A two-dimensional risk model with proportional reinsurance

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Outline

- The bivariate insurance risk model
  - Proportional reinsurance
    - Avram et al. 2008 bivariate risk model
    - A bivariate risk model with two types of claims
    - Geometric arguments
    - Model constraint and absorbing sets
  - Auxiliary results
- Main result: Laplace transform of joint ruin time
The bivariate insurance risk model

- Surplus process \(\{Y^i_t\}_{t\geq 0}\) of \(i\)-th insurer \((i = 1, 2)\):
  \[
  Y^i_t = y^i_i + p^i_i t - C^i_t, \quad t \geq 0
  \]
  - \(y^i_i = Y^i_0 \geq 0\): Initial surplus
  - \(p^i_i > 0\): Net income rate per unit time
  - \(C^i_t\): Aggregate insurance claims until time \(t\)
    * \(\{C^i_t\}_{t\geq 0}\) is non-decreasing process with jumps only
  - \(\tau^i_i = \inf\{t \geq 0|Y^i_t < 0\}\): Time of ruin / default / bankruptcy
  - \(P_{y^i_i}(\tau^i_i < \infty) \equiv \Pr(\tau^i_i < \infty|Y^i_0 = y^i_i)\): Ruin probability
• Interested in joint ruin behaviour of the two risk processes

• Claims processes \( \{C_t^1\}_{t \geq 0} \) and \( \{C_t^2\}_{t \geq 0} \) are generally correlated
  – Due to ‘common shocks’ arising from catastrophes such as earthquakes or tsunamis
  – Due to ‘cost sharing’ via reinsurance contract (i.e. one of the insurers is a reinsurer)
Various ruin concepts for bivariate risk process \{ (Y^1_t, Y^2_t) \}_{t \geq 0} (Chan, Yang & Zhang (2003))

- \( \tau_{or} = \inf \{t \geq 0 | \min \{Y^1_t, Y^2_t\} < 0\} = \min(\tau_1, \tau_2) \)
  \( \Rightarrow \) First time when (at least) one process is below zero

- \( \tau_{and} = \max(\tau_1, \tau_2) \)
  \( \Rightarrow \) First time when ruin has occurred in both processes

- \( \tau_{sim} = \inf \{t \geq 0 | \max \{Y^1_t, Y^2_t\} < 0\} \)
  \( \Rightarrow \) First time both processes are below zero simultaneously

- \( \tau_{sum} = \inf \{t \geq 0 | Y^1_t + Y^2_t < 0\} \)
  \( \Rightarrow \) First time the sum of two processes becomes negative
• Analysis of $\tau_{\text{sum}}$ can usually be reduced to one-dimensional ruin problem (Chan, Yang & Zhang (2003) and Cai & Li (2005))

• Ruin probabilities for $\tau_{\text{or}}$, $\tau_{\text{sim}}$ or $\tau_{\text{and}}$ consist mainly of bounds, approximations and asymptotics (Chan, Yang & Zhang (2003), Cai & Li (2005, 2007), Yuen, Guo & Wu (2006), Li, Liu & Tang (2007), Avram, Palmowski & Pistorius (2008b), Dang, Zhu & Zhang (2009), Rabehasaina (2009) and Gong, Badescu & Cheung. (2010))

• In this talk, we focus on $\tau_{\text{or}}$, denoted by $\tau = \min(\tau_1, \tau_2)$
Proportional reinsurance

• Recall

\[ Y_t^i = y_i + p_i t - C_t^i \]

• Avram, Palmowski & Pistorius (2008)'s model:

- The 2nd insurer is a reinsurer engaged in proportional reinsurance with the 1st insurer, so that

\[
\begin{cases}
    C_t^1 = a L_t \\
    C_t^2 = (1 - a) L_t
\end{cases}
\]

- \( \{L_t\}_{t \geq 0} \) is a compound Poisson process with arrival rate \( \lambda_L \) and i.i.d. exponential secondary claim distribution
\[
\begin{align*}
\{ \quad & Y_t^1 = y_1 + p_1 t - a L_t \\
& Y_t^2 = y_2 + p_2 t - (1 - a) L_t \\
\{ \quad & (1 - a) Y_t^1 = (1 - a) y_1 + (1 - a) p_1 t - a (1 - a) L_t \\
& a Y_t^2 = a y_2 + a p_2 t - a (1 - a) L_t \\
\{ \quad & Y_{t*}^1 = y_{1*} + p_{1*} t - L_{t*} \\
& Y_{t*}^2 = y_{2*} + p_{2*} t - L_{t*} \\
\text{NOTE} \quad & \tau^* = \min(\tau_{1*}, \tau_{2*}) = \tau
\end{align*}
\]
\[ Y_{t}^{1*} \rightarrow Y_{t}^{2*} \]

\[ y_{1}^{*} \rightarrow y_{2}^{*} \]

\[ T = \frac{y_{1}^{*} - y_{2}^{*}}{p_{2}^{*} - p_{1}^{*}} \]
• Assume without loss of generality that \( y_1^* > y_2^* \) and \( p_1^* < p_2^* \).

• Letting \( T = \frac{y_1^* - y_2^*}{p_2^* - p_1^*} \)

  – if \( \tau \leq T \Rightarrow \tau = \tau_2 \)

  – if \( \tau > T \Rightarrow \tau = \tau_1 \)

  – we separate the problem in two univariate cases !!!!

• They obtained the Laplace transform of \( \tau \), namely

\[
E(y_1,y_2) \left[ e^{-\beta \tau} \mathbb{1}_{\{\tau < \infty\}} \right]
\]
A bivariate risk model with two types of claims

- We generalize Avram et al. (2008) in two ways:
  - 1st insurer faces an additional class of claims \( \{S_t\}_{t \geq 0} \)
    \[
    \begin{aligned}
    C_t^1 &= aL_t + S_t \\
    C_t^2 &= (1-a)L_t
    \end{aligned}
    \]
    * Only claims arising from \( \{ L_t \}_{t \geq 0} \) is reinsured proportionally
    * \( \{ S_t \}_{t \geq 0} \) is a compound Poisson process with arrival rate \( \lambda_S \), independent of \( \{ L_t \}_{t \geq 0} \)
  - The secondary claim distributions arising from both \( \{ S_t \}_{t \geq 0} \) and \( \{ L_t \}_{t \geq 0} \) are left arbitrary
Geometric arguments

• Plot evolution of $\{(Y^1_t, Y^2_t)\}_{t \geq 0}$ on Cartesian Plane

• Ruin time $\tau = \min(\tau_1, \tau_2)$ is the first exit from positive quadrant

• Define the line $\Delta$ whose equation is $Y^2_t = \frac{1-a}{a} Y^1_t$

• Let $\{X_t\}_{t \geq 0}$ be a process such that

\[
X_t = < (1-a, -a), (Y^1_t, Y^2_t) > \\
= x - ct - (1-a)S_t
\]

- $x = X_0 = (1-a)y_1 - ay_2$

- $c = ap_2 - (1-a)p_1$
\[ \Delta : Y_t^2 = \frac{1-a}{a} Y_t^1 \]

\( (y_1, y_2) \)

\( S_t \) claims

\( L_t \) claims
• $X_t$ is the relative distance between $(Y_t^1, Y_t^2)$ and $\Delta$ at time $t$

• $\Delta$ splits $\mathbb{R}^2$ in two disjoint sets:

  $\mathcal{A}^+ = \{ \vec{x} \in \mathbb{R}^2 | \langle \vec{x}, (1 - a, -a) \rangle > 0 \}$ is such that $X_t > 0$ is equivalent to $(Y_t^1, Y_t^2) \in \mathcal{A}^+$

  $\mathcal{A}^- = \{ \vec{x} \in \mathbb{R}^2 | \langle \vec{x}, (1 - a, -a) \rangle < 0 \}$ is such that $X_t < 0$ is equivalent to $(Y_t^1, Y_t^2) \in \mathcal{A}^-$

• Define ‘time of ruin’ of $\{X_t\}_{t \geq 0}$ by $\tau_X = \inf\{t \geq 0 | X_t < 0\}$

• Assume $\{(Y_t^1, Y_t^2)\}_{t \geq 0}$ starts in $\mathcal{A}^+$ (i.e. $x > 0$)
• The process \( \{(Y^1_t, Y^2_t)\}_{t \geq 0} \)
  - Drifts upwards as long as no claim occurs
  - Moves downwards in parallel to the line \( \Delta \) when a claim from \( \{L_t\}_{t \geq 0} \) occurs
  - Moves horizontally to the left when a claim from \( \{S_t\}_{t \geq 0} \) occurs

• Note a competition between \( \tau_X \) and \( \tau_2 \)
  \[ \Rightarrow \text{If } \tau_2 < \tau_X, \text{ then } \tau = \tau_2 \]
Additionally assume the condition

\[
\frac{p_2}{p_1} > \frac{1 - a}{a}
\]  

(1)

\[
\Rightarrow A^- \text{ is an absorbing set}
\]

\[
\Rightarrow \text{If } \tau_X < \tau_2, \text{ then } \tau = \tau_1
\]

Condition (1) holds when \( E[L_t] \gg E[S_t] \)

\[
\Rightarrow \text{Makes sense to reinsure claims arising from } \{L_t\}_{t \geq 0}
\]
• Recall

\[
\begin{aligned}
X_t &= x - ct - (1 - a)S_t \\
Y_t^2 &= y_2 + p_2t - (1 - a)L_t
\end{aligned}
\]

\[\Rightarrow \{X_t\}_{t \geq 0} \text{ and } \{Y_t^2\}_{t \geq 0} \text{ are independent}\]

\[\Rightarrow \tau_X \text{ and } \tau_2 \text{ are independent}\]

\[\Rightarrow \text{Easy to compare whether } \tau_2 < \tau_X \text{ or } \tau_X < \tau_2\]
Auxiliary results

- To compare $\tau_2$ and $\tau_X$, we need their distributions

- Distribution of $\tau_2$, namely $P_{y_2}(\tau_2 \in dt)$, is available from Dickson & Willmot (2005)

- Note $c = ap_2 - (1 - a)p_1 > 0$ because of condition (1)
  \[ \Rightarrow X_t \equiv x - ct - (1 - a)S_t \] is a strictly decreasing and $\tau_X \leq \frac{x}{c}$

- Realization of $\tau_X$ can be due to
  - Continuity, i.e. $X_{\tau_X} = 0$
  - Claims from $\{S_t\}_{t \geq 0}$, i.e. $X_{\tau_X} < 0$
• For $\tau_X < \tau_2$ (implying $\tau = \tau_1$), it is important to keep track of the level of $\{Y^1_t\}_{t \geq 0}$ at time $\tau_X$:

$$Y^1_{\tau_X} = \frac{a}{1 - a} Y^2_{\tau_X} - \frac{1}{1 - a} |X_{\tau_X}|$$

⇒ Requires

1. Joint distribution of $(|X_{\tau_X}|, \tau_X)$ pertaining to $\{X_t\}_{t \geq 0}$

2. Distribution of $\{Y^2_t\}_{t \geq 0}$ avoiding ruin enroute
Part 1 : Joint distribution of \((|X_{\tau_X}|, \tau_X)\)

- **Due to continuity** (i.e. \(|X_{\tau_X}| = 0\))
  
  - Point mass of \(P_x(\tau_X = \frac{x}{c}, |X_{\tau_X}| = 0) = e^{-\lambda_S \left(\frac{x}{c}\right)}\)
  
  - Density of \(\tau_X\) at \(t\) is, for \(0 < t < \frac{x}{c}\) :
    \[
    h_C(t|x) = c \sum_{n=1}^{\infty} e^{-\lambda_S t} \left(\frac{\lambda_S t}{n!}\right)^n f_{a,S}(x - ct)
    \]

- **Due to jumps** (i.e. \(|X_{\tau_X}| > 0\))

  - Density of \((|X_{\tau_X}|, \tau_X)\) at \((z, t)\) is, for \(0 < t < \frac{x}{c}\) and \(z > 0\) :
    \[
    h_J(z,t|x) = \lambda_S e^{-\lambda_S t} f_{a,S}(z + x - ct)
    + \lambda_S \sum_{n=1}^{\infty} e^{-\lambda_S t} \left(\frac{\lambda_S t}{n!}\right)^n \int_{0}^{x-ct} f_{a,S}(z + y) f_{a,S}(x - ct - y) \, dy
    \]
Part 2 : Distribution of \( \{Y_t^2\}_{t \geq 0} \) avoiding ruin enroute

- A point mass of \( Y_t^2 \) at \( u = y_2 + p_2 t \):

\[
P_{y_2} \left( \inf_{s \leq t} Y_s^2 > 0, Y_t^2 = u \right) = e^{-\lambda_L t}
\]

- Density of \( Y_t^2 \) at \( u < y_2 + p_2 t \):

\[
P_{y_2} \left( \inf_{s \leq t} Y_s^2 > 0, Y_t^2 \in du \right) = \zeta(y_2, t, u) \, du
\]

where \( \zeta(y_2, t, u) \) can be obtained from Landriault & Willmot (2009)
Main result

- Laplace transform of $\tau = \min(\tau_1, \tau_2)$ is $E_{(y_1,y_2)}[e^{-\beta \tau} 1_{\{\tau < \infty\}}]$.
  - If $\beta = 0$, then reduces to ruin probability $P_{(y_1,y_2)}(\tau < \infty)$.

- Useful to decompose as
  
  $$E_{(y_1,y_2)}[e^{-\beta \tau} 1_{\{\tau < \infty\}}] = E_{(y_1,y_2)}[e^{-\beta \tau_2} 1_{\{\tau_2 < \tau_X\}}]$$
  $$+ E_{(y_1,y_2)}[e^{-\beta \tau_1} 1_{\{\tau_X < \tau_2, \tau_1 < \infty\}}]$$

- 1st term is easy:
  
  $$E_{(y_1,y_2)}[e^{-\beta \tau_2} 1_{\{\tau_2 < \tau_X\}}] = \int_0^\infty e^{-\beta t} P_x(\tau_X > t) P_{y_2}(\tau_2 \in dt)$$
- 2nd term represents the case $\tau_X < \tau_2$ in which $\tau = \tau_1$
  - Requires $E_{y_1} \left[ e^{-\beta \tau_1} 1_{\{\tau_1<\infty\}} \right]$ for $\{Y_{t}^{1}\}_{t \geq 0}$ (Gerber & Shiu (1998))
  - It is the sum of following 6 contributions
    (i.e. 2 cases from $\{X_{t}\}_{t \geq 0}$ times 3 cases from $\{Y_{t}^{2}\}_{t \geq 0}$)
1. Ruin in \( \{X_t\}_{t \geq 0} \) by continuity

(a) No claims at all from both \( \{S_t\}_{t \geq 0} \) and \( \{L_t\}_{t \geq 0} \) within \( (0, \tau_X) \)

(i.e. Point mass of \( \tau_X \) at \( \frac{\theta}{c} \); and point mass of \( Y^2_f \) at \( y_2 + \frac{\beta}{\lambda} \))

\[
e^{-(\beta + \lambda_S + \lambda_L)\tau} \quad E_{\frac{a}{1-a}}(y_2 + \frac{\beta}{\lambda} \frac{x}{c}) \left[ e^{-\lambda_1 \tau} \mathbf{1}_{\{\tau_1 < \infty\}} \right]
\]
(b) No claims from \( \{S_t\}_{t \geq 0} \); at least one from \( \{L_t\}_{t \geq 0} \) within \((0, \tau_X]\)

(i.e. Point mass of \( \tau_X \) at \( \frac{x}{t} \); density of \( Y_{x2}^2 \in du \))

\[
\int_0^{y_2+\frac{x}{t}} e^{-\left(\beta+\lambda_K\right)\frac{x}{t}} E_1 \left( \frac{a}{u} \right) \left[ e^{-\beta \tau_1} 1_{[\tau_1 < \infty]} \right] \zeta \left( y_2, \frac{x}{t}, u \right) du
\]
(c) At least one claim from \( \{S_t\}_{t \geq 0} \); none from \( \{L_t\}_{t \geq 0} \) within \((0, \tau_X]\) (i.e. Density of \( \tau_X \in dt \); point mass of \( Y_t^2 \) at \( y_2 + p_2t \))

\[
\int_0^\infty e^{-(\beta + \lambda L)t} E_{1-a(y_2+p_2t)} \left[ e^{-\beta \tau_1 1_{\{\tau_1 < \infty\}}} \right] h(t|x) \, dt
\]
(d) At least one claim from both \( \{S_t\}_{t \geq 0} \) and \( \{I_t\}_{t \geq 0} \) within \( (0, \tau_X) \)

(i.e. Density of \( \tau_X \in dt \); density of \( Y_t^2 \in du \))

\[
\int_0^\pi e^{-\beta t} \left( \int_0^{2y_2 + y_2^2} E_{\frac{\alpha}{1-\alpha}u} \left[ e^{-\beta t} 1_{\{\tau_1 < \infty\}} \right] \zeta(y_2, t, u) \, du \right) h_C(t \, x) \, dt
\]
2. Ruin in $\{X_t\}_{t \geq 0}$ by jumps

(a) No claims from $\{L_t\}_{t \geq 0}$ within $(0, \tau_X]$

(i.e. Density of $\tau_X \in dt$; point mass of $Y_t^2$ at $y_2 + p_2 t$)

$$\int_0^\infty e^{-(\beta + \chi_t)} \left( \int_0^\infty \left( e^{-\frac{\alpha(y_2 + p_2 t)}{a(y_2 + p_2 t)}} \cdot \frac{1}{1 - a} \cdot \left[ e^{-\beta \tau} \mathbb{1}_{\{\tau < \infty\}} \right] h(z, t, x) \, dz \right) \, dt$$

$$\int_0^\infty e^{-(\beta + \chi_t)} \left( \int_{a(y_2 + p_2 t)}^\infty \left( e^{-\frac{\alpha(y_2 + p_2 t)}{a(y_2 + p_2 t)}} \cdot \frac{1}{1 - a} \cdot \left[ e^{-\beta \tau} \mathbb{1}_{\{\tau < \infty\}} \right] h(z, t, x) \, dz \right) \, dt$$

$\Delta : y_2 \rightarrow \frac{1}{a} \cdot y_1$
(b) At least one claim from \( \{L_t\}_{t \geq 0} \) within \( (0, \tau_X) \)
(i.e. Density of \( \tau_X \in dt \); density of \( Y^2_t \in du \))

\[
\int_0^\infty e^{-\beta t} \int_0^{\nu_2 + \mu_2k} \left( \int_0^{au} \frac{e^{-\frac{z}{\mu t}} - 1}{1 - \frac{z}{\mu t}} \right) e^{-\beta \tau_1 1_{(\tau_1 < \infty)}} h_f(\tau_1, t|\nu) \, dz \\
\left( \int_{au}^{\infty} h_f(\tau_1, t|\nu) \, d\tau_1 \right) \xi(y_2, t, u) \, du \, dt
\]
Concluding remarks

- Once $E_{(y_1,y_2)} \left[ e^{-\beta \tau} 1_{\{\tau < \infty \}} \right]$ has been determined, optimization problem can be performed, e.g. maximization of the joint survival probability.

- Optimization may be done with respect to
  - Parameter $a$ of the proportional reinsurance contract
    $\Rightarrow$ Optimal reinsurance problem

  - Initial capitals $(y_1, y_2)$ of the companies, subject to constant total capital $y_1 + y_2 = K$
    $\Rightarrow$ Optimal capital allocation problem
THE END
Thank you!