PRICE VS. RESERVE REGULATION
CONDITIONED BY SOLVENCY REQUIREMENTS
IN THE COLLECTIVE RISK MODEL

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ABSTRACT
The balance of policy prices vs. reserves conditioned by solvency requirements is considered, aiming analysis of an insurer as subject of price competitive insurance market. The intrinsic relationship between the policy prices and the risk reserves, and the influence of this balance on solvency of individual insurance business are formalized in the framework of the collective risk model. Different approaches to tuning prices vs. reserves conditioned by solvency requirements expressed in terms of the probability of ruin within finite time and of the ultimate probability of ruin, based on (a) exact numerical technique, (b) new approximations, and (c) simulation, are discussed.

KEYWORDS

1. INTRODUCTION
Insurance is a method of coping with risks, while the object of the theory of risk is to give a mathematical analysis of random fluctuations in an insurance business and to discuss the various means of protection against their adverse effects. The insurer is subject of competitive insurance market where the policy prices are among the primary influences. The prices, the reserves, and solvency of each individual business are inseparable. Both practical and theoretical aspects of the solvency of insurer and of the respective price and capital requirements are clearly in evidence (see e.g., Section 14.6 of [9]). Developing the system of solvency testing, the theory of risk is applied, bringing to the forefront either analytical or simulation approach.

The traditional models of the risk theory have certain recognized shortcomings which arise from structural deficiencies rather than merely from the technical restrictions, though the latter attracted more attention in literature. One of these shortcomings is making insufficient allowance for the interdependence between the premium rate and the capital size which makes some important business aspects overlooked, even if the attention is focused on testing the financial position of the insurer at annual intervals and inflation, return on investments, market cycles, and certain other premises are ignored.

In this paper we discuss a generalization of the collective risk model binding together the initial capital and the risk premium rate so that the relative safety loading becomes decreasing, as the initial capital grows (see [12], [13]). Unlike making the risk premium rate variable and dependent on the current value of the risk reserve (see e.g., [3], [4], [5]), our approach reflects the fact that supervision is usually implemented by testing the financial position of each insurer at regular intervals, normally annually. It might rouse the company to scheduled actions such as annual change of prices and reserves. The disadvantages of such a "scheduled" price vs. solvency optimization is substantially compensated by the fact that the solvency requirements are expressed in terms of the finite time ruin probabilities. In a sense it preserves the risk reserve all along the accounting period from being frequently too small.

In Section 2 we revise an example by Seal, applying exact numerical techniques. Our consequential aims are (1) to counter his contention formulated in [15], [16] which regards this example "an emphatic illustration of the poverty of asymptotic numerical approximations for the practical man" and (2) to develop the adequate asymptotic results. In Section 3 we develop the examination by Seal and demonstrate an example of crude and delicate tuning of price vs. reserve conditioned by solvency.

In Section 4 we revise exact numerical technique applied to finite time ruin probabilities.
In Section 5 we suggest an adjustment of the collective risk model where the safety loading $\tau$, depends on the initial capital $u$ and propose new approximations for the probabilities of ruin, as $u$ increases. Unlike the standard Cramér-Lundberg approximations, they are suitable for the framework considered in Sections 2 and 3. It overcomes the shortcoming perceived by Seal. Though approximations are more flexible than the exact numerical technique, we are fully sharing the opinion that any formula approach inevitably has rigid limitations. A reputed contender of the formula approach is simulation. In Section 6 we briefly consider the stochastic bundle approach described in Section 14.6 of [9] and point out several complications which emerge in the modified risk model.

2. AN OBSERVATION BY SEAL

The theory of risk focuses its attention on the reference insurer through the outflow process, looking first at claim numbers, then at the distribution of claim sizes and finally putting these two together into an aggregate claim amounts process. The income process, which is the initial capital plus premium income, is introduced in a rather simple way, growing linearly in time with a constant intensity $c$. The resulting surplus process of the insurance business is generated as initial capital plus premium income minus outflow.

This bird's eye view has been formalized in the notion of the collective risk model which remains up to now one of the main actuarial premises concerned with final business results. Ignoring individual policies, this model views an insurance business as a whole: claims occur from time to time and are settled by the company, while on the other hand the company receives a continuous flow of risk premiums from the policyholders.

Mathematically, the surplus process at any operational time $t$ is described by the risk reserve process $R(t) = u - \sum_{i=1}^{N(t)} Y_i + ct$ starting at time $t = 0$, where $N(t)$ is the number of claims occurred up to time $t$, $u > 0$ is the initial risk reserve. The insurance company is supposed to pay premiums as they are received into a risk reserve and $c > 0$ is the risk premium rate, $\{T_i\}_{i\geq1}$ are (i.i.d.) interclaim times and $\{Y_i\}_{i\geq1}$ are (i.i.d.) amounts of claims.

Supervision is usually implemented by testing the financial position of each insurer at regular intervals, normally annually. In practice, adverse fluctuations often occur in consecutive years, giving rise to considerable accumulation of losses. This may not be revealed by the analysis limited to one calendar year. But the short term time horizon, in most cases one year, is the fundamental building block for the long-term analysis.

Mathematically, the probability of ruin $\psi(t,u) = P\{\inf_{s \leq t} R(s) < 0\}$ within the time interval $(0,t)$, which particular case is the probability of ultimate ruin $\psi(u) = \psi(+\infty,u)$, is an important scientific paradigm: within the collective model the solvency requirements are similar to the constraint to keep the probability of ruin at a certain prescribed level throughout the accounting period.

Two reputed influences are as follows. First, the insurer typically needs to charge loaded premiums sufficient for business to take its normal course over a long time. The amount $\tau = cET_1/\text{E}Y_1 - 1$, called the relative safety loading, reflects this need. Indeed, since $cT_i$ is the premium acquired and $Y_i$ is the claims amount paid out on the $i$-th "step" which is the time between $(i - 1)$-th and $i$-th claims, the condition $\tau > 0$ means that successful "steps" are persistent. The opposite, $\tau < 0$, means that successful "steps" are rare and, in total, the business is trading unfavorably and is liable to ruin. Second, the insurers are required by law to keep the necessary reserves to safeguard solvency and, in particular, to meet early claims.

Operating in a competitive insurance market, the insurer might be interested to establish a balance of price (by decreasing the loading $\tau$) vs. reserve $u$, aiming at the legal or desired level of solvency expressed by the probability $\psi(t,u)$ within the accounting period $(0,t)$.

In our opinion, an early attempt to examine this balance was made by Seal. In [15] he considers the collective risk model where exponential claims are occurring as a Poisson process. He takes unit Poisson intensity, so that the unit of time is the expected interval between claim occurrences, and unit exponential distribution parameter, so that the average individual claim size is the monetary unit. In [16] he sharpens his outlines by considering constant unit claims.

Basing on an exact formula which Seal attributed to Arfwedson [1], for no-loading $\tau = 0$ and for $\tau = 0.1$ which is a 10% risk loading in this model, he calculates numerically the probability of non-ruin within the interval $(0,t)$, $U(t,u) = P\{\inf_{0 \leq s \leq t} R(s) \geq 0\}$.

Seal's analysis (see [16], pp. 128-129; we adapted it to suit the model in [15]) of the calculations which we partly reproduced from Tables 2 and 3 in [15] in our Tables 1 and 2 respectively, is as follows: "with no risk loading — which is known to lead to ultimate ruin whatever (finite) value $u$ has — and an initial risk-reserve of as little as ten times the mean unit claim there is still
an 70.5% chance of not being ruined during an interval within which 50 claims are expected."

"One sees how far 50 is from infinity so far as the probability of ruin is concerned!" exclaimed Seal. Table 2 indicates that with $u$ as large as 110 there is a 99.8% chance of not being ruined in an interval during which 600 claims are expected.

Seal's conclusion is that "these are emphatic illustrations of the poverty of asymptotic numerical approximations for the practical man. The real value of risk theory is, we believe, to the entrepreneur just starting out in business with a casualty insurance company. It is the early claims that worry him not those that occur after he has built a successful business!"

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### Table 1. Probability of non-ruin $U(t, u)$ for the Poisson-exponential model, $\tau = 0.1$

### Table 2. Probability of non-ruin $U(t, u)$ for the Poisson-exponential model and no risk loading

3. PRICE VS. RESERVE BALANCE IN THE RISK MODEL: AN EXAMPLE

In our opinion, a sensible extension of the considerations by Seal is to introduce a tuning of price (or rather its safety loading component which is flexible) conditioned by solvency expressed through the ruin probabilities, first in a crude and then in a more refined way. The first approach consists in the gradual reduction of the safety loading, roughly sketched as passing from Table 1 to Table 2, looking back at $\psi(u)$ and at $u$, the second consists in finer
time considerations, since the balance of magnitudes of $u$ and $\tau$ is basically estimated, and applies more insight at $\psi(t, u)$.

Our first concern is an example of what we called crude tuning. As Seal in [15], we consider the Poisson-exponential risk model with unit Poisson intensity and unit parameter of the exponential claim size distribution. Assume rather arbitrarily the starting value of the initial risk reserve to be $u_0 = 10$ and the respective value of the ultimate ruin probability to be $\psi(u_0) = 0.1$ (in practice they could be those values which have been applied on the preceding accounting period). Evidently, using the explicit formula (1) below, the safety loading calculated numerically must be $\tau = 0.26113$.

The actuary is faced with the problem of analyzing the following balance: to decrease the safety loading (i.e., to reduce prices) respecting solvency requirements e.g., keeping the ultimate ruin probability between 0.1 and 0.05 (too small values might be considered superfluous and unrealistic), by means of a sensible increase of the initial risk reserve $u$, say up to the values in between $u = 20$ and $u = 60$.

Once the exact result as in formula (1) below is available, one can easily construct a lower bound for $\tau$ as function of $u$: for each value of $u$ it is a solution of the equation $(1 + \tau)^{-1} \exp(-u\tau (1 + \tau)^{-1}) = 0.05$ w.r.t. $\tau$. But the form of $\tau$ as function of $u$ and in particular the rate of decrease of $\tau$, as $u$ increases, is hard to discover by this procedure. Moreover, the link between the lower bound and the initial values which were $u_0 = 10$ and $\psi(u_0) = 0.1$ in our example, is rather implicit. This approach appears highly sensitive to any deviation from the original model assumptions.

The standpoint might be revised: one may define certain parametric families e.g., $\tau_u = au^{-k}$, or $\tau_u = a(\ln u)^{-k}$, $a, k > 0$, reflecting more or less aggressive price policy and aiming to evaluate the parameters of these families according to certain solvency and capital requirements. In our opinion, this approach sheds more light on the process of decision making.

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**Fig. 1.** Upper family of graphs: constant safety loading $\tau_u^{(0)} = 0.26113$ (thick line), variable safety loadings $\tau_u^{(1)} = 0.31636u^{-1/12}$ (dashed line with long segments), $\tau_u^{(2)} = 0.68158u^{-5/12}$, $\tau_u^{(3)} = 1.46842u^{-9/12}$, $\tau_u^{(4)} = 2.15534u^{-11/12}$ (dotted line). Lower family of graphs: the ultimate ruin probabilities $\psi_k(u) = (1 + \tau_u^{(k)})^{-1} \exp(-u\tau_u^{(k)} (1 + \tau_u^{(k)})^{-1})$, conditioned by $\psi_k(10) = 0.1$, $k = 0, 1, 2, 3, 4$. 

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Fortunately, it fits beautifully with the refined asymptotical approach, as \( u \) increases, which will be developed for the general claim size and interclaim times distributions in Section 5.

In Fig. 1 where the upper family of graphs represents the safety loading whilst the lower represents the ruin probabilities, we present five rates \( \tau_u^{(0)} = 0.26113, \tau_u^{(1)} = 0.31636 u^{-1/12}, \tau_u^{(2)} = 0.68158 u^{-5/12}, \tau_u^{(3)} = 1.46842 u^{-9/12}, \tau_u^{(4)} = 2.15534 u^{-11/12} \) and the respective ultimate ruin probabilities \( \psi_k(u) = (1 + \tau_u^{(k)})^{-1} \exp(-u\tau_u^{(k)}(1 + \tau_u^{(k)})^{-1}) \), conditioned by \( \psi_k(10) = 0.1, k = 0, 1, 2, 3, 4 \). One sees that only the rate \( \tau_u^{(4)} \) is close to satisfying our original requirements. However, if the solvency requirements are lowered to the value 0.025, the attention should be also paid to \( \tau_u^{(3)} \).

Delicate tuning refines these considerations by allowing for the finite nature of the relevant time interval. Indeed, we based our previous considerations on the ultimate ruin probability \( \psi(u) \) which is merely an upper bound for \( \psi(t, u) \), whenever \( t \) is finite. This idea was basic in Section 2 where the safety loading was assumed zero but non-ruin remained rather probable for certain finite \( t \).

In Fig. 2 below we present the probabilities \( \psi(t, u) \) of ruin within finite time \( t \) for the rates \( \tau_u^{(2)} = 0.68158 u^{-5/12} \) and \( \tau_u^{(3)} = 1.46842 u^{-9/12} \), and for \( u = 20 \) and 30, calculated numerically by applying the exact formula (2) below. Evidently, the values \( \psi_2(20) = 0.0317, \psi_3(20) = 0.0122, \psi_3(30) = 0.0589 \) are a considerable undershoot as \( \psi_2(t, 20), \psi_2(t, 30) \) and \( \psi_3(t, 20), \psi_3(t, 30) \) for e.g., \( t = 200 \) are concerned.

4. EXACT NUMERICAL METHODS

The implementations by Seal and the examples of Sections 2 and 3 are crucially based on the exact numerical methods. Seal's was based on a numerical integration worked out for the Poisson claims incidence. It is described in details in his books [14], [17]. The calculations of Section 3 apply the famous Cramér exact formula (1), and the formula (2) which can be found for \( \mu = c = 1 \) in [2], [7], and in [11].

ASSERTION. Let the i.i.d. sizes of claims \( \{Y_i\}_{i \geq 1} \) and the i.i.d. interclaims times \( \{T_i\}_{i \geq 1} \) be mutually independent, \( Y_1 \) be exponential with a positive parameter \( \mu \) and \( T_1 \) be exponential with a positive parameter \( \lambda \). In this Poisson-exponential risk model

\[
\psi(u) = \frac{\lambda}{c\mu} \exp(-u(c\mu - \lambda)/c)
\]

and

\[
\psi(t, u) = \psi(u) - \frac{1}{\pi} \int_0^\pi f(x; t, u)dx
\]

for any \( u > 0 \), where

\[
f(x; t, u) = (\lambda/c\mu)(1 + \lambda/c\mu - 2\sqrt{\lambda/c\mu \cos x})^{-1} \\
\times \exp\left(u (\sqrt{\lambda/c\mu \cos x} - 1) + t\mu \left(2\sqrt{\lambda/c\mu \cos x} - \lambda/c\mu - 1\right)\right) \\
\times \left[\cos \left(u\sqrt{\lambda/c\mu \cos x} + 2x\right)\right].
\]

It is worth mentioning that (2) was derived in [11] as a corollary of the following result which supplies an exact numerical technique for a non-Poisson claims arrival.

THEOREM 1. Let the sizes of claims \( \{Y_i\}_{i \geq 1} \) and the interclaims times \( \{T_i\}_{i \geq 1} \) be i.i.d. and mutually independent. Let \( Y_1 \) be exponential with a positive parameter \( \mu \) and the Laplace transform of \( T_1 \) be \( \gamma(\alpha) = E e^{-\alpha T_1}, \alpha > 0 \). Then for any \( u > 0 \)

\[
\alpha \int_0^\infty e^{-\alpha t} \psi(t, u) dt = y(\alpha) \exp\{-u\mu(1 - y(\alpha))\}, \quad \alpha > 0,
\]

where \( y(\alpha) \) is a solution of the equation

\[
y(\alpha) = \gamma(\alpha + c\mu(1 - y(\alpha))), \quad \alpha > 0.
\]
5. APPROXIMATIONS

Any formula approach inevitably has its limitations. The limitations of the exact methods are particularly restrictive. Therefore the risk problem was addressed by asymptotic methods. The most famous are the Cramér-Lundberg approximations like

\[ \lim_{u \to \infty} e^{xu} \psi(u) = C, \]

(3)

where \( C \) is the Cramér-Lundberg constant, and

\[ \lim_{u \to \infty} \sup_{t \geq 0} |\psi(t, u)e^{xu} - C \Phi_{(m_u, D^u)}(t)| = 0, \]

(4)

where \( \Phi_{(m_u, D^u)}(t) \) is the Normal probability distribution function (see e.g., [6]).

Going back to Section 2, the discontent of Seal who declared poverty of the asymptotic numerical approximations clarifies much when we conceive the major restriction of (3) and (4) which in fact is that we must have \( c \) constant, as \( u \) is growing. It means e.g., that we are allowed to apply (3) and (4) for the approximation of the thick line in Fig. 1, but not allowed to apply these results for the approximation of the dashed ones. In our opinion, Seal pointed implicitly to this gap between the asymptotical and the exact numerical methods by an extreme case of no-loading, or extremely small \( c \), and rather large \( u \). However, we are blaming certain deficiency of a particular risk model, rather than the asymptotic approximations approach per se. As a conclusion, we have to refine the collective risk model and to extend the approximations (3) and (4) to make them valid within this new, refined, model.

Introduce \( \tau_u > 0 \) depending on \( u \) and such that \( \tau_u \to 0 \) monotonically, as \( u \) grows to infinity, starting from a certain positive value \( u_0 \). The particular choice of \( \tau_u \) depends on external factors and the motivation deserves a separate discussion outside the scope of this paper.

For i.i.d. random vectors \((T_i, Y_i), i = 1, 2, \ldots\), define a series of the risk reserve processes \( R_u(t) = u - \sum_{i=1}^{N(t)} Y_i + c_u t \), where the premium rate is \( c_u = (1 + \tau_u) \) \( EY_1/ET_1 \). For \( i = 1, 2, \ldots \) introduce i.i.d. random variables \( X_u, T_u \), and put \( S_{n,u} = \sum_{i=1}^{n} X_u, U_n = \sum_{i=1}^{n} T_u \).

For the p.d.f. \( B_u(x, y) = \{X_{u,1} \leq x, T_{1} \leq y\} \), and for a positive solution \( x_u \) of the Lundberg equation

\[ \mathbf{E} \exp(x_u X_{u,1}) = 1 \]

introduce an associate p.d.f. by \( \overline{B}_u(dx, dy) = e^{x_u x} B_u(dx, dy) \). Introduce the associated sequence \( \{(X_{u,1}, T_{u,1})\}_{i \geq 1} \) of i.i.d. random vectors having the p.d.f. \( B_u(x, y) \), and put \( \overline{S}_{n,u} = \sum_{i=1}^{n} X_{u,1}, \overline{U}_{n,u} = \sum_{i=1}^{n} T_{u,1} \).

Put

\[ \nu^{i,j} = \mathbf{E} X_{u,1}^{i} T_{1}^{j}, \nu^{i}_u = \mathbf{E} X_{u,1}^{i}, \nu^{j}_u = \mathbf{E} T_{1}^{j}, i, j = 0, 1 \ldots. \]

As usual, asterisk denotes convolution. Introduce

\[ m_u = \nu^{0,1}/\nu^{1,0}_u, \quad D_u^2 = (\nu^{0,1})^2 \nu^{2,0}_u - 2 \nu^{1,0}_u \nu^{0,1}_u \nu^{1,1}_u + (\nu^{0,1})^2 \nu^{0,2}_u/(\nu^{1,0}_u)^3, \]

\[ C_u = \frac{1}{\nu^{1,0}_u} \exp \left( - \sum_{n=1}^{\infty} \frac{1}{n} [\mathbf{P}(S_{n,u} > 0) + \mathbf{P}(S_{n,u} = 0)] \right). \]

In the aggregate with the approximations \( x_u = a_1 \tau_u + a_2 \tau_u^2 + \ldots, m_u = m_1 \tau_u^{-1} + m_0 + m_1 \tau_u + \ldots, D_u^2 = v_{-3} \tau_u^{-3} + v_{-2} \tau_u^{-2} + v_{-1} \tau_u^{-1} + \ldots, \) and \( C_u = 1 + c_1 \tau_u + \ldots \), as \( u \to \infty \) (see Theorems 3.1, 3.2, 3.3, and 4.1 in [12]), the following "scheme of series" counterpart of the approximations (3) and (4) constitutes the main result of this Section.
Theorem 2. In the risk model with the p.d.f. $B_{Y,T}$ having a bounded density w.r.t. Lebesgue
meaure on $R^2$, assume that $c_u = (1 + \tau_u)EY_1/ET_1$ with $\tau_u \geq u^{-5/12}$ and for a sufficiently
large $u_0 > 0$.

1. $\sup_{u > u_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\beta_u(t_1, t_2)|^p dt_1 dt_2 < \infty,$
2. $\sup_{u > u_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\rho_u(t_1, t_2)|^p dt_1 dt_2 < \infty$ for some $p \geq 1,$
3. $D^2 = \lim_{u \to \infty} D^2_u > 0.$

Then

$$\sup_{t \geq 0} |\psi(t, u) - C_u e^{-\kappa_u u} \Phi(m_u, u, D^2_u)(t)| = O((\tau_u u)^{-1/2} e^{-\kappa_u u}),$$

(5)
as $u \to \infty$.

The proof of Theorem 2 is far from being a straightforward extension of the renewal arguments
used by von Bahr in [6] to obtain (4). Indeed, the proof of (4) was based on a theorem of Stam on
the asymptotics of a two-dimensional renewal measure (r.m.). The behavior of a r.m. generated
by non-identically distributed random variables (r.v.) is known to be much more complicated
(see e.g., [18]). The situation is even worse for the r.m. generated by series of independent r.v.
as in our particular case. The renewal approach encounters therefore huge technical difficulties.
In contrast, the approach developed in [10] can be extended to the scheme of series.
The following result is a corollary of Theorem 2.

Theorem 3. In the Poisson-exponential risk model with $Y_1$ exponential with a positive pa-
rameter $\mu$ and $T_1$ exponential with a positive parameter $\lambda$ assume that $c_u = (1 + \tau_u)\lambda/\mu$ with
$\tau_u \geq u^{-5/12}.$ Then

$$\sup_{t \geq 0} |\psi(t, u) - C_u e^{-\kappa_u u} \Phi(m_u, u, D^2_u)(t)| = O((\tau_u u)^{-1/2} e^{-\kappa_u u}),$$

(6)
as $u \to \infty$, where $\kappa_u = \mu \tau_u/(1 + \tau_u), \ m_u = \mu/(\lambda \tau_u(1 + \tau_u)), \ D^2_u = 2\mu/(\lambda^2 \tau^2_u),$ and $C_u = 1/(1 + \tau_u).$

Unlike (3) and (4), the results (5) and (6) are suitable to approximate the dashed lines in Fig. 1.
Examining further the accuracy of (5) and (6) we present in Fig. 2 below a numerical example
where the finite time ruin probabilities $\psi(t, u)$ are involved. We compare the exact values
calculated numerically by (2) and the approximations calculated by (6) for $\tau_u(2) = 0.68158 u^{-5/12}$
and $\tau_u(3) = 1.46842 u^{-9/12}, u = 20$ and $30$.

It is worth noting that if a better accuracy is desired, as $u$ grows, (5) and (6) should be refined
e.g., by the asymptotic expansions like in [10]. The same should be done to strengthen the approximations while only moderate or even small values of $u$ are available.

We finish this Section by a remark that Theorem 2 is perfectly eligible for various generalizations
of the Poisson-exponential model though there are no exact formulas like (1) and (2), and even
no exact numerical technique as in Theorem 1, or in the books [14] and [17].

Let the (i.i.d.) amounts of claims $\{Y_i\}_{i \geq 1}$ and the (i.i.d.) inter-occurrence times $\{T_i\}_{i \geq 1}$
be mutually independent. Let $Y_1$ be Gamma with the shape parameter $\beta$ and the scale parameter
$\lambda$, all these parameters being positive. Then (see Example 3.1 in [12]) $a_1 = 2\alpha \mu/(\beta(\alpha + \beta)), a_2 = -4\alpha \mu(2\alpha + \beta)/(3\beta^2(\alpha + \beta)^2), a_3 = 2\alpha \mu(14\alpha^2 + 17\alpha \beta + 5\beta^2)/(9\beta^3(\alpha + \beta)^3), m_0 = \alpha \mu/\lambda, m_0 = -2\alpha \mu(2\alpha + \beta)/(3\lambda^2(\alpha + \beta)), m_1 = 2\alpha \mu(2\alpha + \beta)/(3\lambda^2(\alpha + \beta)), m_2 = \alpha \beta(\alpha + \beta)\mu/\lambda^2,$
$v_2 = 0,$ which yields the approximations (as segments of the series expansions mentioned
above) for $\kappa_u, m_u, D^2_u, C_u$. Furthermore, the conditions of Theorem 2 are satisfied and the
approximation (5) is valid.

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6. SIMULATION

The idea of the simulation approach in the standard risk model within the finite time horizon $t$ is based on simulation of the risk reserve as the difference between incoming premiums and outgoing claims (see e.g., [9], Section 1.3, Fig. 1.3.2) and on the resulting derivation of a stochastic bundle (see e.g., [9], Section 13.2, Fig. 13.1.5) comprising $N$ independent risk reserve realizations. This direct approach can be refined e.g., by applying the importance sampling idea (see e.g., [3]), but the essence remains as above.

Daykin et al. [9] described the basic idea as follows: "Once a ruin barrier has been defined, the simulated paths of the course of business which pass below the barrier are counted as ruin. Then the ratio of the number of ruins $n_{\text{ruin}}$ to the total number $N$ of realizations in the simulation gives an estimate of the probability of ruin. A visual inspection of the bundle of simulated paths can provide a good idea of the risk structure." ([9], p. 361).

In our case the ruin barrier is fixed and zero (the event of ruin is defined as $R(s) < 0$ for a certain time moment $s$). Instead of the ruin barrier being defined as stated above, the initial capital $u$ such that the ruin probability $\psi(t, u)$ remains within certain limits for a given $c$ is to be chosen. Evidently, these two approaches are equivalent if the premium rate $c$ is constant and independent of $u$. The equivalence is due to the fact that for such $c$ the shape of the simulated bundle is invariant as the initial capital is shifted.
When the premium rate $c$ depends on the initial capital $u$, as in Sections 3 and 5, shifting $u$ implies shifting $c$, which produces deformation of the entire bundle of simulated paths (see Fig. 3). The evaluation of the solvency margin by the mere "up and down shifting" of a single bundle of simulated paths is no longer possible and each new trial requires a fresh evaluation of the whole bundle of simulated paths. It might dramatically increase the complexity of the simulation.

![Fig. 3. Imitated stochastic bundles for $\gamma_u^{(n)} = 2.15534 u^{-11/12}$, $u = 10$ and 50, $\lambda = \mu = 1$.](image)

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