Measuring the Impact of Inflation
on Undiscounted Loss Reserves

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1 Introduction

The paper develops an approach to measure the impact of inflation on the undiscounted loss reserves of a Property & Casualty insurance company. A case study is presented that examines a book of business comprising two subportfolios, fire and third party liability business. Results on expected value and standard deviation of loss reserves are presented. No margins, neither prudential nor market value margin, are considered. Cash flows are not discounted.

First, nominal paid triangles are converted into real paid triangles. Second, inflation indices and real cash flows (emanating from claims that occurred in the past) are projected into the future. Third, nominal future cash flows are calculated. Finally, results in terms of first and second moments are derived and compared with results of ‘classical’ approaches.

Inflation indices are modeled explicitly by means of a multivariate time series approach. The parameterization of the model spans historical experience of fifty years. The approach to project real runoff payments translates the methods of T. Mack and C. Braun, resp., into a simulation approach. Each path of the projection is the result of randomly drawn, correlated link ratios. The approach presented is relevant for local GAAP and IFRS accounting, as loss reserves for Property & Casualty lines are not discounted apart from certain exceptions like annuity reserves. In line with Solvency II, explicit assumptions about future inflation are integrated into the model.

2 Projection of Inflation

This section deals with modeling and projecting inflation. It starts with univariate models and then switches to multivariate models. Purely statistical
methods are applied to historical data. Macroeconomic theory is not used.

2.1 Univariate Autoregressive Processes

Inflation indices are generally non-stationary, i.e. the first two moments are not time invariant. Indices from the German market have been taken from [7] and are depicted in figure 1. Non-stationary time series can often be transformed into stationary ones by (logarithmic) differencing. Let $I_t$ be an inflation index in year $t$ and $y_t$ be the corresponding rate of inflation defined by

$$y_t = \ln(I_t) - \ln(I_{t-1}).$$

The impact of economic cycles and oil price shocks on the rate of inflation can be observed, see figure 2. Negative values are assumed several times. Wilkie, see [6], modeled consumer prices as autoregressive process of order one (AR(1)), i.e.,

$$y_t = \mu + a (y_{t-1} - \mu) + u_t,$$  \hspace{1cm} (1)
with $\mu$ and $a$ constant. $u_t$ is white noise with Variance $\sigma^2$, i.e. $E(u_t) = 0$, $\text{Var}(u_t) = \sigma^2$ and $\text{Cov}(u_tu_s) = 0$ for $t \neq s$.

Here, it is assumed that $u_t$ is normally distributed.

The process $y_t$ is called stable if $|a| < 1$. A stable process is stationary, that is, the first two moments of $y_t$ are time invariant. The first two moments are given by

$$E y_t = \mu \text{ for all } t$$

(2)

and

$$\gamma(h) := E[(y_t - \mu)(y_{t-h} - \mu)] = a^h \frac{\sigma^2}{1 - a^2} \text{ for all } t \text{ and } h.$$  

(3)

The results of a maximum likelihood estimation, see [5], of the parameters $\mu$, $\phi$ and $\sigma$ are displayed below. The estimation is based on data from 1960 to 2010:

<table>
<thead>
<tr>
<th>Rate of Inflation</th>
<th>$\hat{\mu}$</th>
<th>$\hat{a}$</th>
<th>$\hat{\sigma}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consumer Prices</td>
<td>0.0259</td>
<td>0.8100</td>
<td>0.00964</td>
</tr>
<tr>
<td>Producer Prices</td>
<td>0.0219</td>
<td>0.4271</td>
<td>0.02588</td>
</tr>
<tr>
<td>Labour Costs</td>
<td>0.0494</td>
<td>0.8309</td>
<td>0.01973</td>
</tr>
</tbody>
</table>

On average consumer prices grew by 2.6% per annum, producer prices by 2.2% and labour costs by 5.0%. The variance of labour costs exceeds the
variance of the other rates of inflation (formula 3 with \( h = 0 \)).

Assume \( y_t \) is stable. Let \( t \) be the forecast origin and \( h \) the forecast horizon. Then, see [1], the conditional expected value

\[
E_t (y_{t+h}) := E (y_{t+h} | \{ y_s | s \leq t \}) \tag{4}
\]

1. is an unbiased predictor of \( y_{t+h} \) and
2. minimizes the mean squared error (MSE).

The optimal \( h \)-step predictor of an AR(1) process \( y_t \) is given by

\[
E_t (y_{t+h}) = \mu + a (E_t (y_{t+h-1}) - \mu).
\]

Figure 3 displays the minimum MSE predictor \( E_t (y_{t+h}) \pm \) two standard deviations for a forecast horizon of ten years.

Modeling prices by AR-processes of higher order does not improve the ac-

![Figure 3: Forecast, 2011 - 2020](image)

...
$h = 1, 2, \ldots$, with independent standard normal variates $u_{t+h}$ and starting value $y_t$.

### 2.2 Vector Autoregressive Processes

In the multivariate case let the inflation index $I_t$ and the annual rate of inflation $y_t$ be given by,

$$I_t = (I_{1,t}, I_{2,t}, \ldots, I_{K,t})'$$  \hspace{1cm} (5)

and

$$y_t = (y_{1,t}, y_{2,t}, \ldots, y_{K,t})'$$  \hspace{1cm} (6)

resp., with

$$y_{i,t} = \ln(I_{i,t}) - \ln(I_{i,t-1}).$$  \hspace{1cm} (7)

A $K$-dimensional stochastic process $y_t = (y_{1,t}, \ldots, y_{K,t})'$ is a vector autoregressive process of order 1 ($VAR(1)$), if

$$y_t = \nu + Ay_{t-1} + u_t,$$  \hspace{1cm} (8)

where $A$ is a $(K \times K)$ coefficient matrix, $\nu = (\nu_{1,t}, \ldots, \nu_{K,t})'$ is a $K$-dimensional vector of intercept terms and $u_t = (u_{1,t}, \ldots, u_{K,t})'$ is $K$-dimensional white noise, i.e. $E(u_t) = 0$, $E(u_tu_t') = \Sigma_u$ and $E(u_tu_s') = 0$ for $s \neq t$.

It is assumed that $u_t$ is normally distributed and that the covariance matrix $\Sigma_u$ is nonsingular.

The following two conditions are equivalent, see [3],

1. All eigenvalues of matrix $A$ have modulus less than 1.

2. The reverse characteristic polynomial $\det(I_k - Az)$, $z \in \mathbb{C}$, has no roots in and on the complex unit circle.

If either of these conditions holds true, the sequence $A^i$, $i = 0, 1, \ldots$, is absolutely summable. Thus, the infinite sum $\sum_{i=1}^{\infty} A^iu_{t-i}$ exists in mean square and $y_t$ with

$$y_t = \mu + \sum_{i=0}^{\infty} A^i u_{t-i},$$  \hspace{1cm} (9)

$\mu := (I_K - A)^{-1} \nu$, is well-defined.

A $VAR(1)$-process is called stable if all eigenvalues of $A$ have modulus less
than 1, see [3]. A stable process is stationary, i.e., its first and second moments are time invariant. The first two moments are given by

$$E(y_t) = \mu \text{ for all } t$$

and

$$\Gamma_y(h) := E[(y_t - \mu)(y_{t-h} - \mu)'] = \sum_{i=0}^{\infty} A^{h+i} \Sigma_u (A^i)' \text{ for all } t \text{ and } h.$$  

If the VAR(1)-process $y_t$ is stable, the minimum mean squared error (MSE) predictor for forecast horizon $h$ at forecast origin $t$ is the conditional expected value

$$E_t(y_{t+h}) := E_t(y_{t+h}|\{y_s; s \leq t\}).$$  

This predictor minimizes the MSE in each component of $y_t$. The optimal $h$-step predictor of a VAR(1)-process $y_t$ is

$$E_t(y_{t+h}) = \mu + A (E_t(y_{t+h-1}) - \mu),$$

provided $u_t$ is independent white noise. The conditional expectation is an unbiased predictor, that is, $E[y_{t+h} - E_t(y_{t+h})] = 0$.

The table below shows the results of a maximum likelihood estimation of producer prices and labour costs modeled as a two-dimensional VAR(1)-process based on procedures described in [5]. Again, the estimation is based on years 1960 to 2010:

<table>
<thead>
<tr>
<th>Rate of Inflation</th>
<th>$\dot{\nu}$</th>
<th>$\hat{A}_1$</th>
<th>$\hat{A}_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Producer Prices</td>
<td>0.00687</td>
<td>0.36203</td>
<td>0.15267</td>
</tr>
<tr>
<td>Labour Costs</td>
<td>0.00857</td>
<td>-0.17876</td>
<td>0.87181</td>
</tr>
</tbody>
</table>

and

$$\Sigma_u = \frac{1}{10^4} \begin{pmatrix} 8.01461 & 0.57963 \\ 0.57963 & 3.67662 \end{pmatrix}.$$  

All eigenvalues of matrix $A$ have modulus less than one. Thus, the process is stable and stationary.

**Algorithm 2.2 (Projection of Inflation, Multivariate Model)**

1. Estimate parameters $\nu$, $A$ and $\Sigma_u$.

2. Perform a Cholesky decomposition of matrix $\Sigma_u$ to obtain the Cholesky factor $\Sigma_u^{1/2}$.  

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3. Calculate recursively,

\[ y_{t+h} = \hat{\nu} + \hat{\alpha}y_{t+h-1} + \sum_{u}^{1/2} u_{t+h}, \]

\[ h = 1, 2, \ldots, \] with independent standard normal variates \( u_{t+h} \) and starting value \( y_t \).

3 Projection of Cumulative Claims Paid

This section describes an approach to estimate undiscounted loss reserves. The general methodology is to convert nominal claim triangles into real ones, project real cash flows and inflation rates via simulation techniques into the future and then calculate future nominal cash flows path by path. It is assumed that the price level of a claim payment depends only on the year in which the payment is carried out. Let \( P_{t}^{\text{nom}} \) be a nominal payment made in year \( t \) and \( I_t \) the relevant inflation index of year \( t \). The real payment can be calculated by

\[ P_t^{\text{real}} = P_t^{\text{nom}} / I_t, \]

and vice versa,

\[ P_t^{\text{nom}} = P_t^{\text{real}} \times I_t. \]

This paper analyses the impact of inflation on fire and general third party liability business. Fire is assumed to be exposed to the producer price index and general third party liability (GTPL) business to the labour cost index. In the following sections approaches to project cumulative real claim payments are described.

3.1 Projection of Real Claim Payments

Consider a portfolio comprising two subportfolios. \( C_{ik}^{(j)} \) denotes the cumulative claim payments of accident year \( i \) after \( k \) years of development, \( 1 \leq i, k \leq n \), for subportfolio \( j \), \( 1 \leq j \leq 2 \). The amounts \( C_{ik}^{(j)} \) with \( i + k \leq n + 1 \) can be observed. The Chain Ladder model estimates future cumulative payments recursively by

\[ \hat{C}_{i,k+1} = \hat{C}_{i,k} \times \hat{f}_{k+1} \]
with starting values \( \dot{C}_{i,n+1-i}^{(j)} = C_{i,n+1-i}^{(j)} \) and \( \dot{f}^{(j)}_{k+1} = \frac{\sum_{i=1}^{n-k} C_{i,k+1}^{(j)}}{\sum_{i=1}^{n-k} C_{i,k}^{(j)}} \). Here, the amounts \( C_{i,k}^{(j)}, \ i + k > n + 1, \) are projected by applying simulation techniques. For each path of the projection link ratios are randomly drawn. The approach is based on the Chain Ladder model. According to [2] and [4] underlying stochastic assumptions of the Chain Ladder model are,

1. Accident years are independent
2. \( E \left( F_{i,k+1}^{(j)} | T_{i,k}^{(j)} \right) = f_{k+1}^{(j)} \)
3. \( Var \left( F_{i,k+1}^{(j)} | T_{i,k}^{(j)} \right) = \left( \sigma_{k+1}^{(j)} \right)^2 \frac{\sum_{i=1}^{n-k} C_{i,k}^{(j)}}{\sum_{i=1}^{n-k} C_{i,k}^{(j)}} \)
4. \( Cov \left( F_{i,k+1}^{(1)}, F_{i,k+1}^{(2)} | T_{i,k}^{(1)} \cup T_{i,k}^{(2)} \right) = \frac{\rho_{k+1}^{(1,2)}}{\sqrt{\sum_{i=1}^{n-k} C_{i,k}^{(1)} C_{i,k}^{(2)}}} \),

where

\[
F_{i,k+1}^{(j)} := \frac{C_{i,k+1}^{(j)}}{C_{i,k}^{(j)}} \quad \text{and} \quad T_{i,k}^{(j)} = \left\{ C_{i,j}^{(j)} ; 1 \leq j \leq k \right\}.
\]

Unbiased estimates for \( f_{k+1}^{(j)}, \sigma_{k+1}^{(j)} \) and \( \rho_{k+1}^{(1,2)} \) are, see [2] and [4],

\[
f_{k+1}^{(j)} = \frac{\sum_{i=1}^{n-k} C_{i,k+1}^{(j)}}{\sum_{i=1}^{n-k} C_{i,k}^{(j)}}, \quad \sigma_{k+1}^{(j)} = \frac{1}{n-k-1} \sum_{i=1}^{n-k} C_{i,k}^{(j)} \left( F_{i,k+1}^{(j)} - f_{k+1}^{(j)} \right)^2 \quad \text{and} \quad \rho_{k+1}^{(1,2)} = \frac{1}{n-k-w_{k+1}^2} \sum_{i=1}^{n-k} \sqrt{C_{i,k}^{(1)} C_{i,k}^{(2)}} \left( F_{i,k+1}^{(1)} - f_{k+1}^{(1)} \right) \left( F_{i,k+1}^{(2)} - f_{k+1}^{(2)} \right),
\]

with

\[
C_{i,k}^{(j)} := \sum_{i=1}^{n-k} C_{i,k}^{(j)} \quad \text{and} \quad w_{k+1}^2 = \frac{\sqrt{\sum_{i=1}^{n-k} C_{i,k}^{(1)} C_{i,k}^{(2)}}}{C_{i,k}^{(1)} C_{i,k}^{(2)}}.
\]
3.2 Projection for One Accident Year, One Portfolio

Since this section and the following section concentrate on a single portfolio, the index \((j)\) denoting the subportfolio is omitted.

The prediction error \(\text{mse} \left( \hat{C}_{i,n} \right)\) for the ultimate claims amount of a single accident year is defined as

\[
\text{mse} \left( \hat{C}_{i,n} \right) := E \left( \left( C_{i,n} - \hat{C}_{i,n} \right)^2 | T_n \right)
\]  

where

\[
T_n^{(j)} = \left\{ C_{i,j}^{(j)}; 1 \leq i \leq n, 1 \leq j \leq n, i + j \leq n + 1 \right\}.
\]

The prediction error according to formula 14 can be estimated recursively by, see [4],

\[
\hat{\text{mse}} \left( \hat{\hat{C}}_{i,k+1} \right) = \hat{\text{mse}} \left( \hat{\hat{C}}_{i,k} \right) \times \hat{f}_{k+1}^2 + \hat{\hat{C}}_{i,k}^2 \left( \frac{\sigma^2_{k+1}}{\hat{\hat{C}}_{i,k}} + \frac{\hat{\sigma}^2_{k+1}}{C_{<,k}} \right).
\]  

The algorithm below projects future cumulative claim payments by drawing link ratios randomly. The algorithm maintains the expected value and the prediction error of the stochastic Chain Ladder model.

**Algorithm 3.2: Projection of Cumulative Payments (One Accident Year, One Portfolio)**

1. Estimate parameters \(f_{k+1}\) and \(\sigma_{k+1}\), \(k = n + 1 - i, \ldots, n - 1\).

2. Calculate recursively,

\[
\bar{C}_{i,k+1} = \bar{C}_{i,k} \times \hat{f}_{k+1} + \bar{C}_{i,k} \sqrt{\frac{\sigma^2_{k+1}}{\bar{C}_{i,k}} + \frac{\hat{\sigma}^2_{k+1}}{C_{<,k}}} Z_{k+1},
\]

\(k = n + 1 - i, \ldots, n - 1\), with independent standard normal variates \(Z_{k+1}\) and starting value

\[
\bar{C}_{i,n+1-i} = C_{i,n+1-i}.
\]
3.3 Projection for All Accident Years, One Portfolio

The prediction error $mse\left(\sum_{i=2}^{n} \hat{C}_{i,n}\right)$ of the ultimate claims amount of all accident years is defined as

$$mse\left(\sum_{i=2}^{n} \hat{C}_{i,n}\right) := E\left(\sum_{i=2}^{n} \left(C_{in} - \hat{C}_{in}\right)^2 | T_n\right)$$  \hspace{1cm} (16)$$

with $$T_n = \{C_{i,j}; 1 \leq i \leq n, 1 \leq j \leq n, i + j \leq n + 1\}.$$ Recursively, see [4], the prediction error can be estimated by

$$\hat{mse}\left(\sum_{i=n+1-k}^{n} \hat{C}_{i,k+1}\right) = \hat{mse}\left(\sum_{i=n+2-k}^{n} \hat{C}_{i,k}\right) \times \hat{f}_{k+1}^2 + \left(\hat{C}_{\geq,k}\right)^2 \left(\frac{\hat{\sigma}_{k+1}^2}{\hat{C}_{\geq,k}} + \frac{\hat{\sigma}_{k+1}^2}{\hat{C}_{<,k}}\right).$$

To distribute the prediction error of the total ultimate claims amount to the accident years, the following rearrangement of the second summand of formula above is of interest,

$$\left(\hat{C}_{\geq,k}\right)^2 \left(\frac{\hat{\sigma}_{k+1}^2}{\hat{C}_{\geq,k}} + \frac{\hat{\sigma}_{k+1}^2}{\hat{C}_{<,k}}\right) = \sum_{i=n+1-k}^{n} \left(\hat{C}_{i,k} \hat{\sigma}_{k+1}^2 + \left(\hat{C}_{i,k} \sum_{j=n+1-k}^{n} \hat{C}_{j,k}\right) \frac{\hat{\sigma}_{k+1}^2}{\hat{C}_{<,k}}\right).$$

The algorithm projects the runoff of each accident year. The variance of the accident years sums up to the prediction error for all accident years.

Algorithm 3.3: Projection of Cumulative Payments (All Accident Years, One Portfolio)

1. Estimate parameters $f_{k+1}$ and $\sigma_{k+1}$, $k = 1, \ldots, n - 1$.

2. For each accident year $i$, $i = 2, \ldots, n$, calculate recursively,

$$\tilde{C}_{i,k+1} = \tilde{C}_{i,k} \times \hat{f}_{k+1} + \sqrt{\tilde{C}_{i,k} \hat{\sigma}_{k+1}^2 + \left(\tilde{C}_{i,k} \sum_{j=n+1-k}^{n} \tilde{C}_{j,k}\right) \frac{\hat{\sigma}_{k+1}^2}{\hat{C}_{<,k}}} Z_{i,k+1},$$

$k = n + 1 - i, \ldots, n - 1$, with independent standard normal variates $Z_{i,k+1}$ and starting values

$$\tilde{C}_{i,n+1-i} = C_{i,n+1-i}.$$
3.4 Projection for One Accident Year, Two Portfolios

In section 3.2 the variance of single accident years has been considered. Here, the covariance between different portfolios is taken into account. The covariance $\text{Cov}(\hat{C}_{i,n}^{(1)}, \hat{C}_{i,n}^{(2)})$ can be estimated recursively, see [2], by,

$$\text{Cov}(\hat{C}_{i,k+1}^{(1)}, \hat{C}_{i,k+1}^{(2)}) = \frac{\hat{C}_{i,k}^{(1)} \hat{C}_{i,k}^{(2)}}{\hat{C}_{<,k}^{(1)} \hat{C}_{<,k}^{(2)}} \rho_{k+1}^{(1,2)} \sum_{j=1}^{n-k} \sqrt{\hat{C}_{j,k}^{(1)} \hat{C}_{j,k}^{(2)}}.$$

Algorithm 3.4: Projection of Cumulative Payments (One Accident Year, Two Portfolio)

1. Estimate parameters $f_{k+1}^{(p)}$, $\sigma_{k+1}^{(p)}$ and $\rho_{k+1}^{(1,2)}$ for $k = n + 1 - i, \ldots, n - 1$ and $p \in \{1, 2\}$.

2. For each subportfolio $p \in \{1, 2\}$, calculate $s_{i,k+1}^{(p)}$ and $r_{i,k+1}^{(1,2)}$,

$$s_{i,k+1}^{(p)} = \left(\frac{\hat{C}_{i,k}^{(p)}}{\hat{C}_{i,k}^{(p)}}\right)^{2} \left(\frac{\sigma_{k+1}^{(p)}}{\hat{C}_{i,k}^{(p)}} + \frac{\sigma_{k+1}^{(p)}}{\hat{C}_{<,k}^{(p)}} \right)$$

and

$$r_{i,k+1}^{(1,2)} = \frac{\hat{C}_{i,k}^{(1)} \hat{C}_{i,k}^{(2)}}{\hat{C}_{<,k}^{(1)} \hat{C}_{<,k}^{(2)}} \rho_{k+1}^{(1,2)} \sum_{j=1}^{n-k} \sqrt{\hat{C}_{j,k}^{(1)} \hat{C}_{j,k}^{(2)}}.$$

$k = n + 1 - i, \ldots, n - 1$.

3. Perform a Cholesky decomposition of $\Sigma_{i,k+1}$, $k = n + 1 - i, \ldots, n - 1$, to obtain the Cholesky factor $\Sigma_{i,k+1}^{1/2}$ where

$$\Sigma_{i,k+1} = \begin{pmatrix} s_{i,k+1}^{(1)} & r_{i,k+1}^{(1,2)} \\ r_{i,k+1}^{(1,2)} & s_{i,k+1}^{(2)} \end{pmatrix}. \quad (17)$$

4. Generate independent standard normal vectors $Z_{k+1} = (Z_{k+1,1}, Z_{k+1,2})'$ for $k = n + 1 - i, \ldots, n - 1$. 

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5. Calculate recursively
\[
\begin{pmatrix}
\hat{C}_{i,k+1}^{(1)} \\
\hat{C}_{i,k+1}^{(2)}
\end{pmatrix} =
\begin{pmatrix}
\hat{C}_{i,k}^{(1)} & 0 \\
0 & \hat{C}_{i,k}^{(2)}
\end{pmatrix}
\begin{pmatrix}
\hat{f}_{k+1}^{(1)} \\
\hat{f}_{k+1}^{(2)}
\end{pmatrix} + \Sigma_{i,k+1}^{1/2}
\begin{pmatrix}
Z_{k+1,1} \\
Z_{k+1,2}
\end{pmatrix},
\]
(18)
k = n + 1 - i, \ldots, n - 1, with starting values
\[
\tilde{C}_{i,n+1-i} = C_{i,n+1-i}.
\]

3.5 Projection for All Accident Years, Two Portfolios

The covariance \(\text{Cov}\left(\sum_{i=n+1-k}^{n} \hat{C}_{i,k+1}^{(1)}, \sum_{i=n+1-k}^{n} \hat{C}_{i,k+1}^{(2)}\right)\) between ultimates of all accident years of different portfolios can be calculated recursively,
\[
\sum_{i,j=n+1-k}^{n} \text{Cov}\left(\hat{C}_{i,k+1}^{(1)} \hat{C}_{j,k+1}^{(2)}\right) = \sum_{i,j=n+2-k}^{n} \text{Cov}\left(\hat{C}_{i,k}^{(1)}, \hat{C}_{j,k}^{(2)}\right) \times \hat{f}_{k+1}^{(1)} \hat{f}_{k+1}^{(2)}
\]
\[
+ \frac{\hat{C}_{\geq,k} \hat{C}_{\geq,k}^{(1,2)} \rho_{k+1}}{C_{<,k}^{(1)} C_{<,k}^{(2)}} \sum_{i=1}^{n-k} \sqrt{\tilde{C}_{i,k}^{(1)} \tilde{C}_{i,k}^{(2)}}.
\]
To allocate the covariance of all accident years to the single accident years we rearrange,
\[
\hat{C}_{\geq,k} \hat{C}_{\geq,k}^{(1)} = \sum_{i=n+1-k}^{n} \left(\frac{1}{2} \tilde{C}_{i,k}^{(1)} \tilde{C}_{\geq,k} + \frac{1}{2} \tilde{C}_{\geq,k} \tilde{C}_{i,k}^{(2)}\right).
\]

Algorithm 3.5: Projection of Cumulative Payments (All Accident Years/Two Portfolios)

1. Calculate parameters \(f_{k+1}^{(p)}, \sigma_{k+1}^{(p)}\) and \(\rho_{k+1}^{(1,2)}\) for \(k = n + 1 - i, \ldots, n - 1\) and \(p \in \{1, 2\}\).

2. For each subportfolio \(p \in \{1, 2\}\), and each accident year \(i, i = 2, \ldots, n\) calculate \(s_{i,k+1}^{(p)}\) and \(r_{i,k+1}^{(1,2)}\),
\[
s_{i,k+1}^{(p)} = \left(\tilde{C}_{i,k}^{(p)}\right)^{2} \left(\tilde{\sigma}_{k+1}^{(p)}\right)^{2} + \left(\tilde{C}_{i,k}^{(p)} \sum_{j=n+1-k}^{n} \tilde{C}_{j,k}^{(p)}\right) \left(\tilde{\rho}_{k+1}^{(p)}\right)^{2}
\]
and
\[
r_{i,k+1}^{(1,2)} = \frac{1}{2} \tilde{C}_{i,k}^{(1)} \tilde{C}_{\geq,k}^{(2)} + \frac{1}{2} \tilde{C}_{\geq,k}^{(1)} \tilde{C}_{i,k}^{(2)} \rho_{k+1}^{(1,2)} \sum_{i=1}^{n-k} \sqrt{\tilde{C}_{i,k}^{(1)} \tilde{C}_{i,k}^{(2)}}.
\]
\[ k = n + 1 - i, \ldots, n - 1 \text{ and } \]
\[ \tilde{C}_{i,n+1-i} = C_{i,n+1-i}. \]

3. Perform a Cholesky decomposition of \( \Sigma_{i,k+1}, k = n + 1 - i, \ldots, n - 1, \) to obtain the Cholesky factor \( \Sigma_{i,k+1}^{1/2} \) where
\[
\Sigma_{i,k+1} = \begin{pmatrix}
\sigma_{i,k+1}^{(1)} & \rho_{i,k+1}^{(1,2)} \\
\rho_{i,k+1}^{(2,1)} & \sigma_{i,k+1}^{(2)}
\end{pmatrix},
\] (19)

4. Generate independent standard normal vectors \( Z_{k+1} = (Z_{k+1,1}, Z_{k+1,2})' \) for \( k = n + 1 - i, \ldots, n - 1. \)

5. Calculate recursively
\[
\begin{pmatrix}
\hat{C}_{i,k+1}^{(1)} \\
\hat{C}_{i,k+1}^{(2)}
\end{pmatrix} = \begin{pmatrix}
\hat{C}_{i,k}^{(1)} & 0 \\
0 & \hat{C}_{i,k}^{(2)}
\end{pmatrix} + \Sigma_{i,k+1}^{1/2} \begin{pmatrix}
Z_{k+1,1} \\
Z_{k+1,2}
\end{pmatrix},
\] (20)
\[ k = n + 1 - i, \ldots, n - 1, \text{ with starting values } \]
\[ \tilde{C}_{i,n+1-i} = C_{i,n+1-i}. \]

4 Results

This section compares the results of different methods to estimate the undiscounted loss reserves. The methods considered differ in terms of the treatment of subportfolios (taking covariance into account: yes/no), the approach for inflation (explicitly modeled: yes/no) and the general approach (closed formula/simulation techniques). The methods are:

A1: Treatment of subportfolios individually, no model for inflation, closed formula (prediction error according to T. Mack).

B1: Treatment of subportfolios individually, no model for inflation, simulation approach 3.3.

C1: Treatment of subportfolios individually, modeling inflation explicitly via AR(1)-process, simulation approach 3.3.

A2: Accounting for covariance between subportfolios, no model for inflation, closed formula (prediction error according to C. Braun).

B2: Accounting for covariance between subportfolios, no model for inflation,
C2: Accounting for covariance between subportfolios, modeling inflation explicitly via VAR(1)-process, simulation approach 3.5.

Approaches B1 and B2 project nominal payments into the future. Approaches C1 and C2 combine the projections of inflation rates and real payments. Fire is exposed to producer prices and general third party liability business is exposed to labour costs.

The table below shows the regular Chain Ladder reserve estimates per accident year and in total. Mean and standard deviation of undiscounted loss reserves are listed (method A1).

<table>
<thead>
<tr>
<th>Acc. Year</th>
<th>Fire Mean</th>
<th>Fire Std</th>
<th>GTPL Mean</th>
<th>GTPL Std</th>
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<tbody>
<tr>
<td>2000</td>
<td>0</td>
<td>0</td>
<td>168</td>
<td>14</td>
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<tr>
<td>2001</td>
<td>-0</td>
<td>0</td>
<td>363</td>
<td>23</td>
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<tr>
<td>2002</td>
<td>1</td>
<td>2</td>
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<td>2003</td>
<td>16</td>
<td>24</td>
<td>1,437</td>
<td>64</td>
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<td>2004</td>
<td>-59</td>
<td>152</td>
<td>1,078</td>
<td>62</td>
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<tr>
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<td>1,422</td>
<td>96</td>
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<td>2006</td>
<td>-4</td>
<td>179</td>
<td>3,400</td>
<td>256</td>
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<td>2007</td>
<td>54</td>
<td>251</td>
<td>1,960</td>
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<tr>
<td>2008</td>
<td>371</td>
<td>548</td>
<td>2,363</td>
<td>711</td>
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<td>2009</td>
<td>1,508</td>
<td>749</td>
<td>2,385</td>
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<td>2010</td>
<td>8,217</td>
<td>2,928</td>
<td>2,898</td>
<td>2,865</td>
</tr>
<tr>
<td>Total</td>
<td>10,016</td>
<td>3,202</td>
<td>18,007</td>
<td>3,187</td>
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</table>

The next table compares different methods in terms of mean and standard deviation. The simulated results of method B1 are close to the results of method A1 as expected. Modeling inflation explicitly (method C1) results in a higher standard deviation. For short term business (fire) the impact is moderate. However, for general third party liability (GTPL) the impact is impressive (+39%). The effect on the mean is mixed.
<table>
<thead>
<tr>
<th>Method</th>
<th>Fire Mean</th>
<th>Fire Std</th>
<th>GTPL Mean</th>
<th>GTPL Std</th>
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<td>3,202</td>
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<td>3,187</td>
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<tr>
<td>B1</td>
<td>10,018</td>
<td>3,205</td>
<td>18,009</td>
<td>3,191</td>
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<tr>
<td></td>
<td>(0.0%)</td>
<td>(0.1%)</td>
<td>(0.0%)</td>
<td>(0.1%)</td>
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<tr>
<td>C1</td>
<td>9,735</td>
<td>3,440</td>
<td>20,331</td>
<td>4,433</td>
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<tr>
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<td>(-2.8%)</td>
<td>(7.4%)</td>
<td>(12.9%)</td>
<td>(39.1%)</td>
</tr>
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Taking the covariance between the subportfolios into account and modeling inflation by a VAR(1)-process instead of an AR(1)-process supports the observations made before. Again, the results reveal that GTPL reserves tend to be deficient if inflation rates are not modeled explicitly. The standard deviation of fire and GTPL reserves are underestimated in the 'classical' approach A2.

<table>
<thead>
<tr>
<th>Method</th>
<th>Fire Mean</th>
<th>Fire Std</th>
<th>GTPL Mean</th>
<th>GTPL Std</th>
<th>Total Mean</th>
<th>Total Std</th>
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<td>3,202</td>
<td>18,007</td>
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<td>28,023</td>
<td>5,323</td>
</tr>
<tr>
<td>B2</td>
<td>10,023</td>
<td>3,206</td>
<td>18,020</td>
<td>3,191</td>
<td>28,042</td>
<td>5,317</td>
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<tr>
<td></td>
<td>(0.1%)</td>
<td>(0.1%)</td>
<td>(0.1%)</td>
<td>(0.1%)</td>
<td>(-0.1%)</td>
<td>(-0.1%)</td>
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<tr>
<td>C2</td>
<td>9,693</td>
<td>3,432</td>
<td>19,903</td>
<td>4,309</td>
<td>29,595</td>
<td>6,313</td>
</tr>
<tr>
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<td>(-3.2%)</td>
<td>(7.2%)</td>
<td>(10.5%)</td>
<td>(35.2%)</td>
<td>(5.6%)</td>
<td>(18.6%)</td>
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</table>

The implicit correlation between the subportfolios decreases in this example if inflation is modeled by a VAR(1)-process. It has to be kept in mind that fire and GTPL business are exposed to different inflation indices (producer price index and labour cost index) and that the tail character of the lines considered differ.
<table>
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<th>GTPL Std</th>
<th>Total Std</th>
<th>Implicit Correl</th>
<th>Total Std Correl=0</th>
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<td>5,323</td>
<td>38.8%</td>
<td>4,518</td>
<td>6,389</td>
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<td>38.5%</td>
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<td>(7.2%)</td>
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5 Literature


6 Appendix

6.1 Fire Business, Cumulative Claims Paid

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6.2 Third Party Liability Business, Cumulative Claims Paid

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