Some Applications
of Phase–Type Distributions
to Insurance and Finance

Søren Asmussen\textsuperscript{1}

Aarhus University, Denmark

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\textsuperscript{1}URL: \url{home.imf.au.dk/asmus}
Phase–Type Distributions
(Erlang, 1917, Jensen, 1953, Neuts 1975–)

• Comprises many standard distributions
  – Convolutions of exponentials;
    Gamma’s with integer parameter;
    (Erlang distrn’s)
  – Mixtures of exponentials (hyperexponentials)
  – All series/parallel/loop combinations;
    of exponentials

• Makes many calculations explicit;
  (or algorithmically tractable);
  “right generalization of exp distr’n”

• Are dense:

![Graph](image)

Figure 1: PH approximations to inverse Gaussian distribution
Def’n of PH Distributions

\{i, j, k, \ldots\} = \{\bullet, \bullet, \bullet, \ldots\} Markov states
one \( \Delta = \bullet \) absorbing

\[ \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array} \]

\[ 0 \quad \zeta \]

\( \zeta \) has PH\((\alpha, T)\) distr’n where
\( \alpha_i = \alpha_\bullet = \mathbb{P}(J_0 = \bullet) = \mathbb{P}(J_0 = i) \) initial prob’s
\( T = (t_{i,j}) \) matrix given by
the \( t_{i,j} = t_{\bullet, \bullet} = \) int’s jump \( \bullet \rightarrow \bullet \)
the \( t_i = t_{\bullet, \bullet} = \) int’s jump \( \bullet \rightarrow \Delta \) (absorption)
\[ t_{ii} = - \sum_{j \neq i} t_{ij} - t_i \]

CDF (distr’n fct) \( F(x) = \mathbb{P}(\zeta \leq x) \) where
\[ 1 - F(x) = \mathbb{P}(\zeta > x) = \mathbb{P}(J_x \neq \Delta) = \alpha e^{tx} 1, \]
\[ e^{tx} = \sum_{n=0}^{\infty} \frac{T^n x^n}{n!}. \]

Density \( \alpha e^{tx} t \)
Mean \(-\alpha T^{-1} t\), 2nd moment \( \alpha T^{-2} t\), \ldots
Example: Erlang($n$)

$n$ exponentials in series (same int’y $\lambda$)

$= \text{convolution of } n \text{ exponentials } = \text{Gamma}(n, \lambda)$

density $\frac{\lambda^n x^{n-1}}{(n-1)!} e^{-\lambda x}$

$n = 3$

$$T = \begin{pmatrix} \cdot & \lambda & 0 \\ 0 & \cdot & \lambda \\ 0 & 0 & \cdot \end{pmatrix} \quad t = \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix}$$

$$T = \begin{pmatrix} -\lambda & \lambda & 0 \\ 0 & -\lambda & \lambda \\ 0 & 0 & -\lambda \end{pmatrix}$$

$$\alpha = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

**Figure 2: Erlang densities**
Classical risk model
(compound Poisson = Cramér–Lundberg)

\[ \tau \text{ ruin time} \]
\[ \psi(u) = \mathbb{P}(\tau < \infty) \text{ ruin prob. (inf. horizon)} \]

Classical expression P–K–B–B
Pollaczek–Khintchine–Beekman–Bowers
\( \beta \) Poisson int’y, unit premium rate
\( B \) claim distr’n, \( B_I(x) = \frac{1}{\mu} \int_0^x B(y) \, dy \) integrated tail

\[
\psi(u) = (1 - \rho) \sum_{n=0}^{\infty} \rho^n B_I^n(u) \quad \rho = \beta \mu_B < 1
\]

Phase–type reduction: \( B \) PHT(\( \alpha, T \))
\( t = -T1, \quad \alpha_+ = -\beta \alpha T^{-1}, \quad Q = T + t \alpha_+ \)

\[
\psi(u) = \alpha_+ e^{qu} 1
\]

Neuts 1981 (M/G/1 queue)
Many special cases in risk literature
SA-Rolski IME 1991
Generalizations to many different models
Finite Horizon Problem

SA–Avram–Usabel, Astin Bull. 2003

\[ \psi(u, T) = \mathbb{P}(\tau \leq T) \]

No reduction in PHT case

Idea: replace \( T \) by PHT r.v. \( H \)

Now nice solution

Use denseness: \( H = H_k \) where \( H_k \to T, k \to \infty \)

\[ \psi(u, T) \approx \mathbb{E}\psi(u, H_k) \]

Best choice Erlang(\( k \)) with mean \( T \)

\[ \frac{\lambda^k x^{k-1}}{(k-1)!} e^{-\lambda x}, \quad \lambda = \lambda_k = k/T \]

\[ \psi(u, T) = 0.2\% \]
Description of Algorithm

1. $n = 1$

$$
\mathbb{E}\psi(u, H) = \int_{0}^{\infty} \psi(u, t) f_H(t) \, dt,
$$

$$
E\psi(u, H_1) = \int_{0}^{\infty} \psi(u, t) \lambda e^{-\lambda t} \, dt
= \lambda \times \text{Lapl.transf.}
= \lambda \times \alpha_\lambda e^{q_\lambda u} 1
$$

$$
\alpha_\lambda = \beta \alpha(s_\lambda I - T^{-1}), \quad Q_\lambda = + t\alpha_\lambda
$$

$s_\lambda$ root of $\beta \alpha(-s I - T)^{-1} t = \beta + \lambda - s$
(Lundberg equation)

2. $n \longrightarrow n + 1$

$$
\alpha_\lambda^{(n+1)} = \left(s_\lambda \alpha_\lambda^{(n)} + \sum_{j=2}^{n} \alpha_\lambda^{n+2-j} t\alpha_\lambda^{(j)}\right)(s_\lambda - T - t\alpha_\lambda^{(1)})^{-1}
$$
Improvement by Extrapolation

\[ \psi(u, T) = 2.28\% \]

\[ \psi(u, T) \times 100\% \]

\[ \psi(u, T) \times 90\% \]
Richardson extrapolation:
Compute $x$ accurately using sequence $x_k \to x$
Assume convergence rate known: $x - x_k = \frac{c}{k} + \frac{d}{k^{1+\epsilon}} + \cdots$
c is unknown but can be eliminated
Improved approximation $(k + 1)x_{k+1} - kx_k$

Implementation:

$$x = \psi(u, T) \approx \mathbb{E}\psi(u, H_k) = x_k$$

$$\mathbb{E}\psi(u, H_k)$$
$$= \mathbb{E} \left[ \psi(u, T) + \psi_T(u, T)(H_k - T) + \psi_{TT}(u, T)(H_k - T)^2/2 + \cdots \right]$$
$$= \psi(u, T) + 0 + \psi_{TT}(u, T)\text{Var}(H_k) + \cdots$$
$$= \psi(u, T) + \frac{c}{k} + \cdots$$

Figure 3: Erlang smoothing
Applications to Option Pricing

Barrier options

\[
\text{Pay–out } \quad e^{-aT}(S_T - K)^+ I(\inf_{t \leq T} S_t > L)
\]

First passage problems occur also for other types of options

**Perpetual American Put**

Pay–out \( e^{-a\tau}(K - S_\tau)^+ \)

\( \tau \) stopping time at your disposal

Price \( \mathbb{E}^* \inf_{\tau} e^{-a\tau}(K - S_\tau)^+ \)

*: risk–neutral measure

\( \inf \) attained for \( \tau^# = \inf\{t : S_t \leq k^#\} \)

**Russian option**

Similar but more complicated; details later
Traditional model (Black–Scholes):
\[ X_t = \log S_t \] Brownian motion
First approx but does not fit data too well
Alternative Lévy models (hyperbolic, NIG, ...)
Price calculations much harder than for BM
For barrier options, feasible with PHT models
No restriction because of denseness
Role of PH Assumptions in Level Crossing Problems

(barrier options; classical ruin problem)

Overshoot at $\tau(K)$? (deficit at ruin)

$S_T | \tau(K) < T$? $\tau(K)$ and overshoot needed

PH assumptions control overshoot

only phase at level crossing needed

often finite set of linear equations
Implementation for Russian Options


Introduced by Shepp & Shiryaev 1991

Approximation to perpetual American option

Pay-out at stopping time $\tau$  
$e^{-a\tau}\max(m, \max_{t\leq\tau} S_t)$

Shepp & Shiryaev 1991: solution for geometric BM

$\tau(k^\#) = \inf\{t : \max_{v\leq t} S_v - S_t \geq k^\#\}$ optimal

AAP: solution in dense class of Lévy processes

$$\log S_t = B_t + \sum_{i=1}^{N_t^+} U_i^+ - \sum_{i=1}^{N_t^-} U_i^-$$

independent compound Poisson, PHT jump $\uparrow$ and $\downarrow$
Solution

1. Risk-neutral measure:
choose as Esscher transform
again, \( \log S_t = BM + CP(PHT)^+ - CP(PHT)^- \)
(changed parameters)

2. Same optimal stopping time
\( \tau(k^\#) = \inf\{t : \max_{v \leq t} S_v - S_t \geq k^\# \} \) as SS
Can be written as \( \tau(k^\#) = \inf\{t : V_t \geq k^\# \} \)
\( V_t = \log S_t \) reflected at maximum

3. \( 2 + p^+ + p^- \) linear equations:
\( \mathbb{P}(\text{upcrossing}), \mathbb{P}(\text{downcrossing}), \ldots \)
\( \mathbb{E}(\text{upcrossing}), \mathbb{E}(\text{downcrossing}), \ldots \)
+2 for BM
obtained by martingale optional stopping