochastic exposure, follows either a discrete or continuous distribution. $X$, the market returns, is supposed in this section to follow a normal distribution with mean $\mu_x$ and volatility $\sigma_x$. 
Discrete Exposure

Let’s suppose that the exposure \( B = \begin{cases} 0 & \text{with probability } p_0 \\ b_1 & \text{with probability } p_1 \\ b_2 & \text{with probability } p_2 \\ \vdots & \text{with probability } p_n \end{cases} \)

with \( \sum_{i=0}^{n} p_i = 1, \ p_i \geq 0, \ i = 0,...,n \) and \( b_i \neq 0, \ i = 1,...,n \)

In this case, the performance returns \( Z = BX \) satisfy:

\[
Prob\{ Z < z \} = \sum_{i=1}^{n} p_i \Phi( \frac{z-b_i \mu_x}{|b_i| \sigma_x} ) \quad \text{if } z < 0
\]

\[
Prob\{ Z = 0 \} = p_0
\]

\[
Prob\{ Z < z \} = p_0 + \sum_{i=1}^{n} p_i \Phi( \frac{z-b_i \mu_x}{|b_i| \sigma_x} ) \quad \text{if } z > 0
\]

where \( \Phi \) is the cumulative function of a normal distribution \( N(0,1) \).

Two examples are given below. Their practical relevancy and economic justification are postponed to Section 2.

\[
B = \begin{cases} -1 & \text{with probability } 0.363879 \\ -1/3 & \text{with probability } 0.136121 \\ +1/3 & \text{with probability } 0.136121 \\ +1 & \text{with probability } 0.363879 \end{cases} \quad [1]
\]

\[
B = \begin{cases} 0 & \text{with probability } 0.5 \\ 1 & \text{with probability } 0.5 \end{cases} \quad [2]
\]

Continuous Exposure

If a trader follows a very large number of strategies, the resulting exposure may well be approximated by a normal distribution. This could also include the case of corporates tendering (in?) the markets. A good example is provided by the car industry where the sale of a car can be assimilated as a “mini” tender to market. We still assume that the market returns \( X \) follow a normal distribution with mean \( \mu_x \) and volatility \( \sigma_x \) and the exposure \( B \) follows a normal distribution with mean \( \mu_b \) and volatility \( \sigma_b \). Cornwell et al (1978) provides an algorithm to numerically evaluate the distribution of the product of two normal variables \( Z = BX \). Their findings take into account possible non-zero correlation between the two variables \( B \) and \( X \). It’s worthwhile noting that when both \( \mu_x = \mu_b = 0 \), the variable \( Z \) is nothing else than the covariance between two normal variables over a sample of two observations. The exact distribution can be found in Johnson, Kotz and
Balakrishnan (1995, p 600, formula 36.120). When $\sigma_x = \sigma_b = 1$, this is a modified Bessel function of the second kind.

Figure 1 highlights the cumulative function of the $Z$ variable for both types of exposure $B$: discrete as given by formulations [1] and [2], or normal with zero mean and variance equal to 0.5 (Example [3]). This also assumes that the market returns follow a standardized normal distribution with mean zero and unit standard deviation.

![Figure 1: Cumulative Function of Performance Returns](image)

3) **Implications for VaR Calculations**

The performance returns $Z = BX$ do not usually follow a normal distribution when the exposure $B$ is stochastic even if the market returns, $X$, are generated by a normal distribution. This has potentially large implications on the way VaR is calculated. VaR is a single number estimate of how much a trader can lose due to the price volatility of the instrument he holds. VaR is usually reported at the 95% level of confidence meaning that there is only a 5% chance that the portfolio will fall by more than the VaR. Let’s recall that the "Exact" VaR at the critical level of $\alpha$ is given by the quantile $c_\alpha^E$ which is deduced from the equality $Pr\{ Z < c_\alpha^E \} = 1 - \alpha$.

When the theoretical quantile is not known, risk managers tend to use either empirical estimates or crude calculations labelled as the normal assumption or raw method. With the normal assumption,
the risk manager simply believes that the distribution of performance returns follow a normal distribution with mean zero and variance \( \sigma^2 \). Then the VaR at the critical level of \( \alpha \) is equal to:

\[
c_a^N = \sigma \cdot q_{1-\alpha} \quad \text{where} \quad q_{1-\alpha} \quad \text{is the (1-} \alpha \text{) quantile of a standardized normal distribution. For instance if} \quad \alpha = 95\%, \quad q_{1-0.95} = q_{0.05} = -1.645. \quad \text{With the raw method, the risk manager calculates every day the VaR as being proportional to the market’s exposure. Over long period of time and assuming constant market volatility, the average VaR is just:} \quad c_a^R = E[|B|] q_{1-\alpha} \sigma_x.
\]

VaR can always be formulated as a coefficient of proportionality to the underlying market volatility and that is the convention adopted in this article. If we look at an investor being exposed to the Euro against Dollar exchange rate and the underlying currency market’s volatility is 10%, a VaR of –1.2 at a confidence level of 95% will mean that there is only a 5% chance that the portfolio will fall by more than –12% (=–1.2*10%).

Table 1 compares the three VaR, Exact, Normal and Raw when the markets \( X \) follow a standardized normal distribution, with zero drift and unit variance, and the exposure \( B \) is stochastic as given by examples [1] to [3]. It’s interesting to note that the raw calculations systematically underestimate the true VaR because this fails to capture the stochastic nature of the exposure. The normal assumption is far less damaging but still systematically overstates risk especially when the exposure follows a continuous distribution. Note that approximating the performance returns by a normal distribution makes indistinguishable the binomial exposure of type [2] and the continuous exposure of type [3].

<table>
<thead>
<tr>
<th>Table 1: Analytical Value at Risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>Confidence</td>
</tr>
<tr>
<td>95%</td>
</tr>
<tr>
<td>96%</td>
</tr>
<tr>
<td>97%</td>
</tr>
<tr>
<td>98%</td>
</tr>
<tr>
<td>99%</td>
</tr>
</tbody>
</table>

The Bank for International Settlements’ requirement for calculation of regulatory capital is a 99% confidence interval (Basel Committee, 1996). In practise, most organizations use a confidence level of 95% or/and 99% for their in-house requirements (Hawkins, 2000). This is why the rest of this article will concentrate on only these two VaR numbers.

4) **Actively trading the Futures Markets**

Futures contracts are probably the best financial markets to investigate the effect of stochastic weights. Low transaction costs allow high frequency of trades. The ability to short the market is also a key feature of active timers. We have chosen to restrain ourselves to the currency markets for the numerous real-life examples of uncertain exposure, which can be found there: active hedging,
technical trading, and tendering situations among other. We now detail the trading process and dataset used to illustrate both discrete and continuous exposures.

**Discrete Exposure**

According to Managed Account Reports, a tracking agency which reports the performance of alternative investments, most of the Futures funds are managed by systematic traders (around two third) whilst discretionary traders constitute the remainder. Systematic traders primarily rely on trading programs or models that generate buy and sell signals.

The simplest rule of this family is the single moving average which says: when the rate penetrates from below (above) a moving average of a given length m, a buy (sell) signal is generated. If the current price is above the m-moving average, then it is left long, otherwise it is held short. Lequeux and Acar (1998) recall that most commodities trading advisors (CTAs) do not trade a single strategy but rather allocates capital to a few. The authors then show that single moving averages of length 32, 61 and 117 can be used to replicate the portfolio of trading rules followed by CTAs. Furthermore, Acar and Lequeux (2001) work out the exposure’s probability under the assumption of a normal random walk without drift using well-known results on orthant probabilities. They find that there is a 36.3879% chance that the price is above (or below) three moving averages, therefore generating a long (short) position of +(-)100%. There is a 13.6121% chance that the price is above only two moving averages out of three, corresponding to a long position of 33.33%=2/3. There is also a 13.6121% chance that the price is above only one moving average out of three, implying a short position of -33.33%=(1-2)/3. If a proprietary trader or currency fund manager applies this portfolio of trading rules in the Futures markets for which underlying returns are denoted \( X \), he will exhibit performance returns \( Z=BX \) where \( B \) is given by Equation [1].

An even simpler example of discrete stochastic exposure is given by an active currency overlay programme. For the sake of clarity, we consider a Yen based investor being long of Dollar assets. The benchmark is unhedged. Then a forecasting strategy is used to predict the Dollar against Yen move. This could be based on technical trading rules or on exogenous information such as fundamental variables. The only thing we know is that up and down forecasts are expected with equal probability. Then if a long dollar position is triggered, the investor sticks to its unhedged benchmark. On the other hand, if a short dollar position is generated, the investor decides to hedge his position and therefore buy the Futures contracts, quoted in reciprocal terms, which earns returns \( X \) for the period. In other words, the relative performance or excess returns over the benchmark is simply equal to \( Z=BX \) where \( B \) is given by Equation [2].

Table 2 displays the summary statistics of the daily Futures contracts. The first available contract has been chosen and rollover implemented on the day before last expiration. More precisely, we study the main five contracts: Euro (Eur), Japanese Yen (Yen), Pound Sterling (Gbp), Swiss Franc (Chf), and Canadian dollar (Cad). All these contracts are quoted in reciprocal terms; that is, dollar
value of one foreign currency unit. Prior to t10 December 1999, the Deutschmark Futures contracts had been used as a proxy for the Euro.

**Table 2:** Daily returns 21-July-83 to 15-Feb-01

<table>
<thead>
<tr>
<th>Currency Contracts</th>
<th>EUR</th>
<th>JPY</th>
<th>GBP</th>
<th>CHF</th>
<th>CAD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
<td>0.0004%</td>
<td>0.0027%</td>
<td>0.0066%</td>
<td>-0.0040%</td>
<td>-0.0010%</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.712%</td>
<td>0.726%</td>
<td>0.683%</td>
<td>0.778%</td>
<td>0.298%</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.16</td>
<td>0.71</td>
<td>0.07</td>
<td>0.17</td>
<td>-0.24</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>2.20</td>
<td>8.00</td>
<td>3.68</td>
<td>2.03</td>
<td>4.61</td>
</tr>
<tr>
<td>Minimum</td>
<td>-3.31%</td>
<td>-4.21%</td>
<td>-4.48%</td>
<td>-3.99%</td>
<td>-2.25%</td>
</tr>
<tr>
<td>Maximum</td>
<td>4.83%</td>
<td>8.27%</td>
<td>4.55%</td>
<td>4.97%</td>
<td>1.99%</td>
</tr>
</tbody>
</table>

Figures 2 to 7 provide VaR estimates for both trading strategies as a function of the methodology being used. All the results have been standardized by the underlying market volatility such that they can be compared. Our goal here is not to assess how easy or difficult it is to predict market volatility but rather assuming the market volatility to be known, what are the consequences of stochastic exposure? In addition to the empirical estimate, we indicate the raw calculations, the assumption of normal profits and losses as well as the theoretical distribution of the stochastic weights. The latter approach outperforms the others at the 99% confidence level and it is increasingly obvious the more the distribution of profits and losses departs from the normal assumption (binary weights). The raw calculations uniformly underestimate risk.

![Figure 2: Basket of Moving Averages, 95% Confidence Level](image-url)
Figure 3: Basket of Moving Averages, 99% Confidence Level

Figure 4: Binomial Overlay, Foreign Currency Base, 95% Confidence Level
Figure 5: Binomial Overlay, US Dollar Currency Base, 95% Confidence Level

Figure 6: Binomial Overlay, Foreign Currency Base, 99% Confidence Level
Sometimes, traders combine technical trading rules with their fundamental view of the markets and the resulting trading process can no longer be formalized. In other words these money managers do not follow a pre-determined forecasting strategy but rather watch a multitude of indicators including Chartism, economics and flows. Then they might decide to leverage their positions according to the cumulative score reached by adding the individual output/exposure initiated by each of the variables. A good proxy may well be provided by the commitments of traders’ reports. All of a trader's reported futures positions in a commodity are classified as commercial if the trader uses futures contracts in that particular commodity for hedging as defined in the Commission's regulations. Then we may see the total net commercial position as an aggregate of individual overlay programs. The non-commercial activity regroups, among others: proprietary traders, commodity trading advisors and commodity pool operators, many of who apply some kind of trading rules. Therefore this will encompass a complex generalisation of trend-following strategies.

Table 3 indicates summary statistics on the net (long minus short) positions of both commercial and non-commercial traders reported on a weekly basis. The Euro contract is not reported because for over a year the Deutschmark contract was traded in parallel rendering difficult the interpretation of individual volume. Compared to previous simulations, we added the Australian Dollar contract. Distributions are rather normal with very little skewness and kurtosis. Such a cash flow is therefore best modelled by continuous stochastic exposures.
The VaR generated by the open positions of both commercial and non-commercial traders are indicated in Figures 8 to 11. All the figures have been standardized by dividing by the product of market and quantity volatility. Empirical values give the quantiles actually observed whereas the raw and normal profit and loss illustrate alternative VaR calculations as explained in Section 3. “Product of Normal” stands for the exact quantile value under the assumption that both market and exposure follow independent normal distributions with known means and variances. To understand how the shapes of individual distributions, flows and markets, affect the overall VaR, we used a Bootstrap methodology. Both flows and market returns were bootstrapped without replacement and independently. The VaR was then averaged over two hundred similar simulations. These results are given for completeness purpose only since they are only marginally closer to the empirical observations. This may well be due to the "insight" nature of the bootstrap methodology and could exhibit little predictive power.

The raw methodology as expected by our theoretical results underestimates the true VaR. At the 99% level, empirical VaR are on average 60% higher than the crude estimates and up to double. Considering that the profit and losses follow a normal distribution provides mixed results: underestimation of risk at the 99% level, reversing to slight overestimation at the 95% level. The product of two normal distributions tends to overestimate risk across the board.

Whereas it is clear that the raw methodology is unacceptable, choosing between the other calculations is not straightforward. When the number of observations is low, the tails of the distribution cannot be easily estimated and the “empirical” quantiles may be inaccurate given the lack of historical information. It may then be tempting to suppose that the profits and losses follow a normal distribution and extrapolate the corresponding quantile. Standard deviation of returns requires far less observations to be properly measured than extreme quantiles. In other words, the confidence interval of the standard deviation estimator is a lot smaller than the extreme quantiles. The analytical results assuming that both market and exposure follow a normal distribution are also attractive but this supposes that the parameters are known. In real life such a methodology will require the estimation of four parameters against one for the normal assumption of profits and losses. It is therefore possible that degree of accuracy is inversely related to the number of parameters having to be estimated.
Figure 8: Non Commercial Positions, 95% Confidence Level

Figure 9: Commercial Positions, 95% Confidence Level
Figure 10: Non Commercial Positions, 99% Confidence Level

Figure 11: Commercial Positions, 99% Confidence Level
5) Conclusion

The purpose of this article has been to highlight the existence of uncertain exposure and its effect on Value at Risk calculations both theoretically and empirically in the Futures markets. In many instances such as active trading or tender’s situations, the exposure itself is uncertain. Raw calculations using the absolute value of end of day positions grossly underestimate value a risk. The error is the largest at the 99% level, where it is the most needed because of lack of historical information. Considering instead that the profits and losses follow a normal distribution provides more accurate calculations. However, treating the cash flow as a stochastic variable has the potential to improve further Value at Risk calculations. Very often, exposure is uncertain and cannot be easily modelled ex-ante, or requires a large number of parameters to achieve a sufficient fit. In that instance, the difficulty in precisely estimating the parameters specifying the uncertain exposure may well overcome the potential gains. Nevertheless this paper has shown that there are real life cases where the exposure follows a simple discrete distribution and parameters can be accurately estimated ex-ante. The clearest example has been given by the use of popular trading rules. In that case, the distribution of the uncertain exposure can be worked out using analytical results independent of the underlying markets. Incorporating such information improves drastically VaR calculations while reducing the degrees of freedom. The results are especially conclusive when using the Bank for International Settlements requirement of a 99% confidence interval. The rarer the extreme events, the harder it will be to make accurate predictions using empirical observations. The importance of theoretical modelling grows the closer the confidence interval is to 100%.
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http://www.marhedge.com/

For illustrative purposes we have used the single moving average of length 117 days to generate the signals. Under the normal random walk without drift assumption, the VaR numbers should remain the same as long as buy and sell signals are generated with the same probability. It is only if the trading rule generates significant profits that the VaR number is likely to be different (smaller).

http://www.cftc.gov/cftc/cftccotreports.htm

Statistics breaking down non-commercial positions have been analysed in the crude oil, heating oil and gasoline futures markets (Weiner, 1999)