Abstract

The evaluation of the insurance risk economic capital for a portfolio of limited excess-of-loss reinsurance contracts with an inflation stability clause is considered. Based on the classical compound Poisson Pareto aggregate claims model, we determine the reinsurer’s cumulative paid loss and incurred loss development until final settlement of all occurred claims using a claims payment pattern, a reserve deviation pattern, and inflation stability clause ratios. Of fundamental importance for risk margin valuation according to the cost-of-capital method are the distributions of the incremental incurred losses or loss spreads. Their somewhat intricate statistical properties are clarified for the Solvency II compliant situation of consecutive incurred loss random variables following bivariate lognormal distributions. A numerical illustration shows that an extended univariate lognormal approximation of the loss spreads can significantly underestimate the risk margin as compared to the use of a bivariate lognormal approximation.

Keywords

excess-of-loss reinsurance, inflation clause, Solvency II, insurance risk economic capital, VaR, CVaR, bivariate lognormal spread distribution
1. Introduction

We consider evaluation of the insurance risk economic capital for the widespread limited excess-of-loss (XL) reinsurance treaties according to the cost-of-capital method for use in worldwide solvency systems including Solvency II and SST. On the reinsurance market it is commonly observed that higher losses are subject to higher inflation than classical inflation, the so-called superimposed inflation. The latter can be measured using the so-called Masterson Claim Cost Index by Masterson(1968/77/81), which provides “claim costs” for different lines of business. Now, if the deductible of an XL contract is fixed over the claims development duration, and a claim exceeds the deductible, then the reinsurer must cover all future increases due to superimposed inflation. To protect themselves against a possible inflation related moral hazard, reinsurers can stipulate inflation stability clauses in their contract agreements. Then future inflation is shared between the ceding company and the reinsurer. For example, the “date of payment” stability clause stipulates indexing of the deductible and limit using the ratio of the sum of actual payments to the sum of inflation adjusted payments. In this situation actuarial valuation is based on variable deductibles and limits per development period until final settlement of all occurred claims. The paper is organized as follows.

Section 2 is devoted to the stochastic modeling of the reinsurer’s cumulative paid loss and incurred loss. To fix ideas and for simplicity, we assume that the aggregate claims associated to excess-of-loss reinsurance are compound Poisson Pareto distributed. Modeling includes claims development using a claims payment pattern, a reserve deviation pattern, and inflation stability clause ratios. Proposition 2.1 summarizes succinctly our analysis. Section 3 contains all detailed risk calculations required to determine the (re)insurance risk solvency capital requirement (SCR) and the associated risk margin (RM) for of a portfolio of XL reinsurance contracts with an inflation stability clause. Of fundamental importance for risk margin valuation are the distributions of the incremental incurred losses or loss spreads, whose statistical properties may be quite intricate. One encounters two difficulties with them. First, they may take negative means (e.g. in some instances of overstated reserves) and a simple lognormal approximation, as suggested by the Solvency II standard approach, is not directly feasible. To overcome this, one may take its negative version, namely the profit spread, approximate it by a lognormal distribution and use the dual functional property of the VaR and CVaR risk measures presented in Appendix A. However, this “extended” lognormal approximation of the spread random variables usually underestimates risk in the sense that the corresponding VaR and CVaR risk measures are underestimated (see our numerical illustration in Subsection 4.3). To overcome this difficulty and increase accuracy, but still remain close in spirit to the Solvency II standard framework, we propose in Subsection 4.2 a bivariate lognormal approximation of these spreads based on the results of Appendix B, which contains an actuarial pendant of the closed-form approximations for spread option prices used in Deng et al.(2008). The numerical example of Subsection 4.3 shows that the extended lognormal approximation of the spread random variables usually underestimates risk in the sense that the corresponding VaR and CVaR risk measures are underestimated (see our numerical illustration in Subsection 4.3). 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2. Stochastic modeling of the reinsurer’s cumulative paid loss and incurred loss

We assume that the aggregate claims of an insurance risk portfolio over some fixed time period can be represented by a compound Poisson random variable

\[ S = \sum_{i=1}^{N} X_i, \quad (2.1) \]

where the number of claims \( N \) is Poisson(\( \lambda \)) distributed, the non-negative claim sizes \( X_i \)'s are independent, identically distributed and independent from \( N \). The identical random variables are denoted \( X =_d X_i, \quad i = 1, \ldots, N \). The reinsured aggregate claims are denoted by \( S_r \) while the retained aggregate claims of the ceding company are denoted by \( S_c \) and one has the risk decomposition \( S = S_r + S_c \).

An excess-of-loss or XL reinsurance contract with deductible \( \ell > 0 \) and limit \( m > 0 \), denoted by \( m \times \ell \), pays the amount of each and every claim that exceeds the deductible up to the limit on the payment of each claim and is defined by the reinsured amount

\[ S_r = \sum_{i=1}^{N} Y_i, \quad Y_i = \min(m, (X_i - \ell)_+). \quad (2.2) \]

In practice of reinsurance it is rather popular to limit the aggregate liability. The XL reinsurance for the layer \( m \times \ell \) with aggregate layer \( M \times \ell \) covers the aggregate claims to the XL layer \( m \times \ell \) that exceeds \( L \) up to the limit \( M \) and is defined by the reinsurance payment

\[ S_r = \min \left( M, \left( \sum_{i=1}^{N} Y_i - L \right)_+ \right). \quad (2.3) \]

In case the aggregate limit is an integer multiple of the XL limit, say \( M = (K+1) \cdot m \), one speaks of an XL reinsurance for the layer \( m \times \ell \) in the aggregate with \( K \) reinstatements. The reinsurance coverage is given by

\[ S_r = \min \left( (K+1) \cdot m, \left( \sum_{i=1}^{N} Y_i - L \right)_+ \right). \quad (2.4) \]

In this situation, the reinsurance has to be reinstated if the aggregate payment exceeds an integer multiple of the XL limit, that is if \( K \geq 1 \). If one has \( K = 0 \) there are no reinstatements. In practice, there are free and paid reinstatements. Some papers on this important reinsurance contract include Simon(1972), Sundt(1991), Hürlimann(2005) and the references therein.

In the present paper we focus our attention on the XL contract with payment (2.2). To fix ideas we assume that the claim size random variable \( X \) is Pareto distributed with scale parameter \( OP \) (the so-called observation point) and index \( \gamma > 1 \) (hence the mean exists). Its distribution function reads
\[ F_X(x) = 1 - \left( \frac{x}{OP} \right)^{-\gamma}, \quad x \geq OP > 0, \quad \gamma > 1 \] (2.5)

In this situation the random variable \( N \) counts the number of claims exceeding \( OP \). A more detailed discussion of this basic collective model of risk theory, which contains only random variables that are observable by the reinsurer, is found in Hess(2003) and related literature Hess et al.(1995), Franke and Macht(1995), Mack(1997) and Schmidt(1996/2002). The use of the two-parameter Pareto distribution has been a first choice in the practice of XL reinsurance for a long time (see e.g. Schmitter(1978), Schmitter and Bütkofer(1997), Doerr(1980), Schmutz and Doerr(1998)) and it is consistent with the theoretical results from Extreme Value Theory (e.g. Embrechts et al.(1997)).

We assume that the distribution parameters \( \lambda, OP, \gamma \) have been estimated possibly based on past data, but also exposure curves or market benchmarks can be used. A realization \( X_i \) of \( X \) represents a loss assumed to be paid in one installment at the time of premium inception. In practice, especially in long-tail business, this is not realistic and a model for the claims development must be specified. We closely follow Walhín et al.(2001).

Suppose that claims payments occur at times \( t_1, t_2, ..., t_n \) according to a given claims payment pattern \( c(t_1), c(t_2), ..., c(t_n) \), where \( t_n \) is the time of final settlement at which the losses are completely paid out. The origin of time is \( t_0 = 0 \). To adjust losses for inflation, we assume that future payments are subject to an inflation index \( f(t_0), f(t_2), ..., f(t_n) \) such that future payments for a loss \( X_i \) are given by

\[ X_i(t_j) = c(t_j) \frac{f(t_j)}{f(t_0)}, \quad j = 1, ..., n. \] (2.6)

On the reinsurance market it is commonly observed that higher losses are subject to higher inflation than classical inflation, the so-called superimposed inflation. Future payments for \( X_i \) that are adjusted for the superimposed inflation index \( s(t_0), s(t_2), ..., s(t_n) \) are determined by

\[ X_i(t_j) = c(t_j) \frac{s(t_j)}{s(t_0)}, \quad j = 1, ..., n. \] (2.7)

Given a loss \( X_i \) one considers the cumulative paid loss sequence

\[ X_i^C(t_j) = \sum_{k=1}^{j} X_i(t_k), \quad j = 1, ..., n, \] (2.8)

and the reinsurer’s cumulative paid loss sequence

\[ Y_i^C(t_j) = \min\left\{ m_i \left(X_i^C(t_j) - \ell_i\right) \right\}, \quad j = 1, ..., n. \] (2.9)
The *exact reserve* sequence of a loss are defined by

\[ R_i(t_j) = X_i^C(t_n) - X_i^C(t_j), \quad j = 1, \ldots, n. \] (2.10)

To take into account systematic deviations from the exact reserve (over- or underestimation and the inclusion of possible IBNR reserves) one considers the *reserve deviation pattern* \( d(t_1), d(t_2), \ldots, d(t_n) \) with the convention that \( d(t_j) = 100\% \) if there is no deviation at time \( t_j \).

From this one obtains the *incurred loss* (paid + outstanding claims reserve)

\[ X_i^I(t_j) = X_i^C(t_j) + d(t_j)R_i(t_j), \quad j = 1, \ldots, n, \] (2.11)

and the *reinsurer’s incurred loss*

\[ Y_i^I(t_j) = \min\left\{ m, \left( X_i^I(t_j) - \ell \right)_+ \right\}, \quad j = 1, \ldots, n. \] (2.12)

Now, if the deductible of the XL contract is fixed over the claims development duration, and a claim exceeds the deductible, then the reinsurer must cover all future increases due to inflation or superimposed inflation. To protect themselves against a possible inflation related moral hazard, reinsurers can stipulate *inflation stability clauses* in their contract agreements. With such a clause future inflation is shared in some way between the ceding company and the reinsurer (see Gerathewohl(1980) or Liebwein(2009) for further information). For illustration we consider the “date of payment” stability clause, which stipulates indexing of the deductible and limit using the ratio of the sum of actual payments to the sum of inflation adjusted payments, called *stability clause ratio*, and defined by

\[ r(t_j) = \frac{X_i^C(t_j)}{\sum_{i=1}^{j} X_i(t_k) \cdot \frac{f(t_k)}{f(t_j)}}, \quad j = 1, \ldots, n. \] (2.13)

The XL contract with stability clause is based on the following variable deductibles and limits

\[ \ell(t_j) = r(t_j) \cdot \ell, \quad m(t_j) = r(t_j) \cdot m, \quad j = 1, \ldots, n. \] (2.14)

With this the reinsurer’s cumulative paid loss and incurred loss under the inflation stability clause are given by

\[ Y_i^C(t_j) = \min\left\{ m(t_j), \left( X_i^C(t_j) - \ell(t_j) \right)_+ \right\}, \]
\[ Y_i^I(t_j) = \min\left\{ m(t_j), \left( X_i^I(t_j) - \ell(t_j) \right)_+ \right\}, \quad j = 1, \ldots, n. \] (2.15)

For computational purposes consider some auxiliary deterministic quantities, which will provide important simplification. Define the *inflation adjusted factors* \( a(t_1), a(t_2), \ldots, a(t_n) \) for the cumulative paid loss and \( b(t_1), b(t_2), \ldots, b(t_n) \) for the incurred loss, which are defined by
\[ a(t_j) = \sum_{k=1}^{j} c(t_k) \frac{s(t_k)}{s(t_0)}, \quad b(t_j) = a(t_j) + d(t_j) \cdot (a(t_n) - a(t_j)), \quad j = 1, \ldots, n. \]  

(2.16)

With these factors the cumulative paid loss (2.8) and the incurred loss (2.11) rewrite as

\[ X_i^C(t_j) = a(t_j) \cdot X_i, \quad X_i^I(t_j) = b(t_j) \cdot X_i, \quad j = 1, \ldots, n. \]  

(2.17)

Similarly, the stability clause ratios (2.13) rewrites as

\[ r(t_j) = \frac{a(t_j)}{\sum_{k=1}^{j} c(t_k) \frac{s(t_k)}{s(t_0)} f(t_0)}, \quad j = 1, \ldots, n. \]  

(2.18)

We are ready to show the following quite useful reinsurance liability formulas.

**Proposition 2.1.** Given is the compound Poisson Pareto reinsurance model \( S_r = \sum_{i=1}^{N} Y_i \) with individual losses \( Y_i = \min\{m, (X_i - \ell)_+\} \), \( i = 1, \ldots, N \), and distribution parameters \( \lambda, \OP, \gamma \). The reinsurer’s cumulative paid losses and incurred losses for the XL contract with deductible \( \ell > 0 \) and limit \( m > 0 \) under the “date of payment” stability clause are random functions of the original losses \( X_i \) and the deterministic sequences \( a(t_j), b(t_j), r(t_j), j = 1, \ldots, n \), and are given by

\[ Y_i^C(t_j) = \min\{r(t_j)m, (a(t_j) \cdot X_i - r(t_j)\ell)_+\}, \]  

\[ Y_i^I(t_j) = \min\{r(t_j)m, (b(t_j) \cdot X_i - r(t_j)\ell)_+\}, \quad j = 1, \ldots, n. \]  

(2.19)

**Proof.** Inserting (2.17) in the formulas (2.15) yields the result. \( \Diamond \)

3. **Solvency Capital Requirement and Risk Margin**

We begin with risk calculations for a single XL contract, and use them to determine the solvency capital requirement and risk margin for a portfolio of XL contracts.

3.1. **Risk calculations for a single XL contract**

Given is a single XL contract whose reinsurance liabilities have been characterized in Proposition 2.1. From now on, in order to simplify notations, we assume that claims payments occur at equally spaced times at the end of each year, that is we set \( t_j = j, j = 1, \ldots, n \). Consider the sequences of annual reinsurer’s aggregate cumulative paid losses and aggregate incurred losses defined by (the index \( r \) indicating the reinsurer’s point of view is removed for simplicity)
\[ S^C(j) = \sum_{i=1}^{n} Y^C_i(j), \quad S^I(j) = \sum_{i=1}^{n} Y^I_i(j), \quad j = 1, ..., n. \] (3.1)

Associated to these quantities one obtains the sequences of annual incremental aggregate cumulative paid losses and incurred losses, as well as of the loss reserves and incremental loss reserves defined respectively by

\[ \Delta S^C(1) = S^C(1), \quad \Delta S^C(j) = S^C(j) - S^C(j-1), \quad j = 2, ..., n, \]
\[ \Delta S^I(1) = S^I(1), \quad \Delta S^I(j) = S^I(j) - S^I(j-1), \quad j = 2, ..., n, \]
\[ R(j) = S^I(j) - S^C(j), \quad j = 1, ..., n, \]
\[ \Delta R(1) = S^I(1) - S^C(1), \quad \Delta R(j) = \Delta S^I(j) - \Delta S^C(j), \quad j = 2, ..., n. \] (3.2)

In order to perform risk calculations based on these random quantities, it is necessary to know their distributions or at least their moments of order one and two. Of fundamental importance for the insurance risk (see the next Subsection for a definition) are the distributions of the incremental incurred losses \( \Delta S^I(j), j = 1, ..., n \), which implicitly require knowledge about the bivariate distributions of the random vectors \( \{S^I(j-1), S^I(j)\}, j = 2, ..., n \). It is possible to calculate these distributions numerically using the multivariate Panjer algorithm of Sundt(1999) (see also Sundt and Vernic(2009)). For practical purposes, and in the spirit of the current Solvency II specification, we adopt a more pragmatic approach and approximate in Section 4 these distributions analytically using bivariate lognormal distributions. For this one needs formulas for the means and coefficients of variation of \( S^I(j) \) as well as the correlation coefficients between \( S^I(j-1) \) and \( S^I(j) \). These distribution characteristics are denoted and obtained as follows:

\[ \mu S^I_j = E[S^I(j)] = \lambda \cdot \mu Y^I_j, \quad \mu Y^I_j = E[Y^I(j)], \]
\[ kS^I_j = \frac{\sqrt{\text{Var}[S^I(j)]}}{E[S^I(j)]} = \sqrt{\frac{m2Y^I_j}{\lambda \cdot (\mu Y^I_j)^2}}, \quad m2Y^I_j = E[Y^I(j)^2], \quad j = 1, ..., n, \]
\[ \rho S^I_j = \frac{\text{Cov}[S^I(j-1), S^I(j)]}{\sqrt{\text{Var}[S^I(j-1)]} \cdot \sqrt{\text{Var}[S^I(j)]}} = \frac{pmY^I_{j-1,j}}{\sqrt{m2Y^I_{j-1,j} m2Y^I_j}}, \]
\[ pmY^I_{j-1,j} = E[Y^I(j-1) \cdot Y^I(j)], \quad j = 2, ..., n. \] (3.3)

The expression for the covariance in (3.3) follows from the law of total covariance, which applied to compound Poisson random variables, yields

\[ \text{Cov}[S^I(j-1), S^I(j)] = E[\text{Cov}[S^I(j-1), S^I(j)|N]] + \text{Cov}[E[S^I(j-1)|N], E[S^I(j)|N]] \]
\[ = E[N] \cdot \text{Cov}[Y^I(j-1), Y^I(j)] + \text{Var}[N] \cdot E[Y^I(j-1)] \cdot E[Y^I(j)] = \lambda \cdot E[Y^I(j-1) \cdot Y^I(j)] \] (3.4)
One notes that the expressions in (3.3) depend besides the Poisson parameter upon the first two moments of the incurred loss claim size \( Y^I(j) \) and the product moment between \( Y^I(j-1) \) and \( Y^I(j) \). Under the Pareto distribution (2.5) for the original claim size, these moments are obtained in closed form as follows. Rewrite \( Y^I(j) = \min(r(j)m, (b(j) \cdot X - r(j)\ell)_+) \) as

\[
Y^I(j) = b(j) \cdot \left( (X - Ld)_+ - (X - Lu)_+ \right), \quad Ld_j = \frac{r(j)}{b(j)} \ell, \quad Lu_j = \frac{r(j)}{b(j)} (\ell + m)
\]

to get

\[
\mu Y^I_j = b(j) \cdot \left( \pi_X(Ld_j) - \pi_X(Lu_j) \right),
\]

\[
m_2 Y^I_j = b(j)^2 \cdot \left( \pi_{2,X}(Ld_j) - \pi_{2,X}(Lu_j) - 2 \cdot (Lu_j - Ld_j) \cdot \pi_X(Lu_j) \right), \quad j = 1, \ldots, n,
\]

\[
\mu m Y^I_{j-1,j} = b(j-1) \cdot b(j)
\]

\[
\left\{ \pi_{2,X}(Ld_j) - \pi_{2,X}(Lu_{j-1}) + (Ld_j - Ld_{j-1}) \cdot \left( \pi_X(Ld_j) - \pi_X(Lu_j) \right) \right\}
\]

\[
\left\{ -(Lu_{j-1} - Ld_j) \cdot \left( \pi_X(Lu_{j-1}) + \pi_X(Lu_j) \right) \right\}, \quad j = 2, \ldots, n.
\]

where use is made of the functions (Pareto stop-loss transforms of degree one and two)

\[
\pi_X(x) = \frac{OP}{\gamma - 1} \left( \frac{x}{OP} \right)^{1-\gamma}, \quad \pi_{2,X}(x) = \frac{2 \cdot OP^2}{(\gamma - 1)(\gamma - 2)} \left( \frac{x}{OP} \right)^{2-\gamma}.
\]  

(3.6)

### 3.2. Solvency risk calculations for a portfolio of XL contracts

Towards the ultimate goal of solvency evaluation for an arbitrary reinsurance portfolio, we consider now a set of \( p \) XL contracts at the current time of valuation \( t = 0 \). From Proposition 2.1 one knows that the \( k \)-th contract \( k \in \{1, \ldots, p\} \) is characterized by the following data set:

- current year \( y_k \in \{1, \ldots, n\} \) of claims development
- original deductible \( \ell_k > 0 \) and limit \( m_k > 0 \)
- inflation adjusted factors and stability clause ratios \( a_k(j), b_k(j), r_k(j), j = 1, \ldots, n \)
- compound Poisson Pareto reinsurance model with parameters \( \lambda_k, OP_k, \gamma_k \)

To the \( k \)-th XL contract one associates their future aggregate cumulative paid and aggregate incurred losses described similarly to (3.1) with the corresponding notations by

\[
S^{C,(k)}(j) = \sum_{i=1}^{N_k} Y^{C,(k)}_i (y_k + j - 1), \quad S^{I,(k)}(j) = \sum_{i=1}^{N_k} Y^{I,(k)}_i (y_k + j - 1), \quad j = 1, \ldots, n - y_k + 1.
\]

(3.7)

The index shifts within the loss functions remind us that at the current time of valuation only those individual losses from the claims development year \( y_k \) and beyond must be considered. In the obvious notations random quantities similar to (3.2) are also defined for the \( k \)-th XL
contract. From now on we also need the premiums $P^{(k)}(j)$ of the $k$-th contract, which are earned in the future years $j = 1,\ldots,n - y_k + 1$ associated to the development year $y_k$ and beyond. Summing over all XL contracts one obtains the future portfolio cumulative paid and incurred losses as well as future portfolio earned premiums through

$$
S^C(j) = \sum_{k=1}^{p} 1_{\{j \leq n - y_k + 1\}} S^{C,(k)}(j), \quad S^I(j) = \sum_{k=1}^{p} 1_{\{j \leq n - y_k + 1\}} S^{I,(k)}(j),
$$

(3.8)

$$
P(j) = \sum_{k=1}^{p} 1_{\{j \leq n - y_k + 1\}} P^{(k)}(j), \quad j = 1,\ldots,n.
$$

In the obvious notations portfolio random quantities similar to (3.2) are also defined. Following Solvency II and the Swiss Solvency Test (SST), and leaving out here the catastrophe risk, we consider solvency requirements for the premium risk, the reserve risk and the insurance risk (=combined premium and reserve risk) of the portfolio as follows. The portfolio (re)insurance risk is determined by the sequence of unexpected increases $L(j) - E[L(j)]$ of the technical result in each future development year defined for $j = 1,\ldots,n$ by

$$
L(j) - E[L(j)] = (E[P(j)] - P(j) + [\Delta S^C(j) - E[\Delta S^C(j)]] + [(\Delta R(j) - E[\Delta R(j)])].
$$

(3.9)

The first two brackets on the right side represent the premium risk and the third one the reserve risk. For simplicity, we assume deterministic earned premiums such that $P(j) = E[P(j)]$. Using the definition of the loss reserves in (3.2) it follows that $\Delta S^C(j) + \Delta R(j) = \Delta S^I(j)$, hence (3.8) can be rewritten as

$$
L(j) - E[L(j)] = \Delta S^I(j) - E[\Delta S^I(j)], \quad j = 1,\ldots,n.
$$

(3.10)

Now, the solvency capital requirement (SCR) associated to the portfolio reinsurance risk for the development year $j \in \{1,\ldots,n\}$ can be defined using either the VaR risk measure to the confidence level $\alpha = 99.5\%$ (Solvency II) or the CVaR risk measure to the confidence level $\alpha = 99\%$ (SST) (see Appendix A for further details). One has the formulas

$$
SCR^\text{VaR}_\alpha(j) = \text{VaR}_\alpha[\Delta S^I(j) - E[\Delta S^I(j)]],
$$

$$
SCR^\text{CVaR}_\alpha(j) = \text{CVaR}_\alpha[\Delta S^I(j) - E[\Delta S^I(j)]],
$$

(3.11)

Furthermore, a risk margin (RM) for the portfolio reinsurance risk, whose contribution to the total risk margin must be included in the technical provisions, is defined to be equal to the cost-of-capital of all future SCR’s beyond the first year. Taking into account a risk-free discount factor $v_f$ and a cost-of-capital rate $i_{\text{CoC}}$ one obtains

$$
RM^\text{VaR}_\alpha = i_{\text{CoC}} \sum_{j=2}^{n} v_f^j \cdot SCR^\text{VaR}_\alpha(j), \quad RM^\text{CVaR}_\alpha = i_{\text{CoC}} \sum_{j=2}^{n} v_f^j \cdot SCR^\text{CVaR}_\alpha(j).
$$

(3.12)
Finally, the economic capital (EC), which is equal to the sum of the target capital (SCR for the first year discounted at the risk-free rate) and the risk margin, is determined by

\[
EC_{\alpha}^{VaR} = v_f \cdot SCR_{\alpha}^{VaR} (I) + RM_{\alpha}^{VaR}, \quad EC_{\alpha}^{CVaR} = v_f \cdot SCR_{\alpha}^{CVaR} (I) + RM_{\alpha}^{CVaR}.
\]  (3.13)

4. Extended and bivariate lognormal distribution approximations

Subsection 4.1 begins with a discussion of the univariate lognormal approximation. We argue that it has to be modified according to Proposition A.2 to take into account the possibility of negative mean losses in determination of the SCR’s for RM calculation. However, the extended lognormal approximation of the spread random variables \( \Delta S^I(j) = S^I(j) - S^I(j-1), j \geq 2 \), usually underestimates risk in the sense that the corresponding VaR and CVaR risk measures are underestimated (see the numerical illustration). To overcome this difficulty and increase accuracy, but still remain close in spirit to the Solvency II standard framework, we propose in Subsection 4.2 a bivariate lognormal approximation of these spreads based on the results of Appendix B. Subsection 4.3 presents a numerical illustration and some conclusions.

4.1. Extended lognormal approximation

The current Solvency II specification of the first year non-life insurance risk SCR is based on a simple univariate lognormal approximation of the current year’s portfolio incurred loss \( S^I(I) \). Since this random variable has a positive mean the formulas (A.3) can be used without loss of generality. However, the calculation of SCR’s for future years, which is based on the loss spreads \( \Delta S^I(j), j \geq 2 \), may take negative means. This often happens when loss reserves are overstated, as encountered in the practice of XL reinsurance with an inflation stability clause (e.g. Wallin et al.(2001) or the examples below). In this situation the loss spread cannot be directly approximated by a lognormal distribution. If its negative version, namely the profit spread, is approximated by a lognormal distribution, then the formulas (A.4) can be used. The extended lognormal approximation only requires calculation of the mean and standard deviation of a spread \( \Delta S^I(j), j \in \{2,...,n\} \), which are defined and denoted by (portfolio version of (3.3))

\[
\mu \Delta S^I_j = \mu S^I_j - \mu S^I_{j-1},
\]

\[
\sigma \Delta S^I_j = \sqrt{(k S^I_{j-1} \mu S^I_{j-1})^2 - 2 \rho S^I_j (k S^I_{j-1} \mu S^I_{j-1}) (k S^I_j \mu S^I_j) + (k S^I_j \mu S^I_j)^2}.
\]  (4.1)

For a single XL contract these summary statistics are evaluated using the formulas (3.5)-(3.6).

4.2. Bivariate lognormal approximation

We assume that the random vector defined by \( (e^{X(j-1)}, e^{X(j)}) = (S^I(j-1), S^I(j)), j \in \{2,...,n\} \), follows a bivariate lognormal distribution. This means that the random pairs \( (X(j-1), X(j)) \) are bivariate normally distributed with normal marginal distributions \( X(j) \sim N(\mu, \nu) \) and
correlation coefficient $\rho_j$ between $X(j-1)$ and $X(j)$. The parameters $\mu_j, \nu_j, \rho_j$ are determined by the formulas (use (3.3)-(3.6) for a single XL contract)

$$
\nu_j = \sqrt{\ln(1 + kS_j^f)}, \quad \mu_j = \ln(\mu S_j^f) - \frac{1}{2} \nu_j^2, \quad j = 1, \ldots, n,
$$

$$
\rho_j = \frac{\ln(1 + \rho S_j^f kS_j^f kS_j^f)}{\nu_j - \nu_j}, \quad j = 2, \ldots, n. \quad (4.2)
$$

To determine the VaR and CVaR risk measures of the bivariate lognormal spreads $\Delta S^f(j) = S^f(j) - S^f(j-1), j = 2, \ldots, n$, we use one of the three numerical methods presented in Appendix B to determine their distribution functions and stop-loss transforms required in the general formulas of Appendix A.

4.3. Numerical illustration

To illustrate the proposed reinsurance contract valuation for solvency purposes, let us concentrate on a portfolio of identical XL contracts with inflation stability clause starting at the current time of valuation $t = 0$. It is characterized by the original deductible $\ell = 100$, limit $m = 100$, and the compound Poisson parameters $(\lambda, \gamma, \gamma) = (10, 100, 1.5)$. We determine VaR and CVaR SCR’s for the current and all future years up to the final settlement year $n = 7$. The claims payment pattern and the reserve deviation pattern are given by the following vectors

$$
c = (0.25, 0.2, 0.1, 0.1, 0.1, 0.05), \quad d = (1.25, 1.2, 1.15, 1.1, 1.05, 1, 1). \quad (4.3)
$$

We assume constant inflation and superimposed inflation such that the corresponding indices are given by $f(j) = 1.03^j, s(j) = 1.045^j, j = 0, 1, \ldots, 7$. Table 4.1 lists the inflation adjusted factors and the stability clause ratios required in risk calculations. Table 4.2 contains the distribution parameters used in the extended and bivariate lognormal approximations of the incurred loss spreads. The Tables 4.3 and 4.4 present the obtained SCR values for the different approximation methods. Since the mean of the future spreads beyond the current year are negative, it is necessary to make use of the Case 2 in Appendix A.

**Table 4.1**: inflation adjusted factors and stability clause ratios

<table>
<thead>
<tr>
<th>development time</th>
<th>inflation adjusted factors</th>
<th>stability clause ratios</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>cumulative paid</td>
<td>incurred</td>
</tr>
<tr>
<td>1</td>
<td>0.261</td>
<td>1.372</td>
</tr>
<tr>
<td>2</td>
<td>0.480</td>
<td>1.284</td>
</tr>
<tr>
<td>3</td>
<td>0.708</td>
<td>1.216</td>
</tr>
<tr>
<td>4</td>
<td>0.827</td>
<td>1.182</td>
</tr>
<tr>
<td>5</td>
<td>0.952</td>
<td>1.160</td>
</tr>
<tr>
<td>6</td>
<td>1.082</td>
<td>1.150</td>
</tr>
<tr>
<td>7</td>
<td>1.150</td>
<td>1.150</td>
</tr>
</tbody>
</table>
Table 4.2: means, coefficients of variation, and correlation coefficients for incurred losses

<table>
<thead>
<tr>
<th>development time</th>
<th>mean</th>
<th>coefficient of variation</th>
<th>correlation coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>927.806</td>
<td>0.303</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>834.297</td>
<td>0.322</td>
<td>0.99396</td>
</tr>
<tr>
<td>3</td>
<td>763.579</td>
<td>0.339</td>
<td>0.99548</td>
</tr>
<tr>
<td>4</td>
<td>728.639</td>
<td>0.348</td>
<td>0.99867</td>
</tr>
<tr>
<td>5</td>
<td>704.402</td>
<td>0.356</td>
<td>0.99915</td>
</tr>
<tr>
<td>6</td>
<td>691.358</td>
<td>0.362</td>
<td>0.99960</td>
</tr>
<tr>
<td>7</td>
<td>689.061</td>
<td>0.364</td>
<td>0.99996</td>
</tr>
</tbody>
</table>

Table 4.3: insurance risk SCR approximations for the VaR risk measure

<table>
<thead>
<tr>
<th>development time</th>
<th>extended lognormal method</th>
<th>relative deviation</th>
<th>bivariate lognormal methods</th>
<th>numerical integration</th>
<th>second-order approximation</th>
<th>first-order approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>extended lognormal</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>978.389</td>
<td>0.0%</td>
<td>978.389</td>
<td>978.389</td>
<td>978.389</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>56.810</td>
<td>24.5%</td>
<td>75.217</td>
<td>75.745</td>
<td>73.344</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>45.078</td>
<td>28.6%</td>
<td>63.149</td>
<td>63.427</td>
<td>62.062</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>23.032</td>
<td>31.5%</td>
<td>33.615</td>
<td>33.665</td>
<td>33.389</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>16.860</td>
<td>38.7%</td>
<td>27.519</td>
<td>27.401</td>
<td>28.379</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>10.001</td>
<td>51.2%</td>
<td>20.509</td>
<td>20.074</td>
<td>24.330</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>2.122</td>
<td>74.0%</td>
<td>8.152</td>
<td>7.111</td>
<td>15.824</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.4: insurance risk SCR approximations for the CVaR risk measure

<table>
<thead>
<tr>
<th>development time</th>
<th>extended lognormal method</th>
<th>relative deviation</th>
<th>bivariate lognormal methods</th>
<th>numerical integration</th>
<th>second-order approximation</th>
<th>first-order approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>extended lognormal</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1038.392</td>
<td>0.0%</td>
<td>1038.392</td>
<td>1038.392</td>
<td>1038.392</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>57.721</td>
<td>27.5%</td>
<td>79.637</td>
<td>78.831</td>
<td>76.922</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>45.753</td>
<td>31.9%</td>
<td>67.214</td>
<td>66.649</td>
<td>66.922</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>23.359</td>
<td>34.9%</td>
<td>35.904</td>
<td>35.721</td>
<td>35.812</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>17.077</td>
<td>42.2%</td>
<td>29.542</td>
<td>29.539</td>
<td>29.540</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>10.103</td>
<td>54.4%</td>
<td>22.138</td>
<td>21.899</td>
<td>21.915</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>2.130</td>
<td>75.9%</td>
<td>8.833</td>
<td>9.570</td>
<td>8.099</td>
<td></td>
</tr>
</tbody>
</table>

This numerical example shows that the extended lognormal approximation clearly underestimates the SCR for future years when compared to the bivariate lognormal approximation. As shown in the Tables 4.5 and 4.6 below, it leads to a relative deviation of nearly 29% for the risk margin (with a 6% cost-of-capital rate and a 3% risk-free interest).
However, comparing figures for the EC only, it turns out that this relative deviation shrinks to negligible 0.14%, which is of course due to the relatively small required risk margin. Further, we observe that the second- and first-order approximations to the bivariate lognormal spread yield good approximations of the risk margin, especially when using the CVaR risk measure. As a future work, it would be interesting to compare the bivariate lognormal and other similar analytical approximations with the SCR and EC values obtained applying the bivariate Panjer algorithm of Sundt (1999). As a conclusion we encourage actuaries to use bivariate spread distribution models for an accurate evaluation of excess-of-loss reinsurance risk margins.

**Table 4.5:** insurance risk EC calculations for the VaR measure

<table>
<thead>
<tr>
<th>development time</th>
<th>discounted SCR values at times of development for extended and bivariate lognormal models</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>extended</td>
</tr>
<tr>
<td>1</td>
<td>949.892</td>
</tr>
<tr>
<td>2</td>
<td>53.549</td>
</tr>
<tr>
<td>3</td>
<td>41.253</td>
</tr>
<tr>
<td>4</td>
<td>20.464</td>
</tr>
<tr>
<td>5</td>
<td>14.544</td>
</tr>
<tr>
<td>6</td>
<td>8.376</td>
</tr>
<tr>
<td>7</td>
<td>1.725</td>
</tr>
<tr>
<td>RM</td>
<td>3.316</td>
</tr>
<tr>
<td>EC</td>
<td>953.209</td>
</tr>
<tr>
<td>abs.dev. RM</td>
<td>28.70%</td>
</tr>
<tr>
<td>abs.dev. EC</td>
<td>0.14%</td>
</tr>
</tbody>
</table>

**Table 4.6:** insurance risk EC calculations for the CVaR measure

<table>
<thead>
<tr>
<th>development time</th>
<th>discounted SCR values at times of development for extended and bivariate lognormal models</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>extended</td>
</tr>
<tr>
<td>1</td>
<td>1008.148</td>
</tr>
<tr>
<td>2</td>
<td>54.408</td>
</tr>
<tr>
<td>3</td>
<td>41.870</td>
</tr>
<tr>
<td>4</td>
<td>20.754</td>
</tr>
<tr>
<td>5</td>
<td>14.731</td>
</tr>
<tr>
<td>6</td>
<td>8.461</td>
</tr>
<tr>
<td>7</td>
<td>1.732</td>
</tr>
<tr>
<td>RM</td>
<td>3.368</td>
</tr>
<tr>
<td>EC</td>
<td>1011.516</td>
</tr>
<tr>
<td>abs.dev. RM</td>
<td>31.74%</td>
</tr>
<tr>
<td>abs.dev. EC</td>
<td>0.15%</td>
</tr>
</tbody>
</table>
Appendix A: VaR and CVaR for loss random variables with positive and negative means

Let $(\Omega, A, P)$ be a probability space such that $\Omega$ is the sample space, $A$ is the $\sigma$-field of events and $P$ is the probability measure. For a measurable real-valued random variable $Z$ on this probability space, that is a map $Z: \Omega \to \mathbb{R}$, the probability distribution of $Z$ is defined and denoted by $F_Z(z) = P(Z \leq z)$, where $z$ is any real number. In the present paper $Z$ represents a loss random variable such that for $\omega \in \Omega$ the real number $Z(\omega)$ is the realization of a loss and profit function with $Z(\omega) \geq 0$ for a loss and $Z(\omega) < 0$ for a profit. The negative random variable $-Z$ represents a profit. The survival distribution and stop-loss transform of $Z$ are defined and denoted by $F_Z(z) = 1 - F_Z(z)$ respectively $\pi_z(z) = E[(Z - z) +]$.

We assume continuous distribution functions and finite means. Given the confidence level $\alpha$ the value-at-risk (VaR) and conditional value-at-risk (CVaR) of $Z$ are defined and denoted by $\text{VaR}_\alpha[Z] = F_Z^{-1}(\alpha)$ and $\text{CVaR}_\alpha[Z] = E[Z|Z > \text{VaR}_\alpha[Z]]$. The VaR solvency capital requirement (SCR) of the loss $Z$ is defined to be the VaR of the unexpected loss random variable $Z - E[Z]$ and is denoted by $\text{SCR}^{\text{VaR}}_\alpha[Z] = \text{VaR}_\alpha[Z - E[Z]]$. Similarly, the CVaR solvency capital requirement of the loss is defined and denoted by $\text{SCR}^{\text{CVaR}}_\alpha[Z] = \text{CVaR}_\alpha[Z - E[Z]]$. The evaluation of SCR’s is usually based on the following result, which provides a duality between SCR’s for losses and profits with non-zero means.

Proposition A.1. Let $Z$ be a loss random variable with continuous distribution function and non-zero finite mean. Let $\varepsilon = 1 - \alpha$ be the loss probability associated to the confidence level $\alpha$. Then one has the following SCR formulas.

Case 1: $E[Z] > 0$

$$\text{SCR}^{\text{VaR}}_\alpha[Z] = \text{VaR}_\alpha[Z] - E[Z]$$
$$\text{SCR}^{\text{CVaR}}_\alpha[Z] = \text{SCR}^{\text{VaR}}_\alpha[Z] + \frac{1}{\varepsilon} \pi_z(\text{VaR}_\alpha[Z]) \quad (A.1)$$

Case 2: $E[Z] < 0$

$$\text{SCR}^{\text{VaR}}_\alpha[Z] = E[-Z] - \text{VaR}_\varepsilon[-Z]$$
$$\text{SCR}^{\text{CVaR}}_\alpha[Z] = \text{SCR}^{\text{VaR}}_\alpha[Z] + \frac{1}{\varepsilon} \{\pi_z(\text{VaR}_\varepsilon[-Z]) - \text{SCR}^{\text{VaR}}_\alpha[Z]\} \quad (A.2)$$

Proof. (A.1) is well-known. The relations (A.2) are obtained using the relationships

$\text{VaR}_\alpha[Z] = -\text{VaR}_\varepsilon[-Z], \quad E[Z|Z > \text{VaR}_\alpha[Z]] = -E[-Z|Z < \text{VaR}_\varepsilon[-Z]]. \quad \Diamond$

The usefulness of Proposition A.1 lies in the fact that it reduces the calculation of SCR’s for losses with negative means to the calculation of SCR’s for profits with positive means. We illustrate for a lognormal approximation of losses and profits, an assumption which for losses
with positive means is compliant with the current Solvency II specification of the first year non-
life insurance risk SCR in the sense that its distribution is assumed to be lognormal.

**Proposition A.2.** Let \( Z \) be a loss random variable with non-zero finite mean and variance. Let \( \varepsilon = 1 - \alpha \) be the loss probability associated to the confidence level \( \alpha \). We distinguish between two cases.

Case 1: \( E[Z] > 0 \). Assume that the loss \( Z \) has a lognormal distribution with mean \( \mu_+ = E[Z] > 0 \) and coefficient of variation \( k_+ = \sqrt{\text{Var}[Z]/\mu_+} \). Then one has the SCR formulas:

\[
\text{SCR}^{\text{Var}}_\alpha [Z] = \left( \frac{\exp\{\Phi^{-1}(\alpha) \cdot \sqrt{\ln(1+k_+^2)}\} - 1}{\sqrt{1+k_+^2}} \right) \cdot \mu_+ \tag{A.3}
\]

\[
\text{SCR}^{\text{CVaR}}_\alpha [Z] = \left( \frac{1 - \exp\{\Phi^{-1}(\alpha) \cdot \sqrt{\ln(1+k_+^2)}\} \cdot \frac{1}{\varepsilon} - \frac{1}{\varepsilon}}{1 - \frac{1}{\varepsilon}} \right) \cdot \mu_+ \tag{A.4}
\]

Case 2: \( E[Z] < 0 \). Assume that the profit \( -Z \) has a lognormal distribution with mean \( \mu_- = E[-Z] > 0 \) and coefficient of variation \( k_- = \sqrt{\text{Var}[Z]/\mu_-} \). Then one has the SCR formulas:

\[
\text{SCR}^{\text{Var}}_\alpha [Z] = \left( 1 - \frac{\exp\{\Phi^{-1}(\varepsilon) \cdot \sqrt{\ln(1+k_-^2)}\} - 1}{\sqrt{1+k_-^2}} \right) \cdot \mu_-
\]

\[
\text{SCR}^{\text{CVaR}}_\alpha [Z] = \left( 1 - \frac{\Phi^{-1}(\varepsilon) - \sqrt{\ln(1+k_-^2)} \cdot \frac{1}{\varepsilon}}{\varepsilon} \right) \cdot \mu_-
\]

**Proof.** In Case 1 write \( Z - E[Z] = (X - 1) \cdot \mu_+ \) with \( X = Z / \mu_+ \). From the lognormal assumption \( \ln X \sim N(\gamma, \beta) \) and the conditions \( E[X] = e^{\gamma + \frac{1}{2} \beta^2} = 1, k_+^2 = e^{\beta^2} - 1 \), one gets the relationships \( \gamma = -\frac{1}{2} \beta^2, \beta^2 = \ln(1+k_+^2) \). Inserting into the SCR VaR functional (A.1), one gets \( \text{SCR}^{\text{Var}}_\alpha [Z] = (\text{VaR}_\alpha[X] - 1) \cdot \mu_+ = \left( \exp\{\beta \cdot \Phi^{-1}(\alpha) - \frac{1}{2} \beta^2 \} - 1 \right) \cdot \mu_+ \), which implies the first formula in (A.3). For the SCR conditional value-at-risk functional one obtains from (A.1) that

\[
\text{SCR}^{\text{CVaR}}_\alpha [Z] = \left( \text{VaR}_\alpha[X] - 1 + \varepsilon^{-1} \cdot \pi_X \left[ \text{VaR}_\alpha[X] \right] \right) \cdot \mu_+.
\]

The second formula in (A.3) is obtained by noting that

\[
\pi_X \left[ \text{VaR}_\alpha[X] \right] = E[X] \cdot \Phi\left( \frac{\gamma - \ln(\text{VaR}_\alpha[X])}{\beta} + \beta \right) - \text{VaR}_\alpha[X] \cdot \Phi\left( \frac{\gamma - \ln(\text{VaR}_\alpha[X])}{\beta} \right)
\]

\[
= \Phi(\beta - \Phi^{-1}(\alpha)) - \Phi(-\Phi^{-1}(\alpha)) \cdot \text{VaR}_\alpha[X] = 1 - \Phi(\Phi^{-1}(\alpha) - \sqrt{\ln(1+k_+^2)}) - \varepsilon \cdot \text{VaR}_\alpha[X].
\]
In Case 2 set \( X = -Z / \mu \). If \( \ln X \sim N(\gamma, \beta) \) one has similarly to the above \( \gamma = -\frac{1}{2} \beta^2, \beta^2 = \ln(1 + k^2) \), hence \( \text{VaR}_x[-Z] = \text{VaR}_x[X] \cdot \mu = \exp\{\beta \cdot \Phi^{-1}(\epsilon) - \frac{1}{2} \beta^2 \} \cdot \mu \). The first formula in (A.4) follows by insertion into (A.2). Similarly, the second formula in (A.4) follows from (A.2) by noting that

\[
\text{SCR}^\text{VaR}_x[Z] = \left\{1 - \text{VaR}_x[X] + \epsilon^{-1} \cdot (\pi_x[\text{VaR}_x[X]] + \text{VaR}_x[X] - 1)\right\} \cdot \mu, \text{ with }
\pi_x[\text{VaR}_x[X]] = 1 - \Phi\left(\Phi^{-1}(\epsilon) - \sqrt{\ln(1 + k^2)}\right) - (1 - \epsilon) \cdot \text{VaR}_x[X].
\]

\[\Diamond\]

Appendix B: distribution and stop-loss transform of the bivariate lognormal spread

Suppose that the random vector \( (S_1, S_2) \) has a bivariate lognormal distribution with parameter vector \( (\mu_1, \nu_1, \mu_2, \nu_2, \rho) \) such that the standardized random vector

\[
(Z_1, Z_2) = \left(\frac{\ln S_1 - \mu_1}{\nu_1}, \frac{\ln S_2 - \mu_2}{\nu_2}\right)
\]

has a bivariate normal distribution with correlation coefficient \( \rho \) and standard normal marginal densities. Our goal is a fast and accurate numerical evaluation of the survival distribution and stop-loss transform of the bivariate lognormal spread, which are defined by

\[
\bar{F}(z) = P(S_1 - S_2 \geq z), \quad \pi(z) = \mathbb{E}[\{(S_1 - S_2 - z)^+\}],
\]

for any real number \( z \). If suffices to consider \( z \geq 0 \). Indeed, the case \( z < 0 \) can be reduced to the case \( z > 0 \) by making use of the identities

\[
1_{[S_1 - S_2 \geq z]} = 1 - 1_{[S_1 - S_2 \leq -z]}, \quad (S_1 - S_2 - z)_+ = S_1 - S_2 - z + (S_2 - S_1 + z)_+.
\]

Starting point is Proposition 1 in Deng et al.(2008).

**Proposition B.1.** The survival distribution and stop-loss transform of the bivariate lognormal spread with \( |\rho| < 1 \) satisfy the following integral representations

\[
\bar{F}(z) = -\pi'(z) = \int_{-\infty}^{\infty} \Phi(A(y, z)) \cdot \varphi(y) dy,
\]

\[
\pi(z) = \exp\{\mu_1 + \frac{1}{2} \nu_1^2\} \cdot I_1(z) - \exp\{\mu_2 + \frac{1}{2} \nu_2^2\} \cdot I_2(z) - z \cdot \bar{F}(z),
\]

\[
I_1(z) = \int_{-\infty}^{\infty} \Phi\left(A(y + \rho \nu_1, z + \sqrt{1 - \rho^2} \cdot \nu_1\right) \cdot \varphi(y) dy, \quad I_2(z) = \int_{-\infty}^{\infty} \Phi\left(A(y + \nu_2, z\right) \cdot \varphi(y) dy,
\]

\[\Box\]
with \( \Phi(z) \) the standard normal distribution, \( \varphi(z) = \Phi'(z) \), and \( A(\cdot, \cdot) \) the auxiliary function

\[
A(y, z) = \frac{\mu_1 + \rho \nu_1 y - \ln(z + \exp(\mu_2 + \nu_2 y))}{\nu_1 \sqrt{1 - \rho^2}}.
\]  

Proposition B.1 yields the formula by Margrabe (1978) (Proposition 2 in Deng et al. (2008)):

\[
\pi(0) = \exp\left[\mu_1 + \frac{1}{2} \nu_1^2 - \frac{1}{2} \nu_2 \rho \nu_1 \nu_2 \right] \cdot \Phi\left( \frac{\mu_1 - \mu_2 - \nu_1^2 + \rho \nu_1 \nu_2}{\sqrt{\nu_1^2 + \nu_2^2 - 2 \rho \nu_1 \nu_2}} \right) - \exp\left[\mu_2 + \frac{1}{2} \nu_2^2 \right] \cdot \Phi\left( \frac{\mu_1 - \mu_2 - \nu_2^2 + \rho \nu_1 \nu_2}{\sqrt{\nu_1^2 + \nu_2^2 - 2 \rho \nu_1 \nu_2}} \right)
\]  

(B.6)

A first method to evaluate (B.2) is numerical integration of one-dimensional infinite integrals. This functionality is available in modern computer algebra systems (e.g. Mathcad 14 from MathSoft) and provides numerical results, which can be used to test two simple closed-form approximations, which are based on a linear and quadratic approximation of the auxiliary function (B.5). Following Deng et al. (2008), Section IV.A, let us expand the auxiliary function \( A(y + c, z) \) around \( y = y_0 - c \) with \( y_0 = 0 \) such that

\[
A(y + c, z) = A(0, z) + \frac{\partial A(y, z)}{\partial y} \bigg|_{y=0} \cdot (y + c) + \frac{1}{2} \frac{\partial^2 A(y, z)}{\partial y^2} \bigg|_{y=0} \cdot (y + c)^2
\]

to get the following quadratic approximation

\[
A(y + c, z) = a(z) + b(z)c + \delta(z)c^2 + \left[ b(z) + 2c \delta(z) \right]y + \delta(z)y^2,
\]

\[
a(z) = A(0, z) = \frac{\mu_1 - \ln(z + \exp(\mu_2))}{\nu_1 \sqrt{1 - \rho^2}}, \quad b(z) = \frac{\partial A(y, z)}{\partial y} \bigg|_{y=0} = \frac{\rho \nu_1 - \nu_2 \exp(\mu_1)}{\nu_1 \sqrt{1 - \rho^2}}, \quad \delta(z) = \frac{1}{2} \frac{\partial^2 A(y, z)}{\partial y^2} \bigg|_{y=0} = -\frac{1}{2\nu_1 \sqrt{1 - \rho^2}} \cdot \frac{\nu_2 \exp(\mu_2) z}{(z + \exp(\mu_2))^2}.
\]  

(B.7)

One shows the following result.

**Proposition B.2.** Let \( z \geq 0 \) and \( \lvert \rho \rvert < 1 \). The survival distribution and stop-loss transform of the bivariate lognormal spread satisfy the second-order closed-form approximation formulas

\[
\bar{F}(z) \equiv IQ_3(z), \quad \pi(z) \equiv \exp\left[\mu_1 + \frac{1}{2} \nu_1^2 \right] - \exp\left[\mu_2 + \frac{1}{2} \nu_2^2 \right] \cdot IQ_2(z) - \exp(\mu_2 + \nu_2 \rho \nu_1) \cdot IQ_{\nu_2}(z), \\
IQ_{\nu_2}(z) = J_0(C_{\nu_2}(z), D_{\nu_2}(z)) + J_1(C_{\nu_2}(z), D_{\nu_2}(z)) \cdot \delta(z) + \frac{1}{2} J_2(C_{\nu_2}(z), D_{\nu_2}(z)) \cdot \delta(z)^2, \quad \delta(z) \equiv \frac{\nu_2 \exp(\mu_2)}{\nu_1 \sqrt{1 - \rho^2}} (z + \exp(\mu_2))^2
\]

(B.8)

with
\[ J_0(u,v) = \Phi\left( \frac{u}{\sqrt{1 + v^2}} \right) \]
\[ J_1(u,v) = \frac{1 + (1 + u^2)v^2}{(1 + v^2)^{5/2}} \cdot \varphi\left( \frac{u}{\sqrt{1 + v^2}} \right) \]
\[ J_2(u,v) = \frac{(6 - 6u^2)v^2 + (21 - 2u^2 - u^4)v^2 + 4(3 + u^2)v^6 - 3}{(1 + v^2)^{5/2}} \cdot \varphi\left( \frac{u}{\sqrt{1 + v^2}} \right) \]
\[ C_1(z) = C_1(z) + D_3(z) \cdot \rho v_1 + \delta(z) \cdot \rho^2 v_1^2 + \sqrt{1 - \rho^2} \cdot v_1, \quad D_1(z) = D_3(z) + 2\delta(z) \cdot \rho v_1, \quad (B.9) \]
\[ C_2(z) = C_3(z) + D_3(z) \cdot v_2 + \delta(z) \cdot v_2^2, \quad D_2(z) = D_3(z) + 2\delta(z) \cdot v_2, \]
\[ C_3(z) = a(z), \quad D_3(z) = b(z). \]

**Proof.** See the proof of Proposition 6 in the Appendix of Deng et al.(2008).

Setting \( \delta(z) = 0 \) into the preceding result one obtains the following simpler approximation.

**Corollary B.3.** The survival distribution and stop-loss transform of the bivariate lognormal spread satisfy the *first-order closed-form approximation* formulas

\[ \overline{F}(z) \equiv \Phi\left( \frac{a(z)}{\sqrt{1 + b(z)^2}} \right) \]
\[ \pi(z) \equiv \exp\left( \mu_1 + \frac{1}{2} v_1^2 \right) \cdot \Phi\left( \frac{a(z) + b(z) \cdot \rho v_1 + \sqrt{1 - \rho^2} \cdot v_1}{\sqrt{1 + b(z)^2}} \right) \]
\[ - \exp\left( \mu_2 + \frac{1}{2} v_2^2 \right) \cdot \Phi\left( \frac{a(z) + b(z) \cdot v_2}{\sqrt{1 + b(z)^2}} \right) - z \cdot \varphi\left( \frac{a(z)}{\sqrt{1 + b(z)^2}} \right) \]

(B.10)

Some remarkable properties are noticed. First, one recovers Margrabe’s formula (B.6) in the special case \( z = 0 \). Second, asymptotically as \( z \to \infty \) these approximations satisfy the property \( \overline{F}(z) \to 0, \pi(z) \to 0 \), which must be fulfilled for VaR and CVaR calculations.

**References**


